New Formal Description of Expert Views of Black-Litterman Asset Allocation Model

Miroslav Vladimirov[1], Todor Stoilov[2], Krasimira Stoilova[2]

[1]*Varna University of Economics, 9002 Varna, Bulgaria*
[2]*Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria*
E-mails: *miro@abv.bg*   *todor@hsi.iccs.bas.bg*   *k.stoilova@hsi.iccs.bas.bg*

Abstract: *The general contribution of this research is the implementation of new formal type of relative view, which has been added to the Black-Litterman Model (BLM) for asset management. It is well known that the BLM integrates both historical data about the assets' returns and subjective views given by experts and investors. Such complicated model is expected to give more realistic assessment about the dynamical behavior of the stock exchanges. The BLM applies both absolute and relative views about the asset returns. The paper proves that the currently applied relative views with equal weights are equivalent to assess the risk of a virtual portfolio with these assets of the view which participate with equal weights. The paper extends this form of views, applying non-equal weights of the assets. This new formal description has been tested on a market, containing ten world known indices for a 10 years period. The calculations which have been provided give benefits to the suggested non-equal weighted form of subjective views. It gives more conservative results and decreases the portfolio risk supporting the same level of returns, provided by the average market behavior.*

Keywords: *optimization of assets allocation, modeling market behavior, assessment of portfolio risks and returns, formal description of subjective views.*

1. Introduction

The Black-Litterman Model (BLM) is assumed as fundamental and valuable contribution to the asset allocation management and optimization of financial investments. The model has been published in 1990 as a report by Goldman Sachs Company. It combines the results of the portfolio theory and Capital Asset Pricing Model. For completeness of the BLM one can refer to [1-4]. The idea of the BLM is to integrate the historical data about the asset returns with subjective views from experts about the future behavior of these returns. Such combinations of current and historical data are used for making investment decisions whose results will be seen in a future period. The integration of historical data with subjective views of experts

has to elaborate the forecasts for the portfolio and asset returns close to their real behavior. The BLM combines the subjective investors' views about their expectations of asset returns with the current state of the market.

The formal description of the BLM suffers from many parameters which have to be identified in a quantitative way by means to provide the next portfolio calculations. The model is not easy for implementation and every attempt to be applied to real cases is a valuable experience for the portfolio theory.

This research makes an inclusion in the estimation of parameters in BLM. For completeness of the presentation the BLM is shortly discussed.

2. Presentation of Black-Litterman formal model

The BLM is based on and applies important results to the Modern Portfolio Theory (MPT) and the Capital Asset Pricing Model (CAPM). The MPT originates from the work of Markowitz [5-7]. Mean variance optimization problem is defined which provides optimal solution for the allocation of investment per assets. The formal optimization problem is defined in quadratic form as

(1)
$$\max_{w}\left(E^{\mathrm{T}}w - \lambda w^{\mathrm{T}}\mathrm{cov}()w\right),$$
$$w^{\mathrm{T}}I = 1, \qquad w \geq 0,$$

where $E^{\mathrm{T}} = |E_1, \dots , E_n|$; E_i, $i=1,\dots, n$, are the average asset returns; n is the number of assets in the portfolio; $w^{\mathrm{T}} = |w_1, \dots , w_n|$, where the component w_i is the weight

of the investment allocated to asset i, $i=1,\dots,n$; I is unity matrix, $I = \begin{Vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{Vmatrix}_{1 \times n}$; $\mathrm{cov}()$

is co-variation matrix which gives values about the relations between each couple of asset returns (E_i, E_j), $i, j=1,\dots,n$; $\lambda \in [0; \infty]$ is a coefficient which gives the value of the ability of the investor to undertake risk: for $\lambda=0$ the investor is willing only to maximize its return; for $\lambda=+\infty$ the investor tries to minimize its risk without considering the portfolio return.

The component $E_{\mathrm{p}} = E^{\mathrm{T}}w^{\mathrm{opt}}$ in the objective function (1) gives value of the portfolio return with the evaluated w^{opt} optimal weights of assets.

The component $\sigma_{\mathrm{p}} = w^{\mathrm{T}\,\mathrm{opt}}\mathrm{cov}()w^{\mathrm{opt}}$ is the risk of the portfolio.

For practical usage of problem (1) the investor has to estimate and forecast the values mean asset returns E_i, $i=1,\dots, n$, and their co-variances between each couple of assets. Problem (1) is used in the BLM for the evaluation of the so-called "implied" assets return, which will be explained below.

An important result from the CAPM [8-13, 19] which is used in BLM is the relation between the market average return E_M and the weights w_M of the assets which comprise the market portfolio. The CAPM defines the so-called Market portfolio which in equilibrium can provide an average return of E_m. The market portfolio contains all risky assets and their weights correspond to the market capitalization w_M

of the assets. Thus, in market equilibrium the values of market return E_m and the asset capitalization can be numerically estimated. Both these sets of parameters are general input data for the BLM. Following the relation, given by CAPM between market return E_m and the market weights $w_m(i)$, $i=1,\ldots, n$, of the assets; n is the number of assets in market portfolio; the BLM introduces the so-called implied average return $\Pi(i)$ for each asset (Fig. 1).
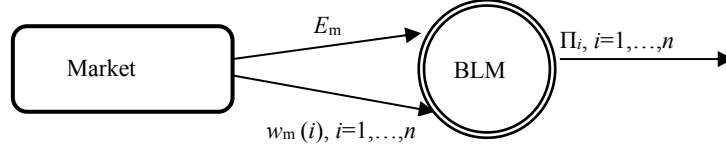

Fig. 1. Definition of implied return

The implied return $\Pi^{\mathrm{T}}=(\Pi_1,\ldots, \Pi_n)$ differs from the average return E_i of each asset. The value E_i is a real one which is presented on the market but it is influenced by noise and random events which take place in the market. The implied return Π_i is theoretically this one which the asset i must have if the market is in equilibrium. Thus, the difference between E_i and Π_i has to be assessed and taken into consideration for the investment process. The BLM applies additional subjective views for the purpose to estimate the future average asset return $E[R]_{\mathrm{BL}}$. The BLM estimation $E[R]_{\mathrm{BL}}$ takes in consideration the current state of the market according to the historical data for the asset average return $E = [E_1, \ldots, E_n]$, co-variances between the assets returns and the subjective predictions for $E[R]_{\mathrm{BL}}$. The formal relation for the forecast of asset returns $E[R]_{\mathrm{BL}}$ is given by

(2) $\qquad E[R]_{\mathrm{BL}} = [(\tau\Sigma)^{-1} + P^{\mathrm{T}}\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P^{\mathrm{T}}\Omega^{-1}Q],$

where n is the number of assets chosen for the portfolio; k is the number of subjective views about the future assets returns; τ is a scalar parameter for tuning the model; Σ is the co-variance matrix evaluated with the historical data about the asset returns; P is the matrix which defines the type of the subjective views (absolute or relative); Ω is a matrix which defines the noise and inaccuracy of the subjective views; Q is a forecast with values defining how the historical assets returns (which are under subjective views) will change.

This paper does not consider the way of deriving this relation. For completeness and theoretical proves of relation (2) one can refer to [15-18]. Following relation (2) and the appropriate theoretical backgrounds in [16] the main results from the inclusion of subjective views influence the values of the historical co-variance matrix from Σ to Σ_{BL}, where

(3) $\qquad \Sigma_{\mathrm{BL}} = (1 + \tau)\Sigma - \tau^2\Sigma P^{\mathrm{T}}(\tau P\Sigma P^{\mathrm{T}} + \Omega)^{-1}P\Sigma.$

Finally, the new weights for the portfolio are

(4) $\qquad w_{\mathrm{BL}} = (\lambda\Sigma_{\mathrm{BL}})^{-1}E[R]_{\mathrm{BL}}.$

This paper targets the inclusion of a new type of presentation of the subjective views. Under consideration will be the definition of matrices P, Ω and Q. Thus, the most important BLM formal relation (2) will be in usage.

The next important parameters in (2) concern the formal definition of the subjective views by P, Ω and Q.

3. Definition of the implied asset returns Π

For the definition of the vector of implied returns Π the BLM applies unconstrained optimization technique known by the term of "inverse optimization". The content of the inverse optimization is based on evaluation of portfolio weights using a utility function

$$(5) \qquad U = w^{\mathrm{T}}\Pi - \frac{1}{2}\lambda w^{\mathrm{T}}\Sigma w.$$

The component $w^{\mathrm{T}}\Pi$ is the excess return of the portfolio (portfolio return E_M reduced the risk free rate P) or

$$w^{\mathrm{T}}\Pi = E_M - r_f.$$

The component $w^{\mathrm{T}}\Sigma w$ is the risk of the portfolio.

The target of the investor is to maximize the utility function (5) by choosing appropriate weights w of the assets in the portfolio. Formally, this target is achieved with lack of considerations by the relations

$$(6) \qquad \max_{w} U\left((w) \to \frac{dU(w)}{dw} = 0 = \Pi - \lambda\Sigma w\right),$$

or

$$w = \Pi(\lambda\Sigma)^{-1}.$$

Considering that Σ is the co-variance matrix of the assets returns estimated by historical data and taking the values $w=w_m$ of the market capitalization of returns, the implied returns are evaluated according to

$$(7) \qquad \Pi = \lambda\Sigma w_m.$$

In this simple relation the risk aversion coefficient λ is not estimated. Because these evaluations concern the market parameters, λ must be evaluated for the market itself. For this case relation (7) is used. Multiplying the both sides of (7) by the vector w_m^{T} it follows

$$(8) \qquad w_m^{\mathrm{T}}\Pi = \lambda w_m^{\mathrm{T}}\Sigma w_m.$$

The left part of (8) presents the excess return of market portfolio or

$$E_m - L = w_m^{\mathrm{T}}\Pi.$$

The right part of (8) represents the market risk σ_m or

$$E_m - L = \lambda\,\sigma_m.$$

Hence,

$$(9) \qquad \lambda = \frac{E_m - L}{\sigma_m},$$

which is not difficult to be evaluated when the market return E_m, market risk σ_m and the risk free rate L are estimated. Substituting (9) in (7) the values of the implied return Π, defined and used in BLM are well estimated

$$(10) \qquad \Pi = \frac{E_m - L}{\sigma_m}\Sigma w_m.$$

For particular investment cases the values for E_m, σ_m are evaluated from the behavior of financial indices, evaluated for each stock exchange and financial markets.

4. Formal presentation of the subjective views

The BLM starts with the evaluation of the implied equilibrium returns. The latter are evaluated applying reverse optimization (7). The next step of BLM is the definition of the subjective views and their combining with the implied returns.

Different schemes for subjective views are given in references. An equal weighting scheme is applied in [18]. The market capitalization of the assets was chosen in [4] to provide different weights to the views. Market capitalization scheme has been use in [17] to provide relative weights for each asset in the P by division of the asset capitalization by the whole market capitalization. As a result the less capitalization of assets receives smaller relative weights in the matrix of views.

The parameters which are used for the definition of the subjective views are the matrices P, Q, Ω and the scalar τ.

4.1. Definition of matrices P and Q

These two matrices are totally defined by the subjective forecasts about the future assets returns. For the BLM the subjective views have absolute and relative forms.

- *Absolute form*: an investor believes that asset 2 will outperform the next year return with 2.5%. Assuming 4 assets for the portfolio, this view is formalized as

$$P = |0 \ 1 \ 0 \ 0|, \ Q = |2.5\%|.$$

For the case of k views P and Q are matrices of dimensions $P_{|k \times n}$ and $Q_{|k \times 1}$, where each row k in P represents next view. $Q(k)$ is the absolute value of the view. For the illustration case the number of assets is $n=4$.

- *Relative form*: an investor believes that the return of asset 2 will outperform with 2.5% this one of asset 3 for the next year. The relative view is formulated as

$$P = |0 \ 1 \ -1 \ 0|, \quad Q = |2,5\%|.$$

In position $P(3)$ the value of the view is (-1) which corresponds to the decrease of view about the return of asset 3. For the case of k views matrix is defined as $P^T = |P_1 \ P_2 \ ... \ P_k|$, where P_i, $i=1,…, k$, are the individual subjective views.

The BLM assumes that all views P_i, $i = 1,…, k$, are independent and not correlated. Relative views which contain more than two assets are also applied [4, 16].

4.2. Definition of matrix Ω

This matrix has to quantify the uncertainty for the views, given by the experts. The value of the variance of each view will give quantitative assessment of the subjective views. For simplification, the BLM assumes that all views are independent and uncorrelated. Thus, matrix Ω representing the co-variances between the views will have diagonal form

$$\Omega = \begin{vmatrix} w_1 & 0 & ... & 0 \\ 0 & w_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & w_k \end{vmatrix},$$

where w_i is the dispersion/uncertainty in the i-th view.

For the case $w_i = 0$ the i-th expert has absolute confidence about his view [18].

The specification of Ω is one of the most difficult tasks in the BLM. The most used form of Ω is [4, 20]

(11) $$\Omega = \tau \times \text{diag}(P\Sigma P^{\text{T}}),$$

where Σ is the co-variance of the asset returns, P is the matrix of subjective views.

Such definition of Ω makes it proportional to the variance of the asset returns.

In [9] this relation is extended in the form

(12) $$\Omega = \frac{1}{c} P\Sigma P^{\text{T}},$$

where $c > 0$ is a newly introduced parameter. But in this form Ω is not diagonal and the evaluation of Ω^{-1} raises computational difficulties. Frequently used value of c is $c = \frac{1}{\tau}$.

In [16] an alternatively developed way of definition of the subjective views is given. It deals with a new parameter, confidence of view. However its inclusion leads to the definition and solution of a new optimization problem, which additionally complicates the utilization of the BLM. This paper will continue with an extension of the Black-Litterman assumption (11).

Finally, the value of parameter τ has to be estimated. An extensive analysis of the role of τ is given in [14]. The idea of this research is to include the influence of τ for the values of the components of matrix Ω. As this paper does not make extensions for τ, here it is that $0 \leq \tau \leq 1$ recommended by [1].

5. Analysis of the matrix Ω

Assuming relation (11) for matrix Ω let's find intuitive interpretation of the components of Ω. For simplicity of the research it has been assumed a portfolio with four assets, $n=4$. From historical data the co-variance matrix Σ is evaluated. Let's assume that two subjective views have been done, $k=2$

$$P = \begin{vmatrix} P_1 \\ P_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix},$$

where the first view P_1 states that asset 1 will outperform asset 4. The second view states that asset 2 will outperform asset 3.

Following (11) the BLM gives values for Ω as

(13) $$\Omega = \tau \times \text{diag}(P\Sigma P^{\text{T}}) = \tau \times \text{diag}\left[\begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{vmatrix} |\Sigma| \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{vmatrix} \right],$$

where Σ is a given co-variance matrix

$$\Sigma = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{vmatrix}.$$

Σ is symmetric matrix and

$$\sigma_{ij} = \sigma_{ji}, \ i, j = 1, \ldots, n, \ \sigma_{ii} = \sigma_i^2,$$

where σ_i is the standard deviation for asset i.

After some matrix calculations (13) results in

$$\Omega = \begin{vmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{vmatrix},$$

(14)
$$\omega_1 = \tau(P_1 \Sigma P_1^T) = \tau[\sigma_{11} + \sigma_{44} - 2\sigma_{14}],$$
$$\omega_2 = \tau(P_2 \Sigma P_2^T) = \tau[\sigma_{22} + \sigma_{23} - 2\sigma_{23}].$$

For the case when Ω is calculated according to (12) the matrix Ω is not diagonal which means that the views are correlated which contradicts the condition for independence of the views.

The paper makes attempt to interpret relation (14). For that case let's consider a new virtual portfolio which contains only two assets.

We assume that these two assets participate in the portfolio with equal weights $w_1 = w_2$. The formal relation for portfolio risk with two assets is [21]

$$\sigma_p = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_{12},$$

where $w_1 + w_2 = 1$; σ_p is the portfolio risk; σ_i is a standard deviation for each asset return, $i = 1, 2$; σ_{12} is co-variance between the assets returns.

For the case of equal weights

$$w_1 = w_2 = 0.5$$

the portfolio risk is

(15)
$$\sigma_p = 0.25(\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}).$$

Let's make comparison between relation (14), defined by the experts' views and the theoretical relation (15). Following (14), one can claim that the unknown variation of the subjective view w_1 is proportional to the risk of a virtual portfolio which contains two assets under consideration with equal weights. Because the subjective view defines an increase of return for the first asset and decreases in forth, as given in $P_1 = |1 \quad 0 \quad 0 \quad -1|$, the co-variance σ_{14} is negative with value defined from the historical evaluated variance matrix Σ. Hence, the BLM uses for the values ω_i of the uncertainty of views Ω proportional part of the risk of a virtual portfolio containing the two assets of the relative view.

Following this result, this research makes an attempt to extend the form of such virtual portfolio, applying different weights to the assets in the view.

6. New weighted form of the relative views

We assume different weights for the relative subjective views. This means that the view P_1 in matrix P will be in the form $P = |\alpha_1 \quad 0 \quad 0 \quad -\alpha_4|$ for the case $n = 4$. The values α_1 and α_4 satisfy the normalization equation

(16)
$$|\alpha_1| + |\alpha_4| = 1.$$

The description of P means that the first asset is expected to increase $(+\alpha_1)$ and the forth one has to decrease $(-\alpha_4)$. Such modification in P matrix will influence the value of ω in Ω. Using (11) it follows

(17)
$$\omega_1 = \tau \times \text{diag}(P\Sigma P^T) = \tau[\alpha_1^2 \sigma_{11} + \alpha_4^2 \sigma_{44} - 2\alpha_1 \alpha_4 \sigma_{41}],$$

because (16) takes place the value ω_1 is proportional to the risk of a new virtual portfolio containing assets one and four with weights α_1 and α_2. The virtual portfolio has negative co-variation σ_{41} according to the view P about assets 1 and 4.

This new formalization doesn't respect the currently applied constraint for the sum of views in matrix P to be zero for relative views. Here such constraint is substituted by the new equation (16) for the modes of the components of the views.

This new form of the subjective view is applied in a research for analyzing 10 world indices of well-known markets.

7. Experiments and application of weighted relative views

The indices of stock exchanges were chosen to enter in a research portfolio as assets (Fig. 2).

Stock exchange	Index	Number of companies in the index
Deutsche Börse	DAX	30
London Stock Exchange	FTSF	100
Hong Kong	Hang Seng	50
NASDAQ	NASDAQ	100
Tokyo Stock Exchange	Nikkei	225
New York Stock Exchange	NYSE composite	100
Toronto Stock Exchange	S&P/TSX Composite Index	200
Australian Stock Exchange	S&P/ASX composite index	200
Shanghai Stock Exchange	SSE composite index	50
Bulgarian Stock Exchange	SOFIX	15

Fig. 2. Indices, included in the portfolio

The period of investigation is chosen for four years: 2012-2015. The opening and closing rates were used from public available sources [22, 23]. Particular records for DAX index can be found in [24].

An illustration for the initial data for the returns is given in Fig. 3. The closing price is taking into consideration for the next evaluations

D A X			
DAX	Open	Close	%
03.08.2015	11295,50	11490,83	1,608
01.08.2015	11050,32	11308,99	3,326
01.07.2015	11462,97	10944,97	−4,108
01.06.2015	11506,84	11413,82	−4,108

Fig. 3. Example of the input data for the asset returns values

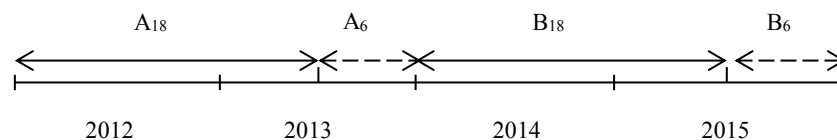The experiments were performed for two periods of two years each (Fig. 4).

Fig. 4. Periods for estimation and prediction

The periods (2012-2013) and (2014-2015) were divided as A_{18} =18 months period and A_6=6 months period, respectively, B_{18} and B_6. The data, acquired for the asset returns for A_{18} period are used as historical data for the evaluation of the average returns E_i, risks σ_i and market capitalization of each index w_m^i, i=1,…,10. Having data for A_{18} period, a subjective view is made applying classical (13) and weighted (17) forms of relative views. The investments with cases (13) and (17) later are compared with the real results for average returns according to the period A_6. Thus, numerical comparison is done to find benefits from the new form (17) of subjective views.

Numerical results. Using the initial data defined for each index which plays role as asset in the portfolio for the period 2014-2015 is shown in Fig. 5. The calculations are illustrated in the same figure.

The co-variance matrix Σ, the average returns E_i^{18} the implied returns Π_i are initially evaluated before the application the BLM. For the evaluation of Π_i, the market return E_m was assumed from the Global Dow index. It integrates the data from 150 companies all over the world. The data for Global Dow index were taken from [24]. Thus, for relation (9), E_m and σ_m are evaluated. For the value of risk free rate L the value of LIBOR were considered. The Thomson Reuters Company publishes the LIBOR value every day. These data are available in [25]. The LIBOR values are used for their average value used in (9) like parameter L.

The subjective views have been defined also by assessing the differences between real and implied returns $E_i^{18} - \Pi_i^{18}$. It is illustrated in Fig. 5 and it is

$$E_{DAX}^{18} - \Pi_{DAX}^{18} = +0.01322,$$

which is the maximal differences for all assets. For SSE this difference is algebraically minimal

$$E_{SSE}^{18} - \Pi_{SSE}^{18} = -0.0063.$$

COVMAT	DAX	FTSE	Hang-Seng	NASDAQ	Nikkey	NYSE	S&P_TSX	S&P_ASE	SSE	SOFIX
DAX	0.001473	0.000849	0.001542	0.001074	0.001028	0.000892	0.000802	0.000633	0.000868	0.000193
FTSE	0.000849	0.000958	0.001202	0.000735	0.001006	0.000786	0.000613	0.000575	0.000721	0.00023
Hang-Seng	0.001542	0.001202	0.002822	0.000985	0.001341	0.001233	0.001067	0.001227	0.001616	-2.9E-05
NASDAQ	0.001074	0.000735	0.000985	0.001263	0.001212	0.000897	0.000584	0.000415	0.000316	0.000108
Nikkey	0.001028	0.001006	0.001341	0.001212	0.003148	0.001228	0.000521	0.000901	0.000685	-4E-05
NYSE	0.000892	0.000786	0.001233	0.000897	0.001228	0.000898	0.000591	0.000629	0.000443	-2.3E-06
S&P_TSX	0.000802	0.000613	0.001067	0.000584	0.000521	0.000591	0.000646	0.000504	0.000822	0.000379
S&P_ASE	0.000633	0.000575	0.001227	0.000415	0.000901	0.000629	0.000504	0.001174	0.000516	0.000469
SSE	0.000868	0.000721	0.001616	0.000316	0.000685	0.000443	0.000822	0.000516	0.00408	0.0002
SOFIX	0.000193	0.00023	-2.9E-05	0.000108	-4E-05	-2.3E-06	0.000379	0.000469	0.0002	0.002017
Wm, Пазарни тегла	0.04394	0.03234	0.0363	0.13736	0.1376	0.52163	0.05027	0.02293	0.01102	0.00031
E18	0.017521	0.006573	0.008168	0.01559	0.028641	0.011508	0.001131	0.010016	-0.0038	0.018623
E6	0.031234	0.014276	0.019512	0.034983	0.030505	0.022609	0.019624	0.018482	0.011791	0.015511
Min	-0.07348	-0.07266	-0.11684	-0.0719	-0.10274	-0.08069	-0.06341	-0.07285	-0.13969	-0.04475
MAX	0.095037	0.064278	0.106111	0.080107	0.118	0.053437	0.041581	0.050806	0.145956	0.155475
П,подразбираща се	0.0043	0.0026	0.0057	0.0043	0.0064	0.0041	0.0027	0.003	0.0025	0.0003
E18-П	0.013221	0.003973	0.002468	0.01129	0.022241	0.007408	-0.00157	0.007016	-0.0063	0.018323

Fig. 5. Co-variance matrix and differences between historical E_i and implied returns

These results define that E_{DAX} is currently overrated and the investor has to expect decrease in the average return of DAX, E_{DAX}. On the other hand, the SSE asset is under estimated and one has to expect increase in E_{SSE}.

The presentation of this view in terms of P and Q is made by taking into consideration both the average E_i, Π_i and the variances σ_i.

For the case of DAX and SSE the weights in the subjective views are evaluated as

$$\alpha_{\text{DAX}} = \{|[E_{\text{DAX}}^{18} - \Pi_{\text{DAX}}^{18}]/\sigma_{\text{DAX}}^{18}|/[|[E_{\text{DAX}}^{18} - \Pi_{\text{DAX}}^{18}]/\sigma_{\text{DAX}}^{18}| + |[E_{\text{SSE}}^{18} - \Pi_{\text{SSE}}^{18}]/\sigma_{\text{SSE}}^{18}|]\} = 0.78,$$

$$\alpha_{\text{SSE}} = \{|[E_{\text{SSE}}^{18} - \Pi_{\text{SSE}}^{18}]/\sigma_{\text{SSE}}^{18}|/[|[E_{\text{DAX}}^{18} - \Pi_{\text{DAX}}^{18}]/\sigma_{\text{DAX}}^{18}| + |[E_{\text{SSE}}^{18} - \Pi_{\text{SSE}}^{18}]/\sigma_{\text{SSE}}^{18}|]\} = 0.22$$

Because the expectations are E_{DAX}^{6} to decrease and E_{SSE}^{6} to increase, the values in P matrix are

$$P = |{-0{,}78} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.22 \quad 0|,$$

the numerical values of views: $k=1$, n, $n=10$ assets.

The value in Q for this view is assumed to be an increase of SSE by value

$$Q = \left|E_{\text{SSE}}^{18} - \Pi_{\text{SSE}}^{18}\right| = |{-0.0038} - 0.0025| = 0.0063.$$

In this paper the BLM was applied both for the classical case with

$$P = |{-1} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0|,$$

and with the weighted views particularly for this case

$$P = |{-0.78} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.22 \quad 0|.$$

Such evaluation was done also for the period (2012-2013).

8. Comparisons of results

Having data for the first 18 months period, the average returns with BLM were performed with equal weighted views P and weighted P_α views presented earlier. The values of returns with BLM $E_{\text{BL}i}^{P}$ and $E_{\text{BL}i}^{P\alpha}$ $i=1,\ldots,10$, are found for each asset.

For the next 6-months period both the average returns E_i^6 and implied returns Π_i^6, $i=1,\ldots,10$ have been evaluated. These values were compared to the $E_{\text{BL}i}^{P}$ and $E_{\text{BL}i}^{P\alpha}$, $i=1,\ldots,10$. For simplicity of comparison the average values are considered

$$E_{\text{BL}i}^{P,18} = \frac{1}{10}\sum_i E_{\text{BL}i}^{P,18},$$

$$E_{\text{BL}i}^{P\alpha,18} = \frac{1}{10}\sum_i E_{\text{BL}i}^{P\alpha,18}.$$

Comparisons were made for the differences

$\Delta_1 = E_{\text{BL}}^{P,18} - E_i^6$ (classical BLM towards average return E_i^6)

$\Delta_2 = E_{\text{BL}}^{P\alpha,18} - E_i^6$ (BLM with weighted views towards average return E_i^6)

$\Delta_3 = E_{\text{BL}}^{P,18} - \Pi_i^6$ (classical BLM towards implied return Π_i^6)

$\Delta_4 = E_{\text{BL}}^{P\alpha,18} - \Pi_i^6$ (BLM with weighted views towards implied return Π_i^6)

The graphical presentation of the results is given in Fig. 6.

The results of Fig. 6 show that the weighted view P_α generates conservative forecasts and the errors for cases Δ_2 and Δ_4 are lower. These values of the errors for the classical BLM form of views P, case Δ_1 and Δ_3, are higher.
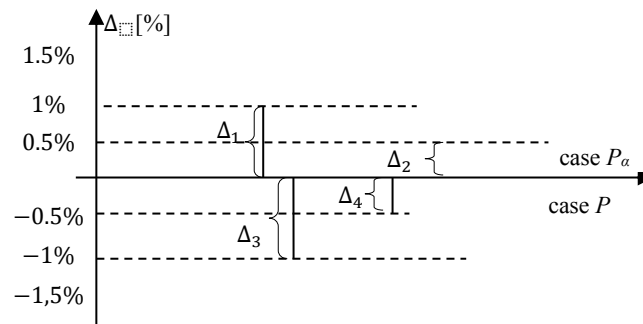
Fig. 6. Graphical presentation of the comparisons results

9. Conclusions

The classical portfolio theory extends its applications even without the complex model of BLM [26-29]. But this paper follows the idea of BLM for adding additional considerations for the assessment of the asset returns. The paper particularly suggests modification for the expert views in BLM for asset allocation. The equal views in matrix P have been modified to weighted values P_α which gives different preferences for the increase or decrease of the asset results. It has been proved that this modification has interpretation of accessing the risk of a virtual portfolio which contains assets with different weights. This modified weighting view can be used for the case when subjective views are missing but only historical data are available. For making a relative view, a couple of assets can be identified using the historical data and estimating the minimal and maximal deviation of the average returns from the implied ones. The experimental evaluations give advantages for the weighted views providing more conservative investment policy in comparison to the classical equal weighted forms of Black-Litterman views. The evaluated average returns with the weighted prediction form generate values closer to the real posterior average asset returns. Such quantitative form of views is recommended for cases with lack of experts' assessments and/or for more precise analysis of the historical behavior of the market dynamics. A continuation of this research will be the usage of newer data from the stock markets and the application of this new formal description of expert views for Black-Litterman asset allocation model.

References

1. B l a c k, F., R. L i t t e r m a n. Asset Allocation: Combining Investor Views with Market Equilibrium. Report Fixed Income Research, Goldman Sachs Company, September 1990.
2. B l a c k, F., R. L i t t e r m a n. Asset Allocation: Combining Investor Views With Market Equilibrium. – The Journal of Fixed Income, Vol. **1**, 1991, No 2, pp. 7-18.
3. B l a c k, F., R. L i t t e r m a n. Global Portfolio Optimization. Financial Analysts, September-October 1992, pp. 28-43.
4. H e, G., R. L i t t e r m a n. The Intuition Behind Black-Litterman Model Portfolios. – Techn. Report, Fixed Income Research, Goldman Sachs Company Investment Management Series, December 1999.
   http://dx.doi.org/10.2139/ssrn.334304

5. M a r k o w i t z, H. Portfolio Selection: Efficient Diversification of Investments. Monograph No 16, Combs Foundation. New York, John Wiley, 1959.

6. M a r k o w i t z, H. Portfolio Selection. – J. of Finance, Vol. **7**, March 1952, No 1, pp. 77-91.

7. M a r k o w i t z, H. Mean Variance Analysis in Portfolio Choice on Capital Markets. John Wiley, 1987.

8. T r e y n o r, J. L. How to Rate the Performance of Mutual Funds. – Harvard Business Review, Vol. **43**, 1965, pp. 63-75.

9. S h a r p e, W. Adaptive Asset Allocation Policies. – J. Financial Analysts, Vol. **66**, 2010, Issue 5, pp. 45-49.

10. C r a i g, F. The Treynor Capital Asset Pricing Model. – J. of Investment management, Vol. **1**, 2003, No 2, pp. 60-72.

11. M a s s i n, J. Theory of Financial Markets. Prentice-Hall, 1973. ISBN 0-13-913699-1.

12. B e v a n, W. Using the Black-Litterman Global Asset Allocation Model. Three Years of Practical Experience. Goldman Sachs Fixed Income. Research Paper, June 1998.

13. J a y, W. The Black-Litterman Model in Details. CFA, Copyright 2007-2014, pp. 1-64. **http://www.blacklitterman.org/Black-Litterman.pdf**

14. J a y, W. The Factor Tau in the Black-Litterman Model. Copyright 2010-2014, pp. 1-13. **http://ssrn.com/abstract=1701467**

15. I d z o r e k, T. A Step by Step Guide to the Black-Litterman Model. Incorporating User Specific Confidence Levels (Working Paper), 2004.

16. I d z o r e k, T. Strategic Asset Allocation and Commodities. IBBOTSON, 2006, pp.1-55. **https://corporate.morningstar.com/ib/documents/MethodologyDocuments/IBBAssociat es/Commodities.pdf**

17. S a t c h e l l, S., A. S c o w c r o f t. A Demystification of the Black-Litterman Model: Managing Quantitative and Traditional Portfolio Construction. – J. of Asset Management, Vol. **1, 2**, 2000, pp. 138-150.

18. D a n i c l, S. Black-Litterman Allocation Model. – Application and Comparison with OMV Stockholm Benchmark PI(OMVSBPI), June 2015, pp. 1-31.

19. M e n c c i, A. The Black-Litterman Approach: Original Model and Extensions. – Bloomberg Alpha Research Education Paper, No 1, 2008.

20. S h a r p e, W. Portfolio Theory and Capital Market. McGraw Hill, 2000. 316 p.

21. W a l t e r s, J. The CFA Factor Tau in the Black-Litterman Model. 9 October 2013. **https://ssrn.com/abstract=1701467 or http://dx.doi.org/10.2139/ssrn.1701467**

22. **https://finance.yahoo.com**

23. **https://www.google.com/finance**

24. **https://www.google.com/finance?q=DAX&xxxxx**

25. **https://www.google.com/finance/historical?q=INDEXDJX%3AGDOW&ei=d4TsWKGeKtS UswHY_5uQBA**

26. G e o r g i e v a, P., I. P o p c h e v, S. S t o y a n o v. A Multi-Step Procedure for Asset Allocation in Case of Limited Resources. – Cybernetics and Information Technologies, Vol. **15**, 2015, No 3, pp. 41-51.

27. P o p c h e v, I., I. R a d e v a. A Decision Support Method for Investment Preference Evaluation. – Cybernetics and Information Technologies, Vol. **6**, 2006, No 1, pp. 3-16.

28. P o p c h e v, I., T. S t o i l o v, K. T e n e k e d j i e v, K. S t o i l o v a, N. N i k o l o v a, Z. I v a n o v a, I. R a d e v a, E. T r i c h k o v a, B. S t o y a n o v, C. B e c h e v, M. P o l i m e n o v, N. D j e d j e v, V. Z h e l e v. Investment Analysis And Portfolio Optimization: Theory and Practice. T. Stoilov, Ed. Publ. House „More", 2010. ISBN 978-954-9722-13-0 (in Bulgarian).

29. I v a n o v a, Z., K. S t o i l o v a, T. S t o i l o v. Portfolio Optimization – Internet Information Service. Sofia, Academic Publishing House "Marin Drinov", 2005. 275 p. ISBN 954-322-021-2.