

Multichannel Modified Covariance Estimator of a Single-Tone Frequency

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Abstract: *The multichannel modified covariance estimator of a single-tone signal frequency is synthesized by using the maximum likelihood method. It is shown that this estimator has an advantage over the estimator averaged by multiple conventional single-channel ones. The particular case of a complex signal is also considered.*

Keywords: *Single-tone harmonic, autoregressive model, modified covariance, multichannel, complex signal, frequency, estimation, maximum likelihood.*

1. Introduction

In many real situations the problem of signal processing is expanded over several data channels, usually from different sensors in the array. Such multichannel signals are typical for hydro acoustics, multistatic radars, seismic measurements, electroencephalography, power systems, etc., [1-6]. The main problem solved in the multichannel systems is data fusion that is usually divided into two adjoining problems of efficient noise removal [2, 4] and multichannel spectral estimation [1, 7]. We will focus on the last one. In the general case the task of the multichannel spectral estimation consists of estimation of the power spectral density matrix, which consists of elements related to auto-spectrums and cross-spectrums for each pair of channels. There are several multichannel implementations of conventional algorithms. In many real situations only a limited number of narrowband signals are presented in the recorded data and the most frequently used methods are AutoRegressive (AR) or all-pole signal model based ones [8, 9].

In the present work attention is paid to the specific case when only the single-tone narrowband signal is presented in each channel and the Least Squares (LS) autoregressive single frequency estimation is considered.

Let us assume that there is a source of the initial real signal

$$s(t) = \text{Re}[\rho_s \exp(j\omega t + j\varphi_s)] \equiv \rho_s \cos(\omega t + \varphi_s),$$

where Re means a real part, $\omega = 2\pi f_s$ is the angular frequency; f_s is a signal frequency, ρ_s is an original signal amplitude, φ_s is an initial phase. It is measured by an array of M sensors (channels) with some measurement noises and amplitude-phase transformations

$$(1) \quad \mathbf{h} = [h_1, h_2, \dots, h_M]^T,$$

where the coefficient for m -th channel is assumed to be constant in time, apriori unknown and it equals

$$h_m = a_m \exp(j\varphi_m),$$

a_m, φ_m are the amplitude transformation coefficient and the respective phase shift due to signal propagation in a medium from the source to each channel.

Such a system can be interpreted as a Single-Input Multiple-Output (SIMO) model [10] (Fig. 1)

$$(2) \quad \zeta_m(t) = \text{Re}[h_m s(t)] + \eta_m(t), \quad m = 1, 2, \dots, M,$$

where the signal measured by the m -th channel $h_m s(t) = \rho_m \cos(\omega t + \varphi_s + \varphi_m)$, has an amplitude $\rho_m = \rho_s a_m$, $\eta_m(t)$ is an additive white Gaussian noise with zero mean and equal variance at each channel $\sigma_m^2 = \sigma^2$. Or, finally, in a matrix form

$$(3) \quad \zeta(t) = \text{Re}[\mathbf{h}s(t)] + \boldsymbol{\eta}(t),$$

where $\zeta(t)$ is a column vector of the recorded impinging signal values in the moment of time t , $\boldsymbol{\eta}(t)$ is a vector of independent values of noise. Therefore, we obtain a set of generally incoherent signals (3) with common frequency.

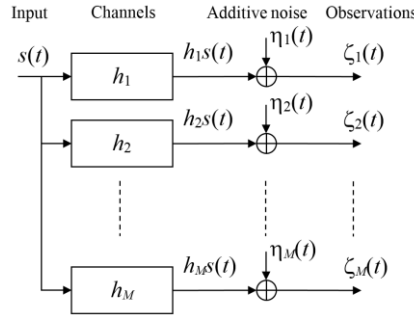


Fig. 1. Scheme of the considered SIMO model of observations

While recording, all channels are evenly sampled with a frequency $F = \tau^{-1}$, and

$$(4) \quad x_m(n) = \int_0^T \zeta_m(t) \delta(t - n\tau) dt, \quad n = 1, 2, \dots, N, \quad m = 1, 2, \dots, M,$$

where τ is a sampling period, N is a sample size, T is an observation time.

It is reasonable to modify the total phase of the sampled signal as
(5)
$$\omega t + (\varphi_s + \varphi_m) \rightarrow \gamma(n-1) + (\varphi_s + \varphi_m),$$

where

$$\gamma = 2\pi f_s / F \equiv \omega \tau,$$

is a normalized frequency that meets Nyquist-criterion $0 < \gamma < \pi$. Taking into account (2), we can represent (4) as

$$(6) \quad x_m(n) = \tilde{x}_m(n) + \eta_m(n),$$

where $\tilde{x}_m(n) = \rho_m \cos(\gamma(n-1) + \varphi_s + \varphi_m)$ is a signal value without noise.

All samples (6) form M input sequences

$$\mathbf{x}_m = [x_m(1), x_m(2), \dots, x_m(N)]^T,$$

which are joined into a $M \times N$ matrix \mathbf{X}

$$\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M]^T \equiv [\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(N)],$$

where the vector $\mathbf{x}(n) = [x_1(n), x_2(n), \dots, x_M(n)]^T$ joins the values in all channels at the moment n .

A clear single sinewave signal at each channel can be written as the next autoregressive model

$$\tilde{x}_m(n) = \alpha \tilde{x}_m(n-1) - \tilde{x}_m(n-2), \quad n = 3, 4, \dots, N.$$

Here the autoregressive parameter depends on the signal frequency as

$$(7) \quad \alpha = 2 \cos(\gamma).$$

Note that here unknown amplitudes and phases of signals in each channel are eliminated from the equation. In other words, such a model is invariant to amplitudes and phases in channels. The only one important parameter, explicitly joined with the signal frequency, is considered. Unfortunately, in real situation, the signal is always corrupted by a noise. Therefore, we consider estimation of the autoregressive parameter α as a linear prediction problem, when the prediction error (residual) $r_m(n)$ between current $x_m(n)$ and predicted $\hat{x}_m(n)$ values is obtained as

$$r_m(n) = x_m(n) - \hat{x}_m(n) \equiv x_m(n) - \alpha x_m(n-1) - x_m(n-2), \quad n = 3, 4, \dots, N.$$

For m -th channel with N samples we can write a vector of $N - 2$ residual values

$$\mathbf{r}_m(\mathbf{x}_m, \alpha) = [r_m(3), r_m(4), \dots, r_m(N)]^T,$$

and for all the channels joining them into $M \times (N - 2)$ matrix

$$\mathbf{R}(\mathbf{X}, \alpha) = [\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_M]^T.$$

In this case the problem is transferred, as it will be later shown, to minimization of the prediction error variance

$$(8) \quad \sigma_p^2 \equiv E\{|\mathbf{R}(\mathbf{X}, \alpha)|^2\}.$$

The objective of this work is obtaining of the autoregressive parameter α estimation for the considered general multichannel case using Maximum Likelihood (ML) approach, It will be shown that the ML approach with some assumptions is equivalent to the Least Squares (LS) one.

2. ML-estimator synthesis

Analytical synthesis of ML-estimator is done on the basis of apriori known Likelihood Function (LF) $L(\cdot)$ by the next general optimization criterion

$$(9) \quad \hat{\alpha} = \arg \max_{\alpha} [L(\mathbf{X} | \alpha, \sigma)].$$

In practice, the estimation by ML-estimator is performed by iterative search due to nonlinearity of the LF.

Let us assume that at high SNR

$$(10) \quad \text{SNR} \equiv \rho^2 / (2\sigma^2) \gg 1.$$

the residuals distribution equals the noise distribution – a white Gaussian with zero mean and variance $\text{var}(\mathbf{r}_m) \approx \sigma^2$.

Taking into account the previously described signal and noise model and independence of all $M \times N$ noise samples, the likelihood function for m -th channel can be written as

$$L_m(\mathbf{x}_m | \alpha, \sigma) = \frac{1}{\sigma^{N-2} 2\pi^{N/2-1}} \exp\left(-\frac{1}{2\sigma^2} \sum_{n=3}^N [r_m(n)]^2\right).$$

Then the joined multichannel LF can be written as

$$L(\mathbf{R} | \alpha, \sigma) = \prod_{m=1}^M L_m(\mathbf{x}_m | \alpha, \sigma) = \frac{1}{\sigma^{M(N-2)} 2\pi^{M(N-2)/2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{m=1}^M \sum_{n=3}^N [r_m(n)]^2\right).$$

The logarithm of the LF is

$$\ln[L(\mathbf{R} | \alpha, \sigma)] = -M(N-2) \ln(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2} \Lambda(\mathbf{R} | \alpha),$$

where only the sufficient statistics is important for α estimation

$$\Lambda(\mathbf{R} | \alpha) = \sum_{m=1}^M \sum_{n=3}^N [r_m(n)]^2.$$

That is equivalent to the equation (8) with a proportional multiplier.

Accordingly to criterion (9), the optimal estimator is received from solution of the next likelihood equation

$$\frac{\partial \Lambda(\mathbf{R} | \alpha)}{\partial \alpha} = 2 \sum_{m=1}^M \sum_{n=3}^N x_m(n-1) [x_m(n) - \alpha \cdot x_m(n-1) + x_m(n-2)] = 0.$$

This after elementary mathematical operations is transformed to

$$\sum_{m=1}^M \sum_{n=3}^N x_m(n-1) [x_m(n) + x_m(n-2)] = \alpha \sum_{m=1}^M \sum_{n=3}^N [x_m(n-1)]^2.$$

The optimal estimator can be written as

$$(11) \quad \hat{\alpha} = \frac{\sum_{m=1}^M \sum_{n=3}^N x_m(n-1) [x_m(n) + x_m(n-2)]}{\sum_{m=1}^M \sum_{n=3}^N [x_m(n-1)]^2}.$$

It is similar (and equal, when $M = 1$) to a single-channel modified covariance

estimator [11, 12], but it calculates joined statistics over all channels.

If we set the next matrices

$$\mathbf{X}_1 \equiv [\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(N-2)], \quad \mathbf{X}_2 \equiv [\mathbf{x}(2), \mathbf{x}(3), \dots, \mathbf{x}(N-1)],$$

$$\mathbf{X}_3 \equiv [\mathbf{x}(3), \mathbf{x}(4), \dots, \mathbf{x}(N)],$$

then the estimator can be rewritten in the next matrix form

$$\hat{\alpha} = \frac{\text{tr}(\mathbf{X}_2 \cdot (\mathbf{X}_3 + \mathbf{X}_1)^T)}{\text{tr}(\mathbf{X}_2 \cdot \mathbf{X}_2^T)}.$$

In a real situation it can be more reasonable to calculate two traces of two products and then add them, if we use only the main diagonal elements without calculation of all elements of the product matrices.

On the final step in accordance to (5) and (7), the frequency estimate is obtained as

$$(12) \quad \hat{f} = \arccos(\hat{\alpha} / 2) / (2\pi\tau).$$

3. Complex signal case

3.1. Asymptotical estimator

So far, we have synthesized the algorithm for estimation of a single common frequency in multiple channels that is invariant to signal amplitudes and phases in these channels. Now let us consider a particular case of a complex signal. The conventional complex signal can be interpreted as a system with two channels that have the same amplitude ρ (apriori unknown) and are shifted in phase by $\pi/2$. If we assume in (1) $\mathbf{h} = [1, \exp(\pi/2)]^T$, then the signal model (3) is written as

$$\zeta(t) = \begin{bmatrix} \rho \cos(\omega t + \varphi_s) + \eta_x(t) \\ \rho \sin(\omega t + \varphi_s) + \eta_y(t) \end{bmatrix}.$$

That actually is a complex signal $\dot{\zeta}(t) = \rho \exp[j(\omega t + \varphi_s)] + \dot{\eta}(t)$ with noise $\dot{\eta}(t) = \eta_x(t) + j \cdot \eta_y(t)$.

After sampling we sign the real and imaginary parts of $\dot{\zeta}$ as

$$\text{Re}(\dot{\zeta}) \equiv \mathbf{x} = \rho \cos(\gamma(n-1) + \varphi_s) + \eta_x(n)$$

$$\text{Im}(\dot{\zeta}) \equiv \mathbf{y} = \rho \sin(\gamma(n-1) + \varphi_s) + \eta_y(n)$$

We can rewrite (11) with $M = 2$ for this case

$$(13) \quad \hat{\alpha} = \frac{\sum_{n=3}^N \{x(n-1)[x(n) + x(n-2)] + y(n-1)[y(n) + y(n-2)]\}}{\sum_{n=3}^N \{x(n-1)^2 + y(n-1)^2\}} \equiv$$

$$\equiv \frac{2 \sum_{n=3}^N \{x(n)x(n-1) + y(n)y(n-1)\} + y(1)y(2) - y(N-1)y(N) + x(1)x(2) - x(N-1)x(N)}{\sum_{n=3}^N \{x(n-1)^2 + y(n-1)^2\}}.$$

Now let us check the asymptotical properties of this frequency estimator for a

complex signal. Even if synthesis was done rigorously enough and the calculation equations are easy to use, let us approximate the Equation (13) for a complex signal with some asymptotic assumptions. Firstly, we consider that the sample size N is big enough, that leads to

$$(14) \quad 2 \sum_{n=3}^N \{x(n)x(n-1) + y(n)y(n-1)\} \gg y(1)y(2) - y(N-1)y(N) + x(1)x(2) - x(N-1)x(N).$$

Hence, the numerator in (13) can be simplified to the term with a sum.

With the assumption about high SNR (10) the distribution of the absolute values $A(n) = \sqrt{x^2(n) + y^2(n)}$ of samples approaches to Gaussian with a low variation coefficient, and $A(n) \approx \rho$. It allows to modify the denominator in (13) in the following manner

$$(15) \quad \sum_{n=3}^N [x^2(n-1) + y^2(n-1)] = \sum_{n=3}^N A^2(n-1) \approx (N-2)\rho^2.$$

On the basis of assumptions (14) and (15) the estimator for a complex signal (13) can be written as asymptotical one:

$$(16) \quad \alpha \approx \frac{2}{N-2} \sum_{n=3}^N \left[\frac{x(n)}{\rho} \cdot \frac{x(n-1)}{\rho} + \frac{y(n)}{\rho} \cdot \frac{y(n-1)}{\rho} \right].$$

Here the fractions can be substituted by the corresponding trigonometric functions, which are the set of estimated instantaneous phases $\{\varphi_n\}$:

$$x(n)/\rho = \cos \varphi_n, \quad y(n)/\rho = \sin \varphi_n,$$

From equation (16), considering (7), we obtain

$$(17) \quad \cos \hat{\gamma} \approx \frac{1}{N-2} \sum_{n=3}^N (\cos \varphi_n \cos \varphi_{n-1} + \sin \varphi_n \sin \varphi_{n-1}) = \frac{1}{N-2} \sum_{n=3}^N \cos(\Delta_n),$$

where $\Delta_n = \varphi_n - \varphi_{n-1}$ is a phase difference between adjacent samples.

After adding the term with an index $n = 2$, the resulting asymptotical frequency estimator is written in the next form:

$$(18) \quad \hat{\gamma}_C \approx \arccos \left[\frac{1}{N-1} \sum_{n=2}^N \cos(\Delta_n) \right].$$

It illustrates the connection between two synthesis principles – ML, that is implemented in this paper, and a differential phases one, that became widely used [13] (and references herein) after the fundamental paper of Kay [14].

3.2. Lank's estimator

It is reasonable to assume that for a complex (quadrature) signal the frequency estimator can be obtained similarly to (18) as

$$(19) \quad \hat{\gamma}_S \approx \arcsin \left[\frac{1}{N-1} \sum_{n=2}^N \sin(\Delta_n) \right].$$

Combining two statistics from (18) and (19) one can synthesize yet another frequency estimator

$$(20) \quad \hat{\gamma}_T \approx \arctan \left[\frac{\sum_{n=2}^N \sin(\Delta_n)}{\sum_{n=2}^N \cos(\Delta_n)} \right].$$

This estimator is equal to the estimator heuristically proposed by Lank, Reed and Polton [15]. It was written with using complex numbers as

$$(21) \quad \hat{\gamma}_L = \arg \left(\sum_{n=2}^N \frac{z_{n-1}^* \dot{z}_n}{|\dot{z}_{n-1}| |\dot{z}_n|} \right),$$

where there are used typical descriptions: z_n^* is a complex conjugate, $\arg(\cdot)$, $|\cdot|$ are an argument and the absolute value of a complex number.

Taking into account the equalities

$$\frac{z_n^*}{|\dot{z}_n|} \equiv \exp(-j \cdot \phi_n), \quad \frac{\dot{z}_{n-1}}{|\dot{z}_{n-1}|} \equiv \exp(j \cdot \phi_{n-1}),$$

the Equation (21) is transformed to the view of (20).

In practice the next modified Lank's estimator also can be used

$$(22) \quad \hat{\gamma}_M \approx \arg \left(\sum_{n=2}^N [x(n)x(n-1) + y(n)y(n-1)] + j \sum_{n=2}^N [y(n)x(n-1) - x(n)y(n-1)] \right).$$

It is obtained by transformations (17) and

$$\sin(\Delta_n) \equiv \sin \varphi_n \cos \varphi_{n-1} - \cos \varphi_n \sin \varphi_{n-1} \approx \frac{y(n)}{\rho} \cdot \frac{x(n-1)}{\rho} - \frac{x(n)}{\rho} \cdot \frac{y(n-1)}{\rho}.$$

Therefore, we can say that equations (20)-(22) represent a frequency estimator of a complex signal in tree equivalent trigonometric, exponential and algebraic forms respectively.

4. Simulation results

4.1. Multichannel data

Computer simulations have been carried out to evaluate the performance of the synthesized method. The mean bias, the standard deviation are used as performance indicators. The synthesized estimator (called "optimal") is compared to the estimator obtained as an average of the multiple single-channel modified covariance estimates [9] (called "averaged") that can be written as

$$\hat{f} = \frac{1}{2\pi\tau\hat{M}} \sum_{m=1}^{\hat{M}} \arccos \left(\frac{\sum_{n=3}^N x_m(n-1)[x_m(n) + x_m(n-2)]}{2 \sum_{n=3}^N [x_m(n-1)]^2} \right),$$

where \hat{M} is a number of terms under arccos that are less than 1.

Statistical simulation by the Monte-Carlo approach was done under the next conditions: signal sample size in the single channel $N = 9$, number of channels $M = 7$, amplitudes at each channel are the same, initial phases are randomized, number of simulations for each plot is 1000.

Fig. 2 shows mean bias, standard deviation of both methods for different normalized frequencies for SNR=20 dB and Fig. 3 shows the same values for different SNR for frequency $\gamma = 1$ rad.

The plots in Fig. 2 are symmetrically relative to point $\pi/2$, but the mean bias has the opposite sign. One can see that both methods have similar precision in the middle frequency range, but the optimal methods become much better at low frequencies. If we look at Fig. 3, we can see that both methods have similar precision with minor advantage of the optimal one at low SNR. It should be noted that the difference between both methods become bigger when the number of samples in each channel approaches to the minimum $N=3$, and vanishes when N is big enough.

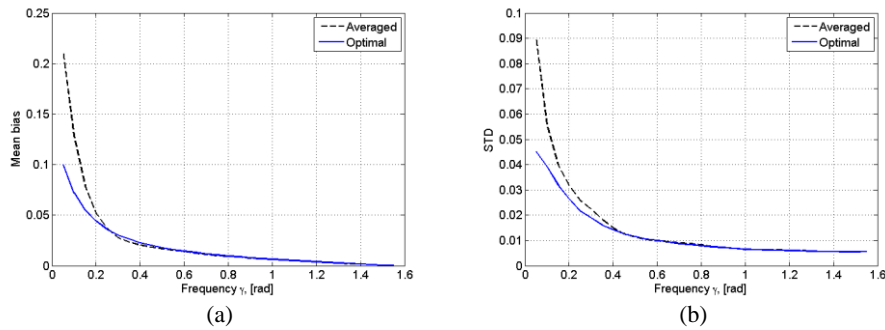


Fig. 2. Plots of mean bias and STD for different frequencies:
Mean bias (a); STD (b);

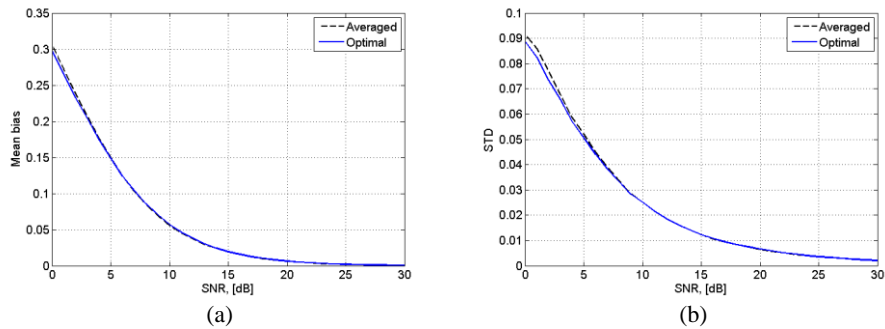


Fig. 3. Plots of mean bias and STD for different SNR:
Mean bias (a); STD (b)

4.2. Complex signal

In the complex signal case all methods mentioned – (12), (18), (19), (21) have been compared in the similar way. Fig. 4 shows the mean bias and standard deviation for different normalized frequencies for SNR=10 dB and Fig. 5 shows the same values for different SNR for frequency $\gamma = 1$ rad. Additionally the plot of Cramer-Rao Bound (CRB) [16] is shown in Figs 4b and 5b.

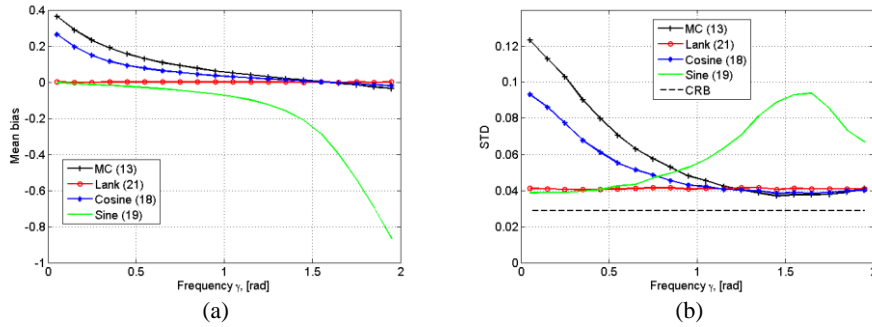


Fig. 4. Plots of mean bias and STD for different frequencies:
Mean bias (a); STD (b)

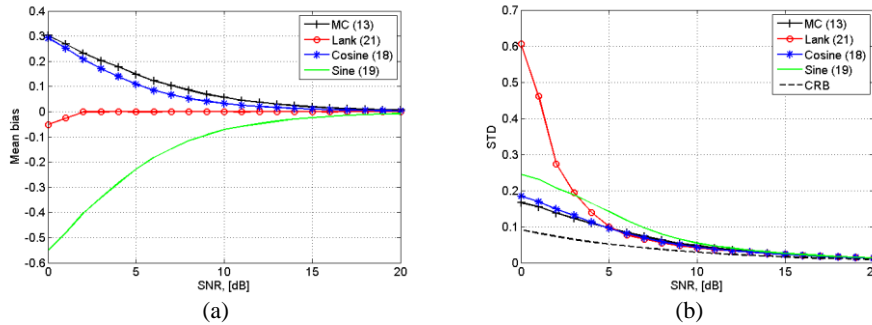


Fig. 5. Plots of mean bias and STD for different SNR:
Mean bias (a); STD (b)

One can see that the optimal estimator (12) is close to (18), the estimator (19) is the worst and the Lank's estimator (21) is the best one almost in all cases except at low SNR.

It should be noted that Lank's estimator is usually better for frequency estimation of a complex signal but it is impossible to use it for multichannel systems or even for a two-channel system with incoherent signals.

5. Conclusion

Using of the autoregressive model of the multichannel input process allows the synthesis by maximum likelihood method of the modified covariance estimator of frequency in the explicit form. The synthesized estimator provides better precision of estimation in comparison to averaging by single channels at low SNR and normalized frequency values less than 0.4 rad. It is shown, that the asymptotical modified covariance estimator has properties of the class of differential phases estimators. In the particular case of a quadrature signal it is reasonable to use Lank's frequency estimator.

Acknowledgments: The research work reported in the paper was partially supported by Project AComIn "Advanced Computing for Innovation", Grant 316087, funded by FP7 Capacity Programme (Research Potential of Convergence Regions).

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