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Analysis of the Chen's and Pham's Software Reliability Models

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Abstract: *In this paper we study the Hausdorff approximation of the shifted Heaviside step function* $h_{t_0}(t)$ *by sigmoidal functions based on the Chen's and Pham's cumulative distribution functions and find an expression for the error of the best approximation. We give real examples with data provided by IBM entry software package and Apache HTTP Server using Chen's software reliability model and Pham's deterministic software reliability model. Some analyses are made.*

Keywords: *Two-parameters Chen's cumulative distribution function* (*2Ccdf*)*, two-parameters Pham's cumulative distribution function* (*2Pcdf*)*, Hausdorff approximation, upper and lower bounds, supersaturation.*

1. Introduction

Within the hierarchical models in the procedure for quantifying the quality of software products, an important role is played by the computational method based on the theoretical and empirical dependencies (usually at an early stage in their development), statistical data accumulated during tests, exploitation and the accompaniment of the program product.

An important measure of reliability assessment (completeness, accuracy and consistency) is the asymptotic metrics.

Over the last decade, researchers have devoted attention to constructing model functions from the field of debugging theory to approximate a wide variety of dataset derived from practical software testing. It turns out that the generally accepted characteristics (confidence intervals and confidential bounds) are not enough to reveal the intrinsic properties of classic and newer models. In this connection, we have placed the modest task of offering the specialists working on this issue the new feature, which we conditionally call "supersaturation". The Hausdorff distance is an appropriate metric for studying the feature. The model task we set up is the approximation of the shifted Heaviside function with some cumulative functions with respect to Hausdorff distance.

In [3], we pay particular attention to both deterministic approaches and probability models for debugging theories. Some of the existing cumulative distributions (Gompertz–Makeham, Yamada-exponential, Yamada-Rayleigh, Yamada-Weibull, transmuted inverse exponential, transmuted Log-Logistic, Kumaraswamy-Dagum and Kumaraswamy Quasi Lindley) are considered in the light of modern debugging theory.

In this note we study the Hausdorff approximation of the shifted Heaviside step function $h_{t_0}(t)$ by sigmoidal functions based on the Chen's [6] and Pham's [2] cumulative distribution functions and find an expression for the value of the best approximation. We propose a software modules (intellectual properties) within the programming environment CAS Mathematica for the analysis.

In Conclusion we illustrate how important is to know the value of the best Hausdorff approximation which compulsory accompanies practical components such as confidence intervals and confidence bounds.

The models have been tested with real-world data.

2. Preliminaries

Many probability distributions haven been introduced to analyze real datasets with bathtub failure rates.

Definition 1 [6]. The Chen probability density function is given as follows:

$$f(t) = \lambda\beta t^{\beta-1}e^{t^{\beta}+\lambda(1-e^{t^{\beta}})},\ t>0,$$

where $\lambda > 0,\ \ \beta > 0.$ The two-parameters Chen's cumulative distribution function (2Ccdf) is

$$M(t) = 1 - e^{\lambda(1-e^{t^{\beta}})}. \tag{1}$$

Definition 2 [2]. The Pham probability density function is given as follows:

$$f_1(t) = \alpha \ln a t^{\alpha-1} a^{t^{\alpha}} e^{1-a^{t^{\alpha}}},\ \ t>0,$$

where $a > 0,\ \ \alpha > 0.$ The two-parameters Pham's cumulative distribution function (2Pcdf) is

$$M_1(t) = 1 - e^{1-a^{t^{\alpha}}}. \tag{2}$$

Definition 3 [4, 5]. The Hausdorff distance (the H-distance) $\rho(f,g)$ between two interval functions f and g on $\Omega \subseteq \mathbb{R}$ is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{\sup_{A\in F(f)}\inf_{B\in F(g)} \|A-B\|, \sup_{B\in F(g)}\inf_{A\in F(f)} \|A-B\|\},$$

wherein $\|.\|$ is any norm in $\mathbb{R}^2$, e.g., the maximum norm $\|(t,x)\| = \max\{|t|,|x|\}$; hence the distance between the points $A=(t_A,x_A),\ \ B=(t_B,x_B)$ in $\mathbb{R}^2$ is $\|A-B\| = \max(|t_A - t_B|, |x_A - x_B|).$

Definition 4. The shifted Heaviside function is defined by

(3) $$h_{t_0}(t) = \begin{cases} 0 & \text{if} \quad t < t_0, \\ [0,1] & \text{if} \quad t = t_0, \\ 1 & \text{if} \quad t > t_0. \end{cases}$$

3. Main results

3.1. A note on the Chen's software reliability model

We consider Chen's Software Reliability Model (CSRM):

(4) $$M^*(t) = 1 - e^{\lambda(1-e^{t^\beta})}$$

with

(5) $$t_0 = \left(\ln\left(1 + \frac{1}{\lambda} \ln 2 \right) \right)^{\frac{1}{\beta}}, \; M^*(t_0) = \frac{1}{2}.$$

The one-sided Hausdorff distance d between the Heaviside step function $h_{t_0}(t)$ and the sigmoid (4)-(5) satisfies the relation

(6) $$M^*(t_0 + d) = 1 - d.$$

The following theorem gives upper and lower bounds for d.

Theorem 1. Let

$$p = -e^{\lambda - e^{t_0^\beta}\lambda} = -\frac{1}{2},$$

$$q = 1 + e^{t_0^\beta + \lambda - e^{t_0^\beta}\lambda} t_0^{\beta-1} \beta\lambda = 1 + \frac{1}{2}\left(1 + \frac{1}{\lambda}\ln 2 \right)\left(\ln\left(1 + \frac{1}{\lambda}\ln 2 \right) \right)^{\frac{\beta-1}{\beta}} \beta\lambda.$$

For the one-sided Hausdorff distance d between h_{t_0} and the sigmoid (4)-(5) the following inequalities hold for:

$$2.1q > e^{1.05},$$

(7) $$d_l = \frac{1}{2.1q} < d < \frac{\ln(2.1q)}{2.1q} = d_r.$$

P r o o f. Let us examine the functions

(8) $$F(d) = M^*(t_0 + d) - 1 + d,$$

(9) $$G(d) = p + qd.$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).

In addition $G'(d) > 0$.

Further, for $2.1q > e^{1.05}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

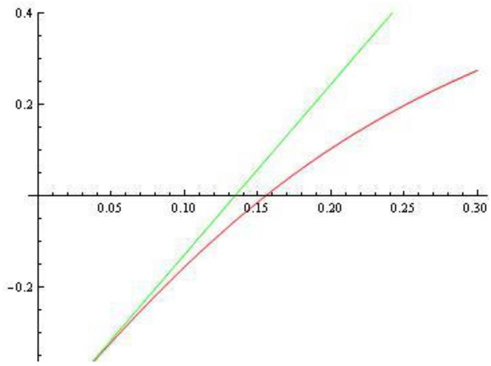


Fig. 1. The functions $F(d)$ and $G(d)$

The model (4)-(5) for $\beta = 2.1$, $\lambda = 10$, $t_0 = 0.276085$ is visualized on Fig. 2.
The model (4)-(5) for $\beta = 10$, $\lambda = 20$, $t_0 = 0.713243$ is visualized on Fig. 3.
The model (4)-(5) for $\beta = 17$, $\lambda = 28$, $t_0 = 0.80389$ is visualized on Fig. 4.

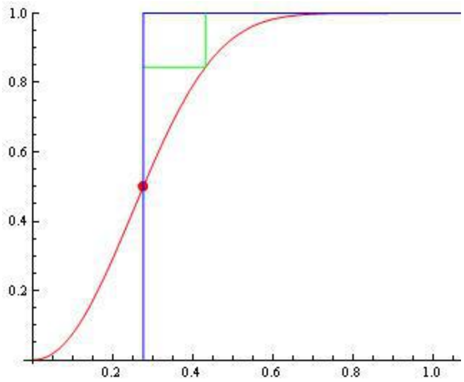


Fig. 2. The model (4)-(5) for $\beta = 2.1$, $\lambda = 10$, $t_0 = 0.276085$; H-distance $d = 0.155083$, $d_l = 0.12782$, $d_r = 0.262943$

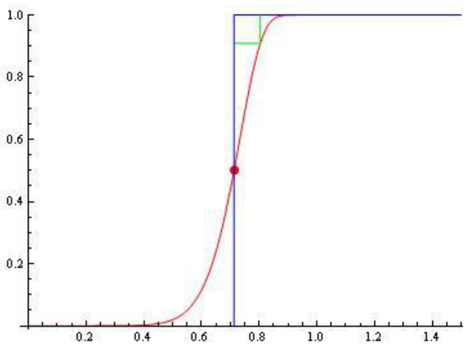


Fig. 3. The model (4)-(5) for $\beta = 10$, $\lambda = 20$, $t_0 = 0.713243$; H-distance $d = 0.090991$; $d_l = 0.0801348$; $d_r = 0.202264$

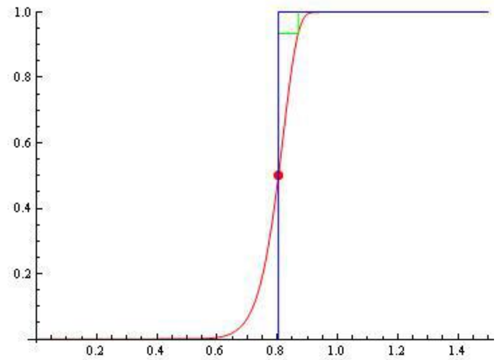


Fig. 4. The model (4)-(5) for $\beta = 17$, $\lambda = 28$, $t_0 = 0.80389$; H-distance $d = 0.065624$; $d_l = 0.0565612$; $d_r = 0.162468$

With above illustrated graphics we show the phenomenon “supersaturation”.

3.2. Numerical example

We examine the following data. (The small on-line data entry software package test data, available since 1980 in Japan [8], is shown in Table 1. For more details, see [2]).

Table 1. On-line IBM entry software package [8]

Testing time (day)	Failures	Cumulative failures
1	2	2
2	1	3
3	1	4
4	1	5
5	2	7
6	2	9
7	2	11
8	1	12
9	7	19
10	2	21
11	1	22
12	2	24
13	2	26
14	4	30
15	1	31
16	6	37
17	1	38
18	3	41
19	1	42
20	3	45
21	1	46

The fitted model (4)-(5)

$$M^*(t) = \omega\left(1 - e^{\lambda(1-e^{t^{\beta}})}\right),$$

based on the data of Table 1 for the estimated parameters $\omega = 46$, $\lambda = 0.01607264$, $\beta = 0.547711$, is plotted on Fig. 5.

```
Clear[λ]
Clear[β]
data = {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 7}, {6, 9}, {7, 11}, {8, 12}, {9, 19}, {10, 21}, {11, 22},
   {12, 24}, {13, 26}, {14, 30}, {15, 31}, {16, 37}, {17, 38}, {18, 41}, {19, 42}, {20, 45}, {21, 46}};
model = 46 (1 - Exp[λ * (1 - Exp[t^β])]);
fit = FindFit[data, model, {λ, β}, t]
= {λ → 0.0160726, β → 0.547711}

= modelf = Function[{t}, Evaluate[model /. fit]]

= Function[{t}, 46 (1 - e^(0.0160726 (1-e^(t^0.547711))))]

= Plot[modelf[t], {t, 0, 21}, Epilog → Map[Point, data], PlotRange → {0, 50}, AxesOrigin → {0, 0}]
```

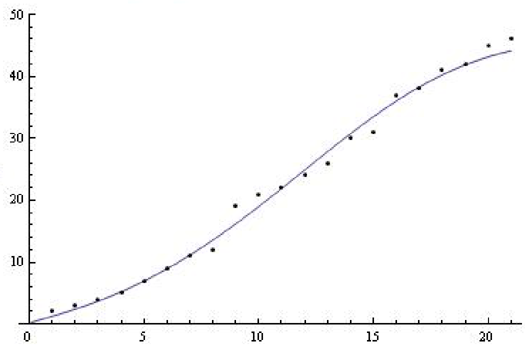


Fig. 5. An example of the usage of dynamical and graphical representation for the function $M(t)$

4. A note on the deterministic Pham's software reliability model

The one-sided Hausdorff distance d between the Heaviside step function $h_{t_0}(t)$ and the sigmoid

$$M_1(t) = \omega\left(1 - e^{1-a^{t^{\alpha}}}\right),$$

satisfies the relation

(10) $$M_1(t_0 + d_1) = 1 - d_1.$$

The reader may formulate corresponding approximation theorem for the model $M_1(t)$.

The fitted model $M_1(t)$ based on the data of Table 1 for the estimated parameters $\omega = 46$, $a = 1.01086$, $\alpha = 1.60447$, is plotted on Fig. 6.

```
Clear[a]
Clear[α]
data = {{1, 2}, {2, 3}, {3, 4}, {4, 5}, {5, 7}, {6, 9}, {7, 11}, {8, 12}, {9, 19}, {10, 21}, {11, 22},
   {12, 24}, {13, 26}, {14, 30}, {15, 31}, {16, 37}, {17, 38}, {18, 41}, {19, 42}, {20, 45}, {21, 46}};
model = 46 (1 - Exp[1 - a^(t^α)]);
fit = FindFit[data, model, {a, α}, t]
{a → 1.01086, α → 1.60447}

modelf = Function[{t}, Evaluate[model /. fit]]
Function[{t}, {46 (1 - e^(1-1.01086^(t^1.60447)))}]

Plot[modelf[t], {t, 0, 21}, Epilog → Map[Point, data], PlotRange → {0, 50}, AxesOrigin → {0, 0}]
```

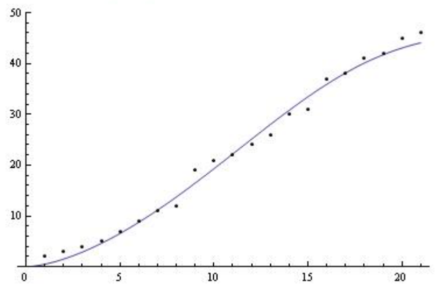


Fig. 6. An example of the usage of dynamical and graphical representation for the function $M_1(t)$

The comparison between $M(t)$ and $M_1(t)$ is visualized on Fig. 7.

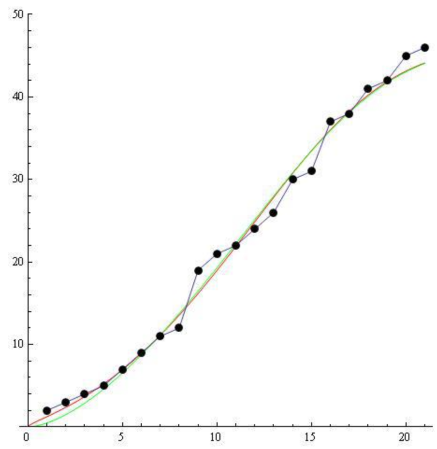


Fig. 7. Comparison between $M(t)$ – red, and $M_1(t)$ – green

5. Remarks

The given comparison shows that the Chen's software reliability model is better than that of Pham's deterministic software reliability model (see the graphics for approximation in the first half of Table 1 data).

One more example to confirm the above mention. Let set $t_0 = 0.27608$ in the approximation of Heaviside function $h_{t_0}(t)$ with the functions $M(t)$ and $M_1(t)$.

For the parameters of the function $M_1(t)$ at fixed t_0 we get $\alpha = 1.4639995$, $a = 32$ and for the Hausdorff distance we have $d_1 = 0.1628$ (see Fig. 8). For the function $M(t)$ we have $d = 0.155083$ (see Fig. 2), i.e., $d < d_1$.

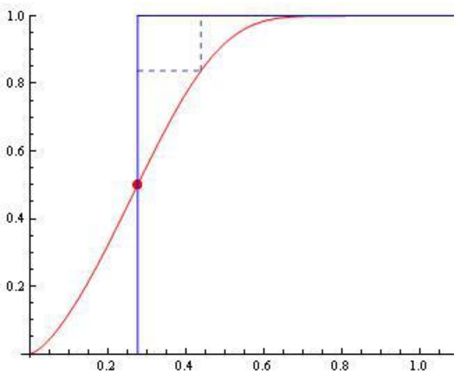


Fig. 8. The model $M_1(t)$ for $a = 32$, $\alpha = 1.4639995$, $t_0 = 0.276085$; H-distance $d_1 = 0.1628$

We will explicitly note that in other cases deterministic Pham's software reliability model provides better results than some much more sophisticated models.

We will illustrate what we have said by approximating the data from the Yamada and Tamura [1] for testing Apache HTTP Server Project which is developed and maintained as an open-source Apache HTTP server for modern operating systems including UNIX and Windows.

Unit time (Days)	Cumulative number of detected faults	Unit time (Days)	Cumulative number of detected faults
0	9	29	37
1	11	30	38
2	11	31	39
3	12	32	39
4	15	33	39
5	17	34	41
6	18	35	41
7	18	36	41
8	18	37	42
9	20	38	42
10	21	39	43
11	24	40	45
12	24	41	45
13	25	42	45
14	26	43	47
15	26	44	49
16	26	45	49
17	27	46	49
18	28	47	50
19	31	48	51
20	31	49	51
21	31	50	53
22	32	51	53
23	32	52	53
24	32	53	55
25	32	54	57
26	33	55	59
27	35	56	60
28	35		

Fig. 9. The actual data in Apache HTTP Server [1]

The fitted model $M_1(t)$ based on the data of Table 2 (see Fig. 9) for the estimated parameters $\omega = 60$, $a = 1.07651$, $\alpha = 0.673929$, is plotted on Fig. 10.

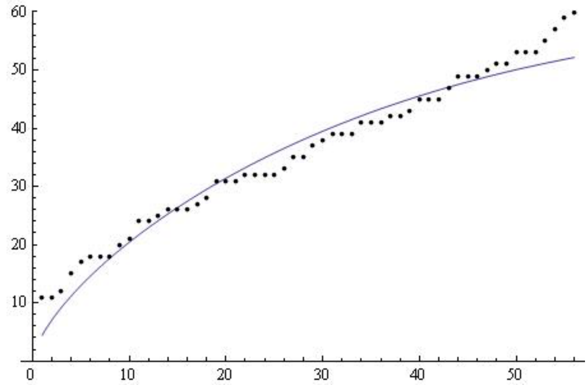


Fig. 10. The model $M_1(t)$

Chaubey and Zhang [7] introduced another extension of Chen's family with cumulative distribution function

$$M_2(t) = \omega\left(1 - e^{\lambda(1-e^{t^{\beta}})}\right)^{\alpha},\ t > 0,$$

where $\omega > 0$, $\lambda > 0$, $\beta > 0$ and $\alpha > 0$.

The fitted model $M_2(t)$ based on the data of Table 1 for the estimated parameters $\omega = 41$, $\lambda = 0.00154907$, $\alpha = 0.48915$, $\beta = 0.694921$ is plotted on Fig. 11.

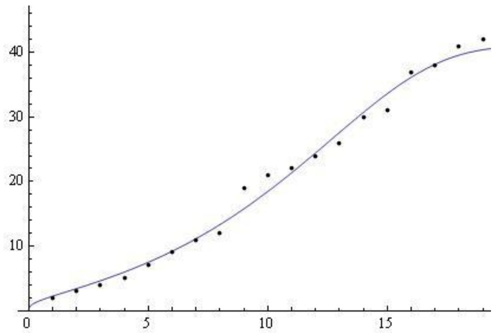


Fig. 11. The model $M_2(t)$ for $\omega = 41$, $\lambda = 0.00154907$, $\alpha = 0.48915$, $\beta = 0.694921$

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered family of functions.

In conclusion, we will note that the determination of compulsory in area of the Software Reliability Theory components, such as confidence intervals and confidence bounds, should also be accompanied by a serious analysis of the value of the best Hausdorff approximation - the subject of study in the present paper.

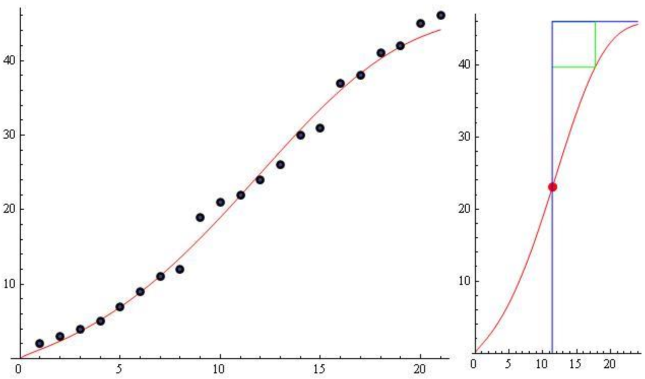
6. Conclusion

With such found parameters ω, λ, β for the value t_0 we have $t_0 = 11.3724$.

The value of the best Hausdorff distance between the function $h_{t_0}(t)$ and sigmoid $M(t)$ satisfies the following nonlinear equation:

$$M(t_0 + d) - \omega + d = 0.$$

In this case, *d* is the length of the side of the square shown on Fig. 12.



(a) (b)

Fig. 12. The model $M(t)$ with $\omega = 46$, $\lambda = 0.01607264$, $\beta = 0.547711$ (a); the Hausdorff approximation of the shifted Heaviside function $h_{t_0}(t)$ by sigmoid $M(t)$ (b)

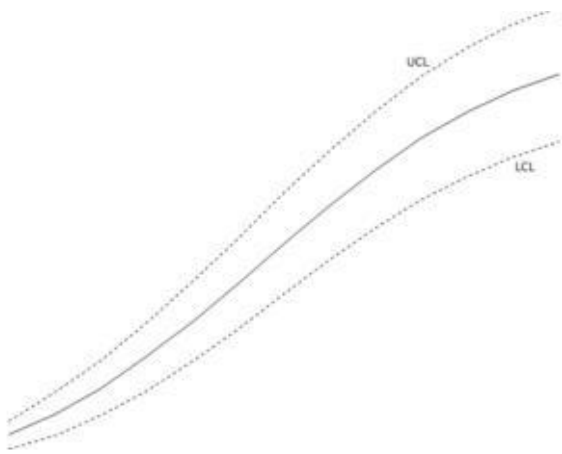


Fig. 13. Typical confidence bounds

The researchs can continue for e-Learning and test theory: Evolutionary development from CBT to e-Learning [9], learning environment [11], DeLC educational portal [10], Virtual Educational Space [13], generation of test questions [12].

The research conducted on Chen's and Pham's models, which are commonly used in practice with offered in this paper Hausdorff apparatus, illustrate the possibility of equitable use of the new characteristic "supersaturation" in combination with established characteristics – confidence intervals and bounds (see Fig. 13).

We hope that the results will be useful for specialists in this scientific area.

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