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Dynamic Multi-Attribute Group Decision Making Model Based on Generalized Interval-Valued Trapezoidal Fuzzy Numbers

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Abstract: *In this paper we investigate the dynamic multi-attribute group decision making problems, in which all the attribute values are provided by multiple decision makers at different periods. In order to increase the level of overall satisfaction for the final decision and deal with uncertainty, the attribute values are enhanced with generalized interval-valued trapezoidal fuzzy numbers to cope with the vagueness and indeterminacy. We first define the Dynamic Generalized Interval-valued Trapezoidal Fuzzy Numbers Weighted Geometric Aggregation* (*DGITFNWGA*) *operator and give an approach to determine the weights of periods, using the probability density function of Gamma distribution, and then a dynamic multi-attribute group decision making method is developed. The method proposed employs the Generalized Interval-valued Trapezoidal Fuzzy Numbers Hybrid Geometric Aggregation* (*GITFNHGA*) *operator to aggregate all individual decision information into the collective attribute values corresponding to each alternative at the same time period, and then utilizes the DGITFNWGA operator to aggregate the collective attribute values at different periods into the overall attribute values corresponding to each alternative and obtains the alternatives ranking, by which the optimal alternative can be determined. Finally, an illustrative example is given to verify the approach developed.*

Keywords: *Dynamic multi-attribute group decision making, generalized interval-valued trapezoidal fuzzy numbers, probability density function.*

1. Introduction

Interval-valued fuzzy sets are characterized by an interval-valued membership function [1, 2], which is an extension of Zadeh's fuzzy set [3]. Interval-valued fuzzy sets permit additional degrees of freedom to represent the uncertainty and fuzziness of the real world [4]. Interval-valued fuzzy sets have been proven to be highly useful in capturing imprecise or uncertain decision information compared to ordinary fuzzy sets in fields that deal with multiple-attribute decision making problems [5-9].

Chen [10] put forth the concept of a generalized trapezoidal fuzzy number and proposed the ranking method.The difference between traditional fuzzy numbers and generalized fuzzy numbers is that the height of a traditional fuzzy number is equal to unity,whereas the height of a generalized fuzzy number is between zero and one [11]. In the work herein presented, generalized fuzzy numbers are extended to Generalized Interval-valued Trapezoidal Fuzzy Numbers (GITFNs). GITFNs are the more general form of fuzzy numbers which cannot only deal with vague information, but also express more abundant and flexible information than other fuzzy numbers. Additionally, GITFNs were used to propound a group decision-making method for handling fuzzy multi-criteria decision analysis problems.

However, most multiple-attribute decision making methods focus on these problems at the same period.When the decision making information is collected at different periods, all the exact methods are unsuitable for dealing with such situations. Therefore, it is necessary to pay attention to solve these problems. Recently, the research on dynamic multi-attribute decision making problems has received some attention. Xu [12] developed a dynamic weighted averaging operator with exact numerical values or interval values for dynamic multi-attribute decision making problems, and some derivation techniques, such as arithmetic series, geometric series, and normal distribution were proposed to elicit the weights in it. Xu and Yager [13] have investigated dynamic multi-attribute decision making problems, where the decision information takes the form of intuitionistic fuzzy information or interval-valued intuitionistic fuzzy information. Wei [14] further developed the dynamic intuitionistic fuzzy weighted geometric operator and uncertain dynamic intuitionistic fuzzy weighted geometric operator to solve dynamic multi-attribute decision making problems. Wei [15] proposed a grey relational analysis model for dynamic hybrid multiple attribute group decision making. Su et al. [16] proposed an internative method to solve the dynamic intuitionistic fuzzy multi-attribute group decision making problems. Thus, it is necessary to develop some new approaches to deal with these issues. In this paper an adaptive dynamic multi-attribute group decision making approach with GITFNs is proposed and a new approach based on the probability density function of Gamma distribution is developed to derive the periods weights.

The remainder of this paper is organized as follows: in Section 2 the contents related to GITFNs are described; in Section 3 three aggregation operators are briefly reviewed, the concepts of argument variable and Dynamic Generalized Interval-valued Trapezoidal Fuzzy Numbers Weights Geometric Aggregation

(DGITFNWGA) operator are defined, and then a method based on Gamma distribution is developed to derive the weights of periods. Section 4 proposes a technique characterized by GITFNHGA and DGITFNWGA operators for dynamic multi-attribute group decision making problems; in Section 5 an illustrative example is presented. The paper is concluded in Section 6.

2. Preliminaries

In this section,we briefly review the contents of the GITFNs.

Definition 1 [17, 18]. Let A be a GITFN defined in the universe of discourse X:

(1) $A=[A^L,\ A^U]=[(a_1^L,\ a_2^L,\ a_3^L,\ a_4^L;\ h_A^L),\ (a_1^U,\ a_2^U,\ a_3^U,\ a_4^U;\ h_A^U)]$,

where

$$A^L=(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L),\quad A^U=(a_1^U, a_2^U, a_3^U, a_4^U; h_A^U),$$

$$0\le a_1^L\le a_2^L\le a_3^L\le a_4^L\le 1 \text{ and } 0\le a_1^U\le a_2^U\le a_3^U\le a_4^U\le 1.$$

Here h_A^L and h_A^U denote the heights of A^L and A^U respectively, $0\le h_A^L\le h_A^U\le 1$, and $A^L\subset A^U$. A GITFN A (as shown in Fig. 1) consists of the lower values of the generalized interval-valued trapezoidal fuzzy number A^L and the upper values of the generalized interval-valued trapezoidal fuzzy number A^U. The membership functions of A^L and A^U are defined as follows:

(2)
$$A^L(x)=\begin{cases}\dfrac{x-a_1^L}{a_2^L-a_1^L}\times h_A^L & x\in(a_1^L, a_2^L),\\ h_A^L & x\in(a_2^L, a_3^L),\\ \dfrac{x-a_4^L}{a_3^L-a_4^L}\times h_A^L & x\in(a_3^L, a_4^L),\\ 0 & \text{otherwise;}\end{cases}$$

(3)
$$A^U(x)=\begin{cases}\dfrac{x-a_1^U}{a_2^U-a_1^U}\times h_A^U & x\in(a_1^U, a_2^U),\\ h_A^U & x\in(a_2^U, a_3^U),\\ \dfrac{x-a_4^U}{a_3^U-a_4^U}\times h_A^U & x\in(a_3^U, a_4^U),\\ 0 & \text{otherwise.}\end{cases}$$

If $h_A^L=h_A^U=1$ then A is called a normal interval-valued trapezoidal fuzzy number. If $h_A^L=h_A^U$ then A is called a generalized trapezoidal fuzzy number. If $a_2^L=a_3^L$ and $a_2^U=a_3^U$ then A is reduced to a generalized interval-valued triangular fuzzy number. If $a_2^L=a_3^L$, $a_2^U=a_3^U$ and $h_A^L=h_A^U=1$ then A becomes a normal interval-valued triangular fuzzy number. If $a_1^L=a_2^L=a_3^L=a_4^L$, $a_1^U=a_2^U=a_3^U=a_4^U$, and $h_A^L=h_A^U=1$ then A becomes a crisp interval.

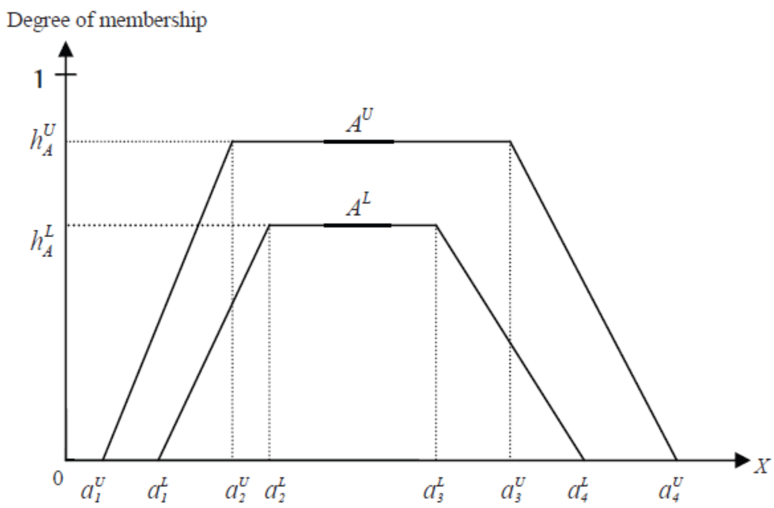

Fig. 1. A generalized interval-valued trapezoidal fuzzy number

Suppose that

$$A=[A^L, A^U]=[(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L), (a_1^U, a_2^U, a_3^U, a_4^U; h_A^U)]$$

and $B=[B^L, B^U]=[(b_1^L, b_2^L, b_3^L, b_4^L; h_B^L), (b_1^U, b_2^U, b_3^U, b_4^U; h_B^U)]$

are two generalized interval-valued trapezoidal fuzzy numbers, where:

$$0 \le a_1^L \le a_2^L \le a_3^L \le a_4^L \le 1,$$

$$0 \le a_1^U \le a_2^U \le a_3^U \le a_4^U \le 1,\ 0 \le h_A^L \le h_A^U \le 1,\ A^L \subset A^U,\ 0 \le b_1^L \le b_2^L \le b_3^L \le b_4^L \le 1,$$

$$0 \le b_1^U \le b_2^U \le b_3^U \le b_4^U \le 1,\ 0 \le h_B^L \le h_B^U \le 1, \text{ and } B^L \subset B^U.$$

Then the operational rules are as follows in next

Definition 2 [19]. The operational rules of generalized interval-valued trapezoidal fuzzy numbers:

(1) addition operation

(4) $$A \oplus B=[(a_1^L+b_1^L, a_2^L+b_2^L, a_3^L+b_3^L, a_4^L+b_4^L;\ \min(h_A^L, h_B^L)),$$
$$(a_1^U+b_1^U, a_2^U+b_2^U, a_3^U+b_3^U, a_4^U+b_4^U;\ \min(h_A^U, h_B^U))];$$

(2) subtraction operation

(5) $$A \ominus B=[(a_1^L-b_4^L, a_2^L-b_3^L, a_3^L-b_2^L, a_4^L-b_1^L;\ \min(h_A^L, h_B^L)),$$
$$(a_1^U-b_4^U, a_2^U-b_3^U, a_3^U-b_2^U, a_4^U-b_1^U;\ \min(h_A^U, h_B^U))];$$

(3) multiplication operation

(6) $$A \otimes B=[(a_1^L \times b_1^L, a_2^L \times b_2^L, a_3^L \times b_3^L, a_4^L \times b_4^L;\ \min(h_A^L, h_B^L)),$$
$$(a_1^U \times b_1^U, a_2^U \times b_2^U, a_3^U \times b_3^U, a_4^U \times b_4^U;\ \min(h_A^U, h_B^U))];$$

(4) division operation

(7) $$A \oslash B=[(a_1^L/b_1^L, a_2^L/b_2^L, a_3^L/b_3^L, a_4^L/b_4^L;\ \min(h_A^L, h_B^L)),$$
$$(a_1^U/b_1^U, a_2^U/b_2^U, a_3^U/b_3^U, a_4^U/b_4^U;\ \min(h_A^U, h_B^U))];$$

(5) multiplication by a positive real number θ

(8) $$\theta A = [(\theta a_1^L, \theta a_2^L, \theta a_3^L, \theta a_4^L; h_A^L), (\theta a_1^U, \theta a_2^U, \theta a_3^U, \theta a_4^U; h_A^U)];$$

(6) exponentiation of a positive real number θ

(9) $$A^\theta = [((a_1^L)^\theta, (a_2^L)^\theta, (a_3^L)^\theta, (a_4^L)^\theta; h_A^L), ((a_1^U)^\theta, (a_2^U)^\theta, (a_3^U)^\theta, (a_4^U)^\theta; h_A^U)].$$

Definition 3 [20]. Let

$$A = [A^L, A^U] = [(a_1^L, a_2^L, a_3^L, a_4^L; h_A^L), (a_1^U, a_2^U, a_3^U, a_4^U; h_A^U)],$$

$$B = [B^L, B^U] = [(b_1^L, b_2^L, b_3^L, b_4^L; h_B^L), (b_1^U, b_2^U, b_3^U, b_4^U; h_B^U)]$$

and $C = [C^L, C^U] = [(c_1^L, c_2^L, c_3^L, c_4^L; h_C^L), (c_1^U, c_2^U, c_3^U, c_4^U; h_C^U)]$

be three generalized interval-valued trapezoidal fuzzy numbers, then the distance of two generalized interval-valued trapezoidal fuzzy numbers is defined as follows:

(10) $$d(A,B) = \frac{1}{8}\times(|a_1^L \times h_A^L - b_1^L \times h_B^L| + |a_2^L \times h_A^L - b_2^L \times h_B^L| + |a_3^L \times h_A^L - b_3^L \times h_B^L| + \\ + |a_4^L \times h_A^L - b_4^L \times h_B^L| + |a_1^U \times h_A^U - b_1^U \times h_B^U| + |a_2^U \times h_A^U - b_2^U \times h_B^U| + \\ + |a_3^U \times h_A^U - b_3^U \times h_B^U| + |a_4^U \times h_A^U - b_4^U \times h_B^U|).$$

Here if A and B are the normalized generalized interval-valued trapezoidal fuzzy numbers, then $d(A, B)$ satisfies the following properties:

(1) $0 \le d(A, B) \le 1$;

(2) $A = B \Leftrightarrow d(A, B) = 0$;

(3) $d(A, B) = d(B, A)$;

(4) $d(A, C) + d(B, C) \ge d(A, B)$.

Definition 4 [20]. Let A and B be two normalized generalized interval-valued trapezoidal fuzzy numbers, $\tilde{1} = [(1,1,1,1;1), (1,1,1,1;1)]$ is the positive ideal value. According to the principle of "closer is better", the ranking of A and B are defined by the distances $d(A, \tilde{1})$ and $d(B, \tilde{1})$:

(1) if $d(A, \tilde{1}) > d(B, \tilde{1})$ then $A \prec B$;

(2) if $d(A, \tilde{1}) = d(B, \tilde{1})$ then A: B;

(3) if $d(A, \tilde{1}) < d(B, \tilde{1})$ then $A \succ B$.

3. Dynamic generalized interval-valued trapezoidal fuzzy numbers weighted geometric aggregation operator

Let $(A_1, A_2, \ldots, A_n)$ be a collection of generalized interval-valued trapezoidal fuzzy numbers

$$A_i = [A_i^L, A_i^U] = [(a_{1i}^L, a_{2i}^L, a_{3i}^L, a_{4i}^L; h_{A_i}^L), (a_{1i}^U, a_{2i}^U, a_{3i}^U, a_{4i}^U; h_{A_i}^U)],$$

$$i = 1, 2, \ldots, n,$$

Q be the set of all generalized interval-valued trapezoidal fuzzy numbers, then we review three aggregation operators as follows in the next definitions.

Definition 5 [20]. Let GITFNWGA: $Q^n \to Q$, if

(11) $$\mathrm{GITFNWGA}_w(A_1, A_2, \ldots, A_n) = \prod_{i=1}^{n} (A_i)^{w_i},$$

where $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of A_i, $i = 1, 2, \ldots, n$, and $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, then the function GITFNWGA is called the generalized interval-valued trapezoidal fuzzy numbers weighted geometric aggregation operator. If $w = (1/n, 1/n, \ldots, 1/n)$, the generalized interval-valued trapezoidal fuzzy numbers weighted geometric aggregation operator is reduced to the generalized interval-valued trapezoidal fuzzy numbers geometric aggregation operator.

Definition 6 [20]. Let GITFNOWGA: $Q^n \to Q$, if

(12) $$\mathrm{GITFNOWGA}_v(A_1, A_2, \ldots, A_n) = \prod_{j=1}^{n} (\delta_j)^{v_j},$$

where δ_j is the *j*-th largest of the arguments A_i, $i = 1, 2, \ldots, n$, $v = (v_1, v_2, \ldots, v_n)$ is the weight vector associated with GITFNOWGA and $v_i \geq 0$, $\sum_{i=1}^{n} v_i = 1$, then the function GITFNOWGA is called the generalized interval-valued trapezoidal fuzzy numbers ordered weighted geometric aggregation operator.

Definition 7 [20]. Let GITFNHGA: $Q^n \to Q$ if

(13) $$\mathrm{GITFNHGA}_{v,w}(A_1, A_2, \ldots, A_n) = \prod_{j=1}^{n} (\sigma_j)^{v_j},$$

where σ_j is the *j*-th largest of the weighted arguments $(A_i)^{nw_i}$, $i = 1, 2, \ldots, n$, $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of A_i, $i = 1, 2, \ldots, n$, n plays the role of balance; $v = (v_1, v_2, \ldots, v_n)$ is the weight vector associated with GITFNHGA and $v_i \geq 0$, $\sum_{i=1}^{n} v_i = 1$. Then the function GITFNHGA is called the generalized interval-valued trapezoidal fuzzy numbers hybrid geometric aggregation operator (in this case if the vector $(w_1, w_2, \ldots, w_n)$ approaches $(1/n, 1/n, \ldots, 1/n)$, then the argument $((A_1)^{nw_1}, (A_2)^{nw_2}, \ldots, (A_n)^{nw_n})$ approaches $(A_1, A_2, \ldots, A_n)$.

It is clear that GITFNHGA operator generalizes the GITFNOWGA and GITFNWGA operators and can reflect the importance of both the given argument and the order position of the argument.

Information aggregation is a necessary step in the process of dynamic multi-attribute decision making, the input arguments collected from different periods and the time series weights should be considered. So we introduce a dynamic generalized interval-valued trapezoidal fuzzy numbers weighted geometric aggregation operator in

Definition 8. Let *t* be a time variable, then we call

$$A_t = [A_t^L, A_t^U] = [(a_{1t}^L, a_{2t}^L, a_{3t}^L, a_{4t}^L;\ h_{A_t}^L),\ (a_{1t}^U, a_{2t}^U, a_{3t}^U, a_{4t}^U;\ h_{A_t}^U)]$$

a generalized interval-valued trapezoidal fuzzy variable, where $0 \le a_{1t}^L \le a_{2t}^L \le a_{3t}^L \le a_{4t}^L \le 1,$ $0 \le a_{1t}^U \le a_{2t}^U \le a_{3t}^U \le a_{4t}^U \le 1,$ $0 \le h_{A_t}^L \le h_{A_t}^U \le 1,$ and $A_t^L \subset A_t^U.$ For a generalized interval-valued trapezoidal fuzzy variable $A_t = [A_t^L, A_t^U],$ if $t = (t_1, t_2, \ldots, t_p),$ then $A_t = (A_{t_1}, A_{t_2}, \ldots, A_{t_p})$ indicate p generalized interval-valued trapezoidal fuzzy numbers collected at p different periods.

Definition 9. Let $A_t = (A_{t_1}, A_{t_2}, \ldots, A_{t_p})$ be a collection of generalized interval-valued trapezoidal fuzzy numbers A_{t_k} collected at different periods $t_k,$ $k = 1, 2, \ldots, p,$ and $\eta(t) = (\eta(t_1), \eta(t_2), \ldots, \eta(t_p))$ be the weight vector of the periods $t_k,$ $k = 1, 2, \ldots, p,$ then

(14)
$$\mathrm{DGITFNWGA}_{\eta(t)}(A_{t_1}, A_{t_2}, \ldots, A_{t_p}) = \prod_{k=1}^{p} (A_{t_k})^{\eta(t_k)} =$$
$$= \prod_{k=1}^{p} [(a_{1t_k}^L, a_{2t_k}^L, a_{3t_k}^L, a_{4t_k}^L; h_{A_{t_k}}^L), (a_{1t_k}^U, a_{2t_k}^U, a_{3t_k}^U, a_{4t_k}^U; h_{A_{t_k}}^U)]^{\eta(t_k)}$$

is called a dynamic generalized interval-valued trapezoidal fuzzy numbers weighted geometric aggregation operator, where

(15)
$$\eta(t_k) \ge 0, \sum_{k=1}^{p} \eta(t_k) = 1, \ \ k = 1, 2, \ldots, p.$$

For DGITFNWGA operator, to determine the weight vector $\eta(t) = (\eta(t_1), \eta(t_2), \ldots, \eta(t_p))$ of the periods $t_k,$ $k = 1, 2, \ldots, p,$ is a key step. Generally $\eta(t)$ can be determined by the decision maker(s), in what follows we introduce the Gamma distribution based method to determine $\eta(t).$

In probability theory and statistics, Gamma distribution is one of the best known and widely used two-parameters family of continuous probability distributions. It has a shape parameter α and a scale parameter β. If α is an integer, then the distribution represents Erlang distribution (the sum of α independent exponentially distributed random variables, each of them has a mean of β). The Gamma distribution is frequently a probability model for waiting times. For instance, in life testing, the waiting time until death is a random variable that is frequently modeled by Gamma distribution. The Gamma distribution has also been used to model the size of insurance claims and rainfalls [21]. In neuroscience, the Gamma distribution is often used to describe the distribution of inter-spike intervals [22]. The probability density function of Gamma distribution is given by

(16)
$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha} \Gamma(\alpha)} \ \ \text{for } x \ge 0 \text{ and } \alpha, \beta > 0.$$

In order to understand more about the probability density function of Gamma distribution, we need to investigate its characteristics.

By taking the first derivative of (16) with respect to x

$$f'(x) = \frac{x^{\alpha-2} e^{-\frac{x}{\beta}} (\beta(\alpha-1)-x)}{\beta^{\alpha+1}\Gamma(\alpha)} \quad \text{for } x \geq 0 \text{ and } \alpha, \beta > 0, \tag{17}$$

Thus:

(1) If $0 < \alpha < 1,$ then $f'(x) < 0,$ i.e., $f(x)$ is a strictly monotonic decreasing function.

(2) If $\alpha = 1,$ the Gamma distribution is reduced to exponential distribution, $f(x)$ is also a strictly monotonic decreasing function. X u [13] has proposed the exponential distribution based method to determine the weights of the periods so that the details about $\alpha = 1$ are omitted here.

(3) If $\alpha > 1,$ $f'(x) = 0,$ when $x = \beta(\alpha-1),$ by taking the second derivative of (16) with respect to x we can obtain

$$f''(\beta(\alpha-1)) = \frac{(\beta(\alpha-1))^{\alpha-3} e^{1-\alpha}(1-\alpha)}{\beta^{\alpha}\Gamma(\alpha)} < 0 \quad \text{for } \alpha > 1,\ \beta > 0, \tag{18}$$

thus $f(x)$ has the maximum value when $x = \beta(\alpha-1),$ then

(3.1) if $0 \leq x \leq \beta(\alpha-1),$ $f(x)$ is a strictly monotonic increasing function;

(3.2) if $x > \beta(\alpha-1),$ $f(x)$ is a strictly monotonic decreasing function.

Therefore, for $\alpha = 1$ the Gamma distribution is reduced to exponential. For $\alpha > 1$ the density functions represent positively skewed curves which start at the origin and which tend to normal distribution as α tends to infinity. For $0 < \alpha < 1$ the axes are asymptotes to the density curves.

The general characteristics of Gamma distribution are more suitable than other probability distributions in generating the weight vector $\eta(t).$ By utilizing the probability density function of the Gamma distribution, (16) can be rewritten as follows:

$$\eta(t_k) = \frac{k^{\alpha-1} e^{-\frac{k}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} \quad \text{for } k = 1, 2, \ldots, p \text{ and } \alpha, \beta > 0. \tag{19}$$

By Equations (15) and (19) we have

$$\eta(t_k) = \frac{k^{\alpha-1} e^{-\frac{k}{\beta}}}{\sum_{k=1}^{p} k^{\alpha-1} e^{-\frac{k}{\beta}}} \quad \text{for } k = 1, 2, \ldots, p \text{ and } \alpha, \beta > 0. \tag{20}$$

By that equation the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, p,$ has the following properties:

(1) If $0 < \alpha \leq 1$ then $\eta(t_k) > \eta(t_{k+1}),$ $k = 1, 2, \ldots, p-1,$ the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, p,$ is a strictly monotonic decreasing sequence.

(2) If $\alpha > 1$ then

(2.1) if $0 < \beta(\alpha-1) \leq 1$ then $\eta(t_k) > \eta(t_{k+1}),$ $k = 1, 2, \ldots, p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, p,$ is a strictly monotonic decreasing sequence;

(2.2) if $\beta(\alpha-1)>1,$ then we discuss the following two cases:

(a) if $\beta(\alpha-1)$ is an integer, let $k_0=\beta(\alpha-1),$ such that

(i) $\eta(t_{k+1})>\eta(t_k),$ $k=1,2,\ldots,k_0-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k=1,2,\ldots,k_0,$ is a strictly monotonic increasing sequence;

(ii) $\eta(t_{k+1})<\eta(t_k),$ $k=k_0,\ldots,p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k=k_0,\ldots,p,$ is a strictly monotonic decreasing sequence;

(b) if $\beta(\alpha-1)$ is a noninteger, then there exist two integers $k_0=\text{int}(\beta(\alpha-1))$ (here,"int" denotes the integer part of $\beta(\alpha-1)$) and $k_0+1,$ such that

(i) $\eta(t_{k+1})>\eta(t_k),$ $k=1,2,\ldots,k_0-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k=1,2,\ldots,k_0,$ is a strictly monotonic increasing sequence;

(ii) $\eta(t_{k+1})<\eta(t_k),$ $k=k_0+1,\ldots,p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k=k_0+1,\ldots,p,$ is a strictly monotonic decreasing sequence.

The distribution of the time series weights can be generated using (20) in the case of $p=10$ and is plotted in Fig. 2 for the shape parameter $\alpha=2$ and the scale parameter $\beta=1,3,5,7,9.$

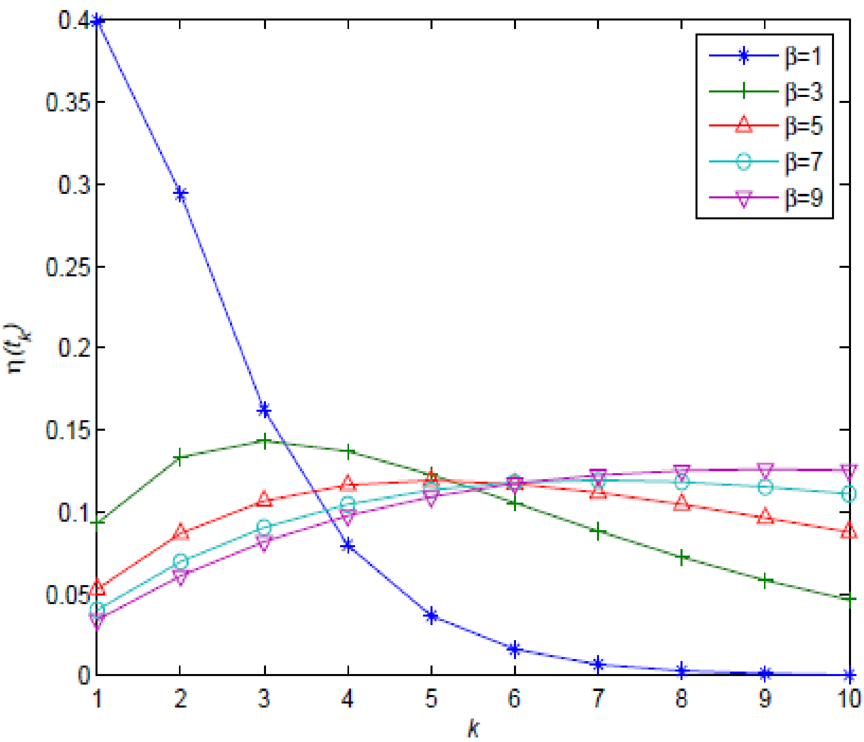

Fig. 2. Time series weights generated by using Gamma distribution (p = 10) for α = 2 and β = 1, 3, 5, 7, 9

If we use the inverse form of Gamma distribution to determine the weight vector, then the following equation is defined:

(21) $$\eta(t_k) = 1/\frac{k^{\alpha-1}e^{-\frac{k}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} = \frac{\beta^{\alpha}e^{\frac{k}{\beta}}\Gamma(\alpha)}{k^{\alpha-1}} \quad \text{for } k = 1, 2, \ldots, p \text{ and } \alpha, \beta > 0.$$

By (15) and (21) we have

(22) $$\eta(t_k) = \frac{e^{\frac{k}{\beta}}}{k^{\alpha-1}} / \sum_{k=1}^{p} \frac{e^{\frac{k}{\beta}}}{k^{\alpha-1}} \quad \text{for } k = 1, 2, \ldots, p \text{ and } \alpha, \beta > 0,$$

which has the following properties:

(1) If $0 < \alpha \le 1,$ then $\eta(t_k) < \eta(t_{k+1}),$ $k = 1, 2, \ldots, p-1,$ the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, p,$ is a strictly monotonic increasing sequence.

(2) If $\alpha > 1,$ then

(2.1) if $0 < \beta(\alpha - 1) \le 1$ then $\eta(t_k) < \eta(t_{k+1}),$ $k = 1, 2, \ldots, p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, p,$ is a strictly monotonic increasing sequence;

(2.2) if $\beta(\alpha - 1) > 1$ then we discuss the following two cases:

(a) if $\beta(\alpha - 1)$ is an integer let $k_0 = \beta(\alpha - 1),$ such that

(i) $\eta(t_{k+1}) < \eta(t_k),$ $k = 1, 2, \ldots, k_0 - 1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, k_0,$ is a strictly monotonic decreasing sequence;

(ii) $\eta(t_{k+1}) > \eta(t_k),$ $k = k_0, \ldots, p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = k_0, \ldots, p,$ is a strictly monotonic increasing sequence;

(b) if $\beta(\alpha - 1)$ is a noninteger, then there exist two integers $k_0 = \mathrm{int}(\beta(\alpha - 1))$ (here "int" denotes the integer part of $\beta(\alpha - 1)$) and $k_0 + 1,$ such that

(i) $\eta(t_{k+1}) < \eta(t_k),$ $k = 1, 2, \ldots, k_0 - 1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = 1, 2, \ldots, k_0,$ is a strictly monotonic decreasing sequence;

(ii) $\eta(t_{k+1}) > \eta(t_k)),$ $k = k_0 + 1, \ldots, p-1,$ i.e., the sequence $\{\eta(t_k)\},$ $k = k_0 + 1, \ldots, p,$ is a strictly monotonic increasing sequence.

The distribution of the time series weights can be generated using (22) in the case of p = 10 and is plotted in Fig. 3 for the shape parameter $\alpha = 2$ and the scale parameter $\beta = 1, 3, 5, 7, 9.$

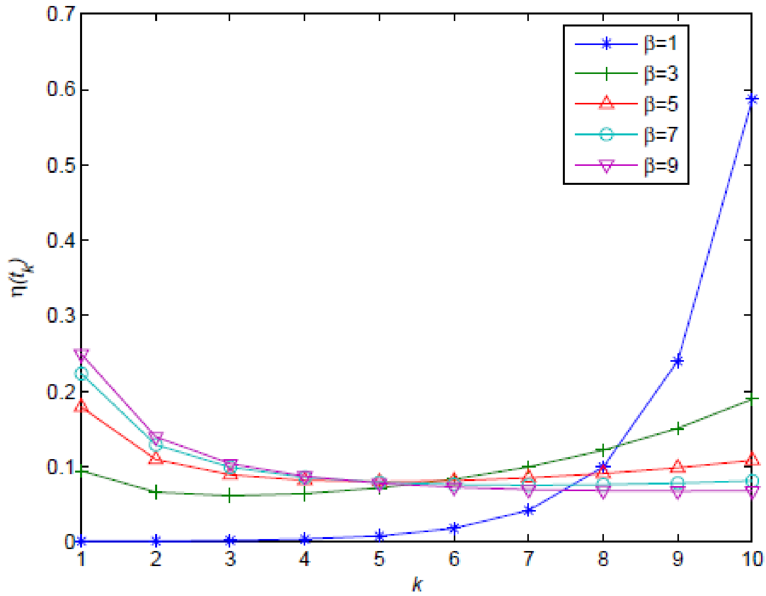

Fig. 3. Time series weights generated using the inverse form of Gamma distribution (p = 10) for $\alpha = 2$ and $\beta = 1, 3, 5, 7, 9$

For convenience in practical applications we take some special values of α and β, then utilize (20) and (22) to determe the weight vectors associated with the DGITFNWGA operator (Table 1).

Table 1. The weight vector $\eta(t)$ with $\beta = 1, 2, \ldots, 9$ and $\alpha = 2$, $p = 2, 3$

Equation	p	$\eta(t)$	$\beta = 1$	$\beta = 2$	$\beta = 3$	$\beta = 4$	$\beta = 5$	$\beta = 6$	$\beta = 7$	$\beta = 8$	$\beta = 9$
(20)	$p = 2$	$\eta(t_1)$	0.576	0.452	0.411	0.391	0.379	0.371	0.366	0.362	0.357
		$\eta(t_2)$	0.424	0.548	0.589	0.609	0.621	0.629	0.634	0.638	0.643
	$p = 3$	$\eta(t_1)$	0.467	0.302	0.252	0.229	0.215	0.207	0.200	0.196	0.192
		$\eta(t_2)$	0.343	0.366	0.361	0.356	0.352	0.350	0.348	0.346	0.346
		$\eta(t_3)$	0.190	0.332	0.387	0.415	0.433	0.443	0.452	0.458	0.462
(22)	$p = 2$	$\eta(t_1)$	0.424	0.548	0.589	0.609	0.621	0.629	0.634	0.638	0.642
		$\eta(t_2)$	0.576	0.452	0.411	0.391	0.379	0.371	0.366	0.362	0.358
	$p = 3$	$\eta(t_1)$	0.207	0.366	0.426	0.456	0.474	0.486	0.495	0.501	0.506
		$\eta(t_2)$	0.282	0.302	0.297	0.293	0.290	0.287	0.285	0.284	0.283
		$\eta(t_3)$	0.511	0.332	0.277	0.251	0.236	0.227	0.220	0.215	0.211

4. Dynamic multi-attribute group decision making method based on GITFNs

In this section we consider the dynamic multi-attribute group decision making problem, in which all the attribute values, provided by the decision makers at different periods, take the form of generalized interval-valued trapezoidal fuzzy numbers. We give a detailed description of the considered dynamic multi-attribute group decision making problems:

Let $X = \{x_1, x_2, \ldots, x_n\}$, be the set of n feasible alternatives, $U = \{u_1, u_2, \ldots, u_m\}$ be the set of the attributes and $E = \{e_1, e_2, \ldots, e_q\}$ be the set of decision makers, whose weight vector is $\pi = (\pi_1, \pi_2, \ldots, \pi_q)$, where $\pi_s > 0$, $s = 1, 2, \ldots, q$, $\sum_{s=1}^{q} \pi_s = 1$. There are p periods t_k, $k = 1, 2, \ldots, p$, whose weight vector is $\eta(t) = (\eta(t_1), \eta(t_2), \ldots, \eta(t_p))$, where $\eta(t_k) \in [0, 1]$, $\sum_{k=1}^{p} \eta(t_k) = 1$; $\omega(t_k) = (\omega_1(t_k), \omega_2(t_k), \ldots, \omega_m(t_k))$ is the weight vector of the attributes $u_i = (u_1, u_2, \ldots, u_m)$ at the period t_k, where $\omega_i(t_k) > 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} \omega_i(t_k) = 1$. The decision makers e_s, $s = 1, 2, \ldots, q$, provide the attribute values $\tilde{a}_{ijt_k}^{s}$ of the alternative $x_j \in X$, $j = 1, 2, \ldots, n$, with respect to the attribute $u_i \in U$, $i = 1, 2, \ldots, m$, at the period t_k, $k = 1, 2, \ldots, p$, and construct the generalized interval-valued trapezoidal fuzzy numbers decision matrices $\tilde{A}_{t_k}^{s} = (\tilde{a}_{ijt_k}^{s})_{m \times n}$, where

$$\tilde{a}_{ijt_k}^{s} = [\tilde{a}_{ijt_k}^{sL}, \tilde{a}_{ijt_k}^{sU}] = [(a_{1ijt_k}^{sL}, a_{2ijt_k}^{sL}, a_{3ijt_k}^{sL}, a_{4ijt_k}^{sL}; h_{\tilde{a}_{ijt_k}^{s}}^{L}),\ (a_{1ijt_k}^{sU}, a_{2ijt_k}^{sU}, a_{3ijt_k}^{sU}, a_{4ijt_k}^{sU}; h_{\tilde{a}_{ijt_k}^{s}}^{U})],$$

and $\tilde{a}_{ijt_k}^{sL} \subset \tilde{a}_{ijt_k}^{sU}$. As above stated, the ratings of the alternative evaluations on various attributes are expressed as generalized interval-valued trapezoidal fuzzy numbers. However, it is difficult to collect the original information in the form of generalized interval-valued trapezoidal fuzzy numbers to give the attribute values directly by the decision makers. So we usually use the form of linguistic terms to overcome the difficulty of data collection. Wei and Chen have utilized the generalized interval-valued trapezoidal fuzzy numbers to represent 9-members linguistic terms [19] (Table 2).

Table 2. A 9-members linguistic term set [19]

Linguistic terms	Generalized interval-valued trapezoidal fuzzy numbers
Absolutely-low	[(0.00, 0.00, 0.00, 0.00; 0.8), (0.00, 0.00, 0.00, 0.00; 1.0)]
Very-low	[(0.00, 0.00, 0.02, 0.07; 0.8), (0.00, 0.00, 0.02, 0.07; 1.0)]
Low	[(0.04, 0.10, 0.18, 0.23; 0.8), (0.04, 0.10, 0.18, 0.23; 1.0)]
Fairly-low	[(0.17, 0.22, 0.36, 0.42; 0.8), (0.17, 0.22, 0.36, 0.42; 1.0)]
Medium	[(0.32, 0.41, 0.58, 0.65; 0.8), (0.32, 0.41, 0.58, 0.65; 1.0)]
Fairly-high	[(0.58, 0.63, 0.80, 0.86; 0.8), (0.58, 0.63, 0.80, 0.86; 1.0)]
High	[(0.72, 0.78, 0.92, 0.97; 0.8), (0.72, 0.78, 0.92, 0.97; 1.0)]
Very-high	[(0.93, 0.98, 1.00, 1.00; 0.8), (0.93, 0.98, 1.00, 1.00; 1.0)]
Absolutely-high	[(1.00, 1.00, 1.00, 1.00; 0.8), (1.00, 1.00, 1.00, 1.00; 1.0)]

Based on the above decision information, we develop a practical method to solve the dynamic multi-attribute group decision making problems. The method involves five steps.

Step 1. Convert the linguistic decision matrices $G_{t_k}^s$ provided by the decision makers e_s, $s = 1, 2, \ldots, q$, into generalized interval-valued trapezoidal fuzzy number matrices $\tilde{A}_{t_k}^s = (\tilde{a}_{ijt_k}^s)_{m \times n}$. In general, there are benefit attributes and cost attributes in dynamic multi-attribute group decision making problems. So we need to normalize the decision matrix $\tilde{A}_{t_k}^s = (\tilde{a}_{ijt_k}^s)_{m \times n}$ into the corresponding decision matrix $\tilde{R}_{t_k}^s = (\tilde{r}_{ijt_k}^s)_{m \times n}$:

$$\tilde{r}_{ijt_k}^s = [(r_{1ijt_k}^{sL}, r_{2ijt_k}^{sL}, r_{3ijt_k}^{sL}, r_{4ijt_k}^{sL}; h_{\tilde{r}_{ijt_k}^s}^L), (r_{1ijt_k}^{sU}, r_{2ijt_k}^{sU}, r_{3ijt_k}^{sU}, r_{4ijt_k}^{sU}; h_{\tilde{r}_{ijt_k}^s}^U =$$

(23)
$$= [(\frac{a_{1ijt_k}^{sL}}{y_{jt_k}^s}, \frac{a_{2ijt_k}^{sL}}{y_{jt_k}^s}, \frac{a_{3ijt_k}^{sL}}{y_{jt_k}^s}, \frac{a_{4ijt_k}^{sL}}{y_{jt_k}^s}; h_{\tilde{a}_{ijt_k}^s}^L), (\frac{a_{1ijt_k}^{sU}}{y_{jt_k}^s}, \frac{a_{2ijt_k}^{sU}}{y_{jt_k}^s}, \frac{a_{3ijt_k}^{sU}}{y_{jt_k}^s}, \frac{a_{4ijt_k}^{sU}}{y_{jt_k}^s}; h_{\tilde{a}_{ijt_k}^s}^U))],$$

for benefit attributes, where

$$y_{jt_k}^s = \max_i (a_{4ijt_k}^{sU}), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, p;$$

(24)
$$\tilde{r}_{ijt_k}^s = [(r_{1ijt_k}^{sL}, r_{2ijt_k}^{sL}, r_{3ijt_k}^{sL}, r_{4ijt_k}^{sL}; h_{\tilde{r}_{ijt_k}^s}^L), (r_{1ijt_k}^{sU}, r_{2ijt_k}^{sU}, r_{3ijt_k}^{sU}, r_{4ijt_k}^{sU}; h_{\tilde{r}_{ijt_k}^s}^U)] =$$

$$[(\frac{z_{jt_k}^s}{a_{4ijt_k}^{sL}}, \frac{z_{jt_k}^s}{a_{3ijt_k}^{sL}}, \frac{z_{jt_k}^s}{a_{2ijt_k}^{sL}}, \frac{z_{jt_k}^s}{a_{1ijt_k}^{sL}}; h_{\tilde{a}_{ijt_k}^s}^L), (\frac{z_{jt_k}^s}{a_{4ijt_k}^{sU}}, \frac{z_{jt_k}^s}{a_{3ijt_k}^{sU}}, \frac{z_{jt_k}^s}{a_{2ijt_k}^{sU}}, \frac{z_{jt_k}^s}{a_{1ijt_k}^{sU}}; h_{\tilde{a}_{ijt_k}^s}^U)],$$

for cost attributes, where

$$z_{jt_k}^s = \min_i (a_{1ijt_k}^{sU}), \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots, p.$$

Step 2. Utilize the GITFNWGA operator to aggregate the attribute values $\tilde{r}_{ijt_k}^s$, $i = 1, 2, \ldots, m$, in the j-th column of $\tilde{R}_{t_k}^s$ provided by the decision maker e_s at period t_k and get the overall attribute value $\tilde{r}_{jt_k}^s$ corresponding to the alternative x_j:

$$\tilde{r}_{jt_k}^s = \text{GITFNWGA}(\tilde{r}_{1jt_k}^s, \tilde{r}_{2jt_k}^s, \ldots, \tilde{r}_{mjt_k}^s) =$$

(25)
$$= \prod_{i=1}^m [(r_{1ijt_k}^{sL}, r_{2ijt_k}^{sL}, r_{3ijt_k}^{sL}, r_{4ijt_k}^{sL}; h_{\tilde{r}_{ijt_k}^s}^L), \ (r_{1ijt_k}^{sU}, r_{2ijt_k}^{sU}, r_{3ijt_k}^{sU}, r_{4ijt_k}^{sU}; h_{\tilde{r}_{ijt_k}^s}^U)]^{\omega_i(t_k)}.$$

Step 3. Utilizing the GITFNHGA operator to aggregate the overall attribute values $\tilde{r}_{jt_k}^s$, $s = 1, 2, \ldots, q$, provided by all the decision makers e_s, $s = 1, 2, \ldots, q$, we get the overall attribute value $\tilde{r}_{jt_k}$ of the alterative x_j at period t_k:

(26)
$$\tilde{r}_{jt_k} = \text{GITFNHGA}(\tilde{r}_{jt_k}^1, \tilde{r}_{jt_k}^2, \ldots, \tilde{r}_{jt_k}^q) = \prod_{l=1}^q (\bar{r}_{jt_k}^l)^{v_l},$$

where $v=(v_1,v_2,\ldots,v_q)$ is the weight vector associated with GITFNHGA operator, $v_l>0$, $\sum_{l=1}^{q}v_l=1$; $\bar{r}_{jt_k}^{l}$ is the l-th largest of the weighted arguments $(\tilde{r}_{jt_k}^{s})^{q\pi_s}$, $s=1,2,\ldots,q$, $\pi=(\pi_1,\pi_2,\ldots,\pi_q)$ is the weight vector of the decision makers e_s, $s=1,2,\ldots,q$, q is not only the number of the decision makers, but also a balancing coefficient,which plays the role of balance. Here we can utilize the normal distribution based method [23] to determine the weight vector associated with GITFNHGA operator because the approach can assign low weights to those “false” or “biased” decision makers, which can make the decision results more scientific and reasonable.

Step 4. Utilizing the DGITFNWGA operator to aggregate the attribute value $\tilde{r}_{jt_k}$ which are provided at different periods t_k, $k=1,2,\ldots,p$, we can get the overall attribute value $\tilde{r}_j$, $j=1,2,\ldots,n$, of the alterative x_j:

(27) $$\tilde{r}_j=\mathrm{DGITFNWGA}(\tilde{r}_{jt_1},\tilde{r}_{jt_2},\ldots,\tilde{r}_{jt_p})=\prod_{k=1}^{p}(\tilde{r}_{jt_k})^{\eta(t_k)}.$$

Step 5. Rank the alternatives $\tilde{r}_j$ in accordance with the ranking method in Section 2 and then select the best one(s).

5. An illustrative example

Suppose that an investment company wants to invest a sum of money in the best option (adapted from [24]). There are five possible companies x_j, $j=1,2,\ldots,5$, in which to invest the money: (1) x_1 is a car company; (2) x_2 is a food company; (3) x_3 is a computer company; (4) x_4 is an arms company; (5) x_5 is a TV company. Three decision makers e_s, $s=1,2,3$, utilize the linguistic terms to evaluate the performance of the five companies in the period 2007-2009 with regard to three attributes u_i, $i=1,2,3$; (1) economic benefit u_1; (2) social benefit u_2 and (3) environmental pollution u_3, and construct the decision matrices $G_{t_k}^{s}$, $k,s=1,2,3$. Here, t_1 denotes the year “2007”, t_2 denotes the year “2008” and t_3 denotes the year “2009” respectively, as listed in Tables 3-11.

Table 3. Decision matrix $G_{t_1}^{1}$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	VH	H	VH	VH	VH
u_2	VH	VH	FH	H	VH
u_3	M	FL	FL	M	FH

Table 4. Decision matrix $G_{t_1}^2$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	H	VH	H	H	H
u_2	H	VH	H	H	H
u_3	FL	M	FL	FL	M

Table 5. Decision matrix $G_{t_1}^3$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	VH	VH	H	VH	H
u_2	H	FH	VH	VH	H
u_3	M	FL	M	M	M

Table 6. Decision matrix $G_{t_2}^1$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	H	VH	VH	H	VH
u_2	H	VH	VH	VH	VH
u_3	FL	FL	M	M	M

Table 7. Decision matrix $G_{t_2}^2$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	VH	VH	VH	H	VH
u_2	H	H	H	VH	VH
u_3	M	FL	FL	M	FH

Table 8. Decision matrix $G_{t_2}^3$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	H	VH	H	VH	VH
u_2	VH	VH	H	H	H
u_3	M	FL	FL	M	FL

Table 9. Decision matrix $G_{t_3}^1$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	H	VH	VH	VH	H
u_2	VH	H	H	H	VH
u_3	FL	FL	FL	M	M

Table 10. Decision matrix $G_{t_3}^2$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	VH	H	VH	H	VH
u_2	H	VH	H	H	H
u_3	FL	FL	M	M	M

Table 11. Decision matrix $G_{t_3}^3$

u_i	x_1	x_2	x_3	x_4	x_5
u_1	VH	H	VH	H	H
u_2	H	VH	H	VH	H
u_3	FL	FL	M	FL	M

Obviously u_1 and u_2 are benefit attributes, u_3 is a cost attribute. Let the weight vector of the decision makers e_s, $s = 1, 2, 3,$ be $\pi = (0.4, 0.3, 0.3)$ and assume that the more the year t_k is closer to the present, the bigger the weight $\eta(t_k)$ of the year t_k is. Without loss of generality, suppose that the weight vectors of the year t_k, $k = 1, 2, 3,$ are $\eta(t) = (0.207, 0.282, 0.511)$ (which is determined by the Gamma distribution based method in (22) with $\alpha = 2$ and $\beta = 1$) and the weight vectors of the attributes u_i, $i = 1, 2, 3,$ at the years t_k, $k = 1, 2, 3,$ are $\omega(t_1) = (0.45, 0.35. 0.20),$ $\omega(t_2) = (0.45, 0.30. 0.25),$ $\omega(t_3) = (0.40, 0.30. 0.30),$ respectively.

In the following five steps we utilize the approach developed to get the desirable alternative(s).

Step 1. Converting the decision matrices $G_{t_k}^s$, $k, s = 1, 2, 3,$ into the generalized interval-valued trapezoidal fuzzy number matrices $\tilde{A}_{t_k}^s$, then utilizing Equations (23) and (24) to normalize the matrices $\tilde{A}_{t_k}^s$ into the corresponding decision matrices $\tilde{R}_{t_k}^s$, $k, s = 1, 2, 3.$

Step 2. Utilizing the GITFNWGA operator in (25) to aggregate the attribute values in the *j*-th column of $\tilde{R}_{t_k}^s$, provided by the decision makers e_s at the period t_k.

Step 3. Utilizing the GITFNHGA operator in (26) (whose weight vector is $v = (0.2429, 0.5142, 0.2429),$ obtained by the normal distribution based method

[23]) to aggregate the values $\tilde{r}^s_{jt_k}$, provided by all the decision makers e_s $s=1,2,3$ and get the overall attribute value $\tilde{r}_{jt_k}$ of the alternative x_j at the period t_k.

Step 4. Employing the DGITFNWGA operator in (27) to aggregate the $\tilde{r}_{jt_k}$ of all the periods t_k, $k=1,2,3,$ into a complex value $\tilde{r}_j$ corresponding to the alternative x_j:

$$\tilde{r}_1 = [(0.6602, 0.7187, 0.8628, 0.9405; 0.8), (0.6602, 0.7187, 0.8628, 0.9405; 1.0)],$$
$$\tilde{r}_2 = [(0.6921, 0.7538, 0.9013, 0.9781; 0.8), (0.6921, 0.7538, 0.9013, 0.9781; 1.0)],$$
$$\tilde{r}_3 = [(0.6385, 0.6929, 0.8255, 0.9001; 0.8), (0.6385, 0.6929, 0.8255, 0.9001; 1.0)],$$
$$\tilde{r}_4 = [(0.6088, 0.6604, 0.7900, 0.8654; 0.8), (0.6088, 0.6604, 0.7900, 0.8654; 1.0)],$$
$$\tilde{r}_5 = [(0.6012, 0.6494, 0.7638, 0.8302; 0.8), (0.6012, 0.6494, 0.7638, 0.8302; 1.0)].$$

Step 5. Calculating the distances $d(\tilde{r}_j, \tilde{1})$ we have $d(\tilde{r}_1, \tilde{1}) = 0.2840$, $d(\tilde{r}_2, \tilde{1}) = 0.2518$, $d(\tilde{r}_3, \tilde{1}) = 0.3122$, $d(\tilde{r}_4, \tilde{1}) = 0.3420$, $d(\tilde{r}_5, \tilde{1}) = 0.3600$ and then rank the alternatives x_j, $j=1,2,\ldots,5,$ according to the distances $d(\tilde{r}_j, \tilde{1})$ is

$$x_2 \succ x_1 \succ x_3 \succ x_4 \succ x_5;$$

thus the best company is x_2.

6. Conclusion

In this paper we have studied the dynamic multi-attribute group decision making problems, in which the attribute values take the form of generalized interval-valued trapezoidal fuzzy numbers. A new operator, called DGITFNWGA operator is introduced, and an approach to determine the time series weights based on the well-known Gamma distribution has been developed. Then we have utilized the GITFNHGA operator, GITFNWGA operator and DGITFNWGA operator to formulate a method and applied it to a dynamic investment decision making example. The application to dynamic investment decision making indicates that the approach presented is able to efficiently handle uncertainty and support dynamic decision making. Furthermore, the proposed method given in this paper should be tested and extended to other domains, in order to develop our understanding and improve the dynamic multi-attribute group decision making approach.

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