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Linear Perturbation Bounds of the Continuous-Time LMI-Based H_∞ Quadratic Stability Problem for Descriptor Systems

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Abstract: *In this paper we present an approach to obtain linear perturbation bounds for the continuous-time linear matrix inequalities (LMI) based H_∞ quadratic stability problem for descriptor systems. Using the approach proposed we are able to obtain tight linear perturbation bounds for the LMIs' solutions to the H_∞ quadratic stability problem for descriptor systems. It is also shown how the estimates of the individual condition numbers of the considered LMIs can be calculated. A numerical example is presented as well.*

Keywords: *Descriptor systems, Linear perturbation bounds, H_∞ Quadratic Stability Problem, LMI based synthesis, Linear systems.*

1. Introduction

Linear Matrix Inequalities (LMIs) are widely used to solve efficiently many fundamental problems in control theory: H_∞ synthesis, the linear quadratic regulator problem, quadratic stability problem, bounded energy problem, etc. [1, 2, 6, 7] and the literature therein.

LMI design is valuable, practical, applicable and useful thanks to the existence of efficient convex optimization algorithms [3] and software [4] plus the MATLAB package Yalmip and SeDuMi solver [5].

Descriptor systems (sometimes also referred to as Differential-Algebraic (DAE) or singular systems), describe a broad class of systems, which are not only of theoretical interest, but also have great practical significance. A considerable amount of studies, concerning linear descriptor systems has been carried out in [10]. The issues of controllability, observability, stability, synthesis of a state feedback have already been considered in [10].

In this paper we propose an approach to obtain linear perturbation bounds of the LMI based H_∞ quadratic stability problem via introducing a suitable right hand part in the considered matrix inequalities. After the considered problem is solved, the results obtained can be applied in the following ways. First, it is possible to estimate the errors in the calculated solution of the H_∞ quadratic stability problem, which are due to rounding errors and parametric disturbances in the considered data. Second, it is possible to study the robust stability and robust performance of the closed loop system with uncertainties in the plant and in the controller. The uncertainties in the controller appear because of the sensitivity of the H_∞ quadratic stability problem.

Further the following notation is used: $R^{m\times n}$ is the space of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n – the identity $n\times n$ matrix; e_n – the unit $n\times 1$ vector; M^{T} – the transpose of M; $M^{\perp}$ – the pseudo inverse of M; $\| M \|_2 = \sigma_{\max}(M)$ – the spectral norm of M, where $\sigma_{\max}(M)$ is the maximum singular value of M; $\mathrm{vec}(M) \in R^{mn}$ – the column-wise vector representation of $M \in R^{m\times n}$; $\Pi_{m,n} \in R^{mn\times mn}$ – the vec-permutation matrix, such that $\mathrm{vec}(M^{\mathrm{T}}) = \Pi_{m,n}\mathrm{vec}(M)$; $M \otimes P$ – the Kroneker product of the matrices M and P. The notation ":=" stands for "equal by definition".

The rest of the paper is structured as follows. Section 2 presents the problem set up and objective. Section 3 reveals the performed linear perturbation analysis of the LMI-based continuous H_∞ quadratic stability problem for descriptor systems. In Section 4 a numerical example is given. And finally in Section 5 we conclude with some final remarks.

2. Problem setup and objective

Linear continuous-time descriptor systems are generally described by the following set of differential-algebraic equations

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0, \qquad (1)$$
$$y(t) = Cx(t),$$

where $x(t) \in R^n$, $u(t) \in R^m$ and $x(t_0) \in R^n$ are the system descriptor state, input and initial conditions, and A, B, C and E are constant matrices of compatible size.

Definition 2.1 (System equivalence). Two systems (E, A, B, C) and $(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ are said to be (system) equivalent, denoted by $(E, A, B, C) \approx (\hat{E}, \hat{A}, \hat{B}, \hat{C})$, if there exist nonsingular transformation matrices $L, R \in R^{n\times n}$ such that the equations

$$\hat{E} = LER, \ \hat{A} = LAR, \ \hat{B} = LB, \ \hat{C} = CR$$

hold true.

Definition 2.2 (Regularity). The system is regular, if the polynomial $\det(sE - A)$ satisfies $\det(sE - A) \neq 0$.

Definition 2.3 (Weierstrass normal form). For any regular system there exist two non-singular matrices $L, R \in R^{n\times n}$ such that by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^{-1}x, x_1 \in R^r, x_2 \in R^{n-r}$$

the following decomposed representation can be obtained

(2)
$$\begin{aligned} \dot{x}_1(t) &= \hat{A}_r x_1(t) + \hat{B}_1 u(t), \\ N\dot{x}_2(t) &= x_2(t) + \hat{B}_2 u(t). \end{aligned}$$

Definition 2.4 (Index of nilpotence). The index of nilpotence ν, i.e., $\nu := \min\{q \mid N^q = 0\}$ is said to be an index of a linear descriptor system. Systems with $\nu \geq 2$ are called high index DAE systems.

The descriptor system (1) has a solution for any initial condition and sufficiently smooth input *u*. It is possible that the solution might show impulsive behavior. That is why, consider the system in Weierstrass normal form under sufficiently smooth input, starting from an initial condition x_0. Then the state evolution can be described according to [10]:

(3)
$$\begin{aligned} x_1(t) &= e^{\hat{A}_r t} x_{01} + \int_0^t e^{\hat{A}_r(t-\tau)} \hat{B}_1 u(t) d\tau, \\ x_2(t) &= -\sum_{i=1}^{\nu-1} \delta^{(i-1)}(t) N^i x_{02} - \sum_{i=1}^{\nu-1} N^i x_{02} \hat{B}_2 u^{(i)}(t). \end{aligned}$$

The expression (3) for state evolution $x_2(t)$ implies that index one descriptor systems $\nu = 1$ and $N = 0$ will have no impulsive solutions. In this case the system (1) is called impulse free and index one.

Consider the linear continuous-time descriptor system (1), where there is no direct relation between the input and the output signal. Throughout the paper we assume the descriptor system (1) is an index one system.

There exists an equivalent system

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_r & 0 \\ 0 & I_{n-r} \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \begin{bmatrix} \hat{C}_1, \hat{C}_2 \end{bmatrix} \right),$$

in Weierstrass canonical form where $A_r \in R^{r\times r}$ is a stable matrix. The transformed system is given as

(4)
$$\begin{aligned} \dot{x}_1(t) &= \hat{A}_r x_1(t) + \hat{B}_1 u(t), \\ y(t) &= \hat{C}_1 x_1(t). \end{aligned}$$

The transformed system (4) is obtained using the expression (3b) for the state evolution $x_2(t)$.

We consider an LMI approach to solve the H_∞ quadratic stability problem for descriptor systems as stated in [11]. For index one descriptor systems we are interested in the solution of the following system of LMIs,

(5)
$$\begin{bmatrix} \hat{A}_r^{\mathrm{T}} P_1 + P_1 \hat{A}_r + \hat{C}_1^{\mathrm{T}} \hat{C}_1 & P_1 \hat{B}_1 \\ \hat{B}_1^{\mathrm{T}} P_1 & -\gamma^2 I \end{bmatrix} < 0, \ P_1 > 0 \cdot$$

This is an Eigenvalue Problem (EVP) with respect to the variables P_1 and γ. Here we assume that the optimal closed-loop performance γ_{opt} of the system (4) is already obtained.

In order to achieve quadratic H_∞ stability and to ensure closed-loop performance γ, it is necessary to design a state-feedback control $u=K_1x_1$. From Schur complement argument [8] the above inequality is equivalent to:

$$\text{(6)}\qquad \begin{bmatrix} (\hat{A}_r+\hat{B}_1K_1)^{\mathrm{T}}P_1+P_1(\hat{A}_r+\hat{B}_1K_1) & 0 & \hat{C}_1^{\mathrm{T}} \\ 0 & -\gamma I & 0 \\ \hat{C}_1 & 0 & -\gamma I \end{bmatrix} < 0,\ P_1>0,$$

with respect to the variables K_1, P_1 and γ. It is obvious that the inequality (6) is not an LMI with respect to the decision variables K_1 and P_1. That is why we perform the substitution $Q_1=P_1^{-1}, Q_1>0,$ and $Y_1=K_1P_1^{-1}$ to obtain the following system of LMIs:

$$\text{(7)}\qquad \begin{bmatrix} \hat{A}_rQ_1+Q_1\hat{A}_r^{\mathrm{T}}+\hat{B}_1Y_1+Y_1^{\mathrm{T}}\hat{B}_1^{\mathrm{T}} & 0 & Q_1\hat{C}_1^{\mathrm{T}} \\ 0 & -\gamma I & 0 \\ \hat{C}_1Q_1 & 0 & -\gamma I \end{bmatrix} < 0,\ Q_1>0.$$

The paper is aimed at obtaining linear perturbation bounds of the LMI system (7) near the optimal value of γ, needed to solve the H_∞ quadratic stability problem for index one descriptor systems.

Suppose that the matrices $\hat{A}_r, \hat{B}_1, \hat{C}_1$ are subject to perturbations $\Delta\hat{A}_r$, $\Delta\hat{B}_1$, $\Delta\hat{C}_1$, and assume that they do not change the sign of the LMI system (7).

3. Linear perturbation bounds calculation

We carry out sensitivity analysis of the LMI (7) for the index one descriptor system (1), given in Weierstrass normal form:

$$\text{(8)}\qquad \begin{bmatrix} \hat{A}_r\hat{B}_1Q_1Y_1 & 0 & Q_1\hat{C}_1^{\mathrm{T}} \\ 0 & -(\gamma I+\Delta\gamma I) & 0 \\ \hat{C}_1Q_1 & 0 & -(\gamma I+\Delta\gamma I) \end{bmatrix} < 0,$$

where

$$\hat{A}_r\hat{B}_1Q_1Y_1 = (\hat{A}_r+\Delta\hat{A}_r)(Q_1+\Delta Q_1)+(Q_1+\Delta Q_1)(\hat{A}_r+\Delta\hat{A}_r)^{\mathrm{T}} + $$
$$+ (Y_1+\Delta Y_1)^{\mathrm{T}}(\hat{B}_1+\Delta\hat{B}_1)^{\mathrm{T}},$$
$$Q_1\hat{C}_1^{\mathrm{T}} = (Q_1+\Delta Q_1)(\hat{C}_1+\Delta\hat{C}_1)^{\mathrm{T}},\ \hat{C}_1Q_1=(\hat{C}_1+\Delta\hat{C}_1)(Q_1+\Delta Q_1).$$

The effect of the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta\hat{C}_1$ and $\Delta\gamma$ on the perturbed LMI solutions $Q_1^*+\Delta Q_1$ and $Y_1^*+\Delta Y_1$ has to be studied. Here Q_1^*, Y_1^* and $\Delta Q_1, \Delta Y_1$ are the nominal solution of the inequality (8) and the perturbations, respectively. After introducing a slightly perturbed suitable right hand part we can obtain:

(9) $$\begin{bmatrix} \hat{A}_r\hat{B}_1Q_1Y_1^* & 0 & Q_1^*\hat{C}_1^{\mathrm{T}} \\ 0 & -(\gamma_{\mathrm{opt}}I+\Delta\gamma_{\mathrm{opt}}I) & 0 \\ \hat{C}_1Q_1^* & 0 & -(\gamma_{\mathrm{opt}}I+\Delta\gamma_{\mathrm{opt}}I) \end{bmatrix} = M_1^* + \Delta M_1 < 0,$$

where

$$\hat{A}_r\hat{B}_1Q_1Y_1^* = (\hat{A}_r + \Delta\hat{A}_r)(Q_1^* + \Delta Q_1) + (Q_1^* + \Delta Q_1)(\hat{A}_r + \Delta\hat{A}_r)^{\mathrm{T}} + \\ + (Y_1^* + \Delta Y_1)^{\mathrm{T}}(\hat{B}_1 + \Delta\hat{B}_1)^{\mathrm{T}},$$

$$Q_1^*\hat{C}_1^{\mathrm{T}} = (Q_1^* + \Delta Q_1)(\hat{C}_1 + \Delta\hat{C}_1)^{\mathrm{T}},\ \hat{C}_1Q_1^* = (\hat{C}_1 + \Delta\hat{C}_1)(Q_1^* + \Delta Q_1),$$

and M_1^* is calculated using the so called nominal LMI:

(10) $$\begin{bmatrix} \hat{A}_rQ_1^* + \hat{B}_1Y_1^* + Q_1^*\hat{A}_r^{\mathrm{T}} + Y_1^{*\mathrm{T}}\hat{B}_1^{\mathrm{T}} & 0 & Q_1^*\hat{C}_1^{\mathrm{T}} \\ 0 & -\gamma_{\mathrm{opt}}I & 0 \\ \hat{C}_1Q_1^* & 0 & -\gamma_{\mathrm{opt}}I \end{bmatrix} = M_1^* < 0.$$

In the matrix ΔM_1 information is taken into account with respect to the data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the considered LMIs.

Applying expression (10), the perturbed equation (9) can be written in the following way

(11) $$\Delta_{Q_1} + \Omega_{Q_1} = \Delta M_1,$$

where

$$\Delta_{Q_1} = \begin{bmatrix} \hat{A}_r\ \Delta Q_1 + \Delta Q_1\ \hat{A}_r^{\mathrm{T}} & 0 & \Delta Q_1\hat{C}_1^{\mathrm{T}} \\ 0 & 0 & 0 \\ \hat{C}_1\Delta Q_1 & 0 & 0 \end{bmatrix},$$

$$\Omega_{Q_1} = \begin{bmatrix} \Delta\hat{A}_rQ_1^* + \hat{B}_1\Delta Y_1 + \Delta\hat{B}_1Y_1^* + Q_1^*\Delta\hat{A}_r^{\mathrm{T}} + \Delta Y_1^{\mathrm{T}}\hat{B}_1^{\mathrm{T}} + Y_1^{*\mathrm{T}}\Delta\hat{B}_1^{\mathrm{T}} & 0 & Q_1^*\Delta\hat{C}_1^{\mathrm{T}} \\ 0 & -\Delta\lambda_{\mathrm{opt}}I & 0 \\ \Delta\hat{C}_1Q_1^* & 0 & -\Delta\lambda_{\mathrm{opt}}I \end{bmatrix}.$$

Due to the fact that we have to calculate linear perturbation bounds, here the terms of second and higher order are neglected. Thus we represent the expression (11) in a vectorized form

(12) $$\mathrm{vec}(\Delta_{Q_1}) + \mathrm{vec}(\Omega_{Q_1}) = \mathrm{vec}(\Delta M_1),$$

where

$$\mathrm{vec}(\Delta_{Q_1}) = \left[I\otimes\hat{A}_r + \hat{A}_r\otimes I, 0, \hat{C}_1\otimes I, 0, 0, 0, I\otimes\hat{C}_1^{\mathrm{T}}, 0, 0\right]^{\mathrm{T}}\mathrm{vec}(\Delta Q) := S_1\Delta q_1,$$

$$\mathrm{vec}(\Omega_{Q_1})=$$

$$=\begin{bmatrix} (Q_1^*\otimes I)+(I\otimes Q_1^*)\Pi_{r^2} & (I\otimes\hat{B}_1)+(\hat{B}_1\otimes I)\Pi_{r\times m} & (Y_1^*\otimes I)+(I\otimes Y_1^{*\mathrm{T}})\Pi_{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (I\otimes Q_1^*)\Pi_{r^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{m^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (Q_1^*\otimes I) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{m^2} \end{bmatrix}\times$$

$$\times\begin{bmatrix} \mathrm{vec}(\Delta\hat{A}_r) \\ \mathrm{vec}(\Delta Y_1) \\ \mathrm{vec}(\Delta\hat{B}_1) \\ \mathrm{vec}(\Delta\hat{C}_1) \\ \Delta\gamma_{\mathrm{opt}} \end{bmatrix}=\begin{bmatrix} S_{t1} & S_{t2} & S_{t3} & S_{t4} & S_{t5} \end{bmatrix}\Delta_{AYBC\gamma}:=S_t\Delta_{AYBC\gamma}.$$

After performing the mathematical transformations we obtain

(13) $$S_1\Delta q_1+S_{t1}\mathrm{vec}(\Delta\hat{A}_r)+S_{t2}\mathrm{vec}(\Delta Y_1)+S_{t3}\mathrm{vec}(\Delta\hat{B}_1)+S_{t4}\mathrm{vec}(\Delta\hat{C}_1)+S_{t5}\Delta\gamma_{\mathrm{opt}}=\mathrm{vec}(\Delta M_1).$$

Finally the relative perturbation bound for the solution Q_1^* of the LMI (7) is obtained

$$\frac{\|\Delta q_1\|_2}{\|\mathrm{vec}(Q_1^*)\|_2}\le$$

(14) $$\le\frac{1}{\|\mathrm{vec}(Q_1^*)\|_2}\left(S_{AYB1}\frac{\|\mathrm{vec}(\Delta\hat{A}_r)\|_2}{\|\mathrm{vec}(\hat{A}_r)\|_2}+S_{AYB2}\frac{\|\mathrm{vec}(\Delta Y_1)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2}+S_{AYB3}\frac{\|\mathrm{vec}(\Delta\hat{B}_1)\|_2}{\|\mathrm{vec}(\hat{B}_1)\|_2}\right)$$

$$\le\frac{1}{\|\mathrm{vec}(Q_1^*)\|_2}\left(S_{AYB4}\frac{\|\mathrm{vec}(\Delta\hat{C}_1)\|_2}{\|\mathrm{vec}(\hat{C}_1)\|_2}+S_{AYB5}\frac{|\Delta\gamma_{\mathrm{opt}}|}{|\gamma_{\mathrm{opt}}|}+M_1\frac{\|\mathrm{vec}(\Delta M_1)\|_2}{\|\mathrm{vec}(M^*)\|_2}\right)$$

where

$$\frac{S_{AYB1}}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S_1^{\perp}\|_2\|S_{t1}\|_2\|\mathrm{vec}(\hat{A}_r)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2},\quad \frac{S_{AYB2}}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S_1^{\perp}\|_2\|S_{t2}\|_2\|\mathrm{vec}(Y_1^*)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2},$$

$$\frac{S_{AYB3}}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S^{\perp}\|_2\|S_{t3}\|_2\|\mathrm{vec}(\hat{B}_1)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2},\quad \frac{M_1}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S^{\perp}\|_2\|\mathrm{vec}(M^*)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2},$$

$$\frac{S_{AYB4}}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S^{\perp}\|_2\|S_{t4}\|_2\|\mathrm{vec}(\hat{C}_1)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2},\quad \frac{S_{AYB5}}{\|\mathrm{vec}(Q_1^*)\|_2}:=\frac{\|S^{\perp}\|_2\|S_{t5}\|_2|\gamma_{\mathrm{opt}}|}{\|\mathrm{vec}(Q_1^*)\|_2}$$

are the estimates of the individual relative condition numbers of LMI (7) with respect to the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta\hat{C}_1, \Delta\gamma$ and ΔY_1.

We apply a procedure like presented in order to compute the relative perturbation bounds for the solution Y_1^* of the LMI (7), i.e.,

$$\Delta_{Y_1} + \Omega_{Y_1} = \Delta M_2, \tag{15}$$

where

$$\Delta_{Y_1} = \begin{bmatrix} \hat{B}_1\Delta Y_1 + \Delta Y_1^{\mathrm{T}}\hat{B}_1^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_{Y_1} = \begin{bmatrix} \hat{A}_r\Delta Q_1 + \Delta\hat{A}_r Q_1^* + \Delta\hat{B}_1 Y_1 * + \Delta Q_1\hat{A}_r^{\mathrm{T}} + Q_1^*\Delta\hat{A}_r^{\mathrm{T}} + Y_1^{*\mathrm{T}}\Delta\hat{B}_1^{\mathrm{T}} & 0 & \Delta Q_1\hat{C}_1^{\mathrm{T}} + Q_1^*\Delta\hat{C}_1^{\mathrm{T}} \\ 0 & -\Delta\gamma I & 0 \\ \hat{C}_1\Delta Q_1 + \Delta\hat{C}_1 Q^* & 0 & -\Delta\gamma I \end{bmatrix}.$$

Due to the fact that we have to calculate linear perturbation bounds, here the terms of second and higher order are neglected. Then relation (15) in a vectorized form will look in the following way:

$$\operatorname{vec}(\Delta_{Y_1}) + \operatorname{vec}(\Omega_{Y_1}) = \operatorname{vec}(\Delta M_2), \tag{16}$$

where

$$\operatorname{vec}(\Delta_{Y_1}) = \left[(I\otimes\hat{B}_1) + (\hat{B}_1\otimes I)\Pi_{r\times m}, 0, 0, 0, 0, 0, 0, 0, 0\right]^{\mathrm{T}} \operatorname{vec}(\Delta Y_1) := Z\Delta y_1,$$

$$\operatorname{vec}(\Omega_{Y_1}) =$$

$$= \begin{bmatrix} (Q_1^*\otimes I) + (I\otimes Q_1^*)\Pi_{r^2} & (I\otimes\hat{A}_r) + (\hat{A}_r\otimes I) & (Y_1^*\otimes I) + (I\otimes Y_1^{*\mathrm{T}})\Pi_{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (\hat{C}_1\otimes I) & 0 & (I\otimes Q_1^*)\Pi_{r^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{r^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (I\otimes\hat{C}_1) & 0 & (Q_1^*\otimes I) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{r^2} \end{bmatrix} \times$$

$$\times\begin{bmatrix}\mathrm{vec}(\Delta\hat{A}_r)\\ \mathrm{vec}(\Delta Q_1)\\ \mathrm{vec}(\Delta\hat{B}_1)\\ \mathrm{vec}(\Delta\hat{C}_1)\\ \Delta\gamma_{\mathrm{opt}}\end{bmatrix}=\begin{bmatrix}Z_{t1} & Z_{t2} & Z_{t3} & Z_{t4} & Z_{t5}\end{bmatrix}\Delta_{AQBC\gamma}:=Z_t\Delta_{AQBC\gamma}.$$

Thus to calculate the linear perturbation bounds, the following relation is used

$$Z\Delta y_1+Z_{t1}\mathrm{vec}(\Delta\hat{A}_1)+Z_{t2}\mathrm{vec}(\Delta Q_1)+Z_{t3}\mathrm{vec}(\Delta\hat{B}_1)+Z_{t4}\mathrm{vec}(\Delta C_1)+Z_{t5}\Delta\gamma_{\mathrm{opt}}=\mathrm{vec}(\Delta M_2). \quad (17)$$

Below the relative perturbation bound for the solution $Y_1{}^*$ of the LMI (7) is shown

$$\begin{aligned}\frac{\|\Delta y_1\|_2}{\|\mathrm{vec}(Y_1^*)\|_2}&\le\frac{1}{\|\mathrm{vec}(Y_1^*)\|_2}\left(Z_{AYB1}\frac{\|\mathrm{vec}(\Delta\hat{A}_r)\|_2}{\|\mathrm{vec}(\hat{A}_r)\|_2}+Z_{AYB2}\frac{\|\mathrm{vec}(\Delta Q_1)\|_2}{\|\mathrm{vec}(Q_1^*)\|_2}+Z_{AYB3}\frac{\|\mathrm{vec}(\Delta\hat{B}_1)\|_2}{\|\mathrm{vec}(\hat{B}_1)\|_2}\right)\\ &\le\frac{1}{\|\mathrm{vec}(Y_1^*)\|_2}\left(Z_{AYB4}\frac{\|\mathrm{vec}(\Delta\hat{C}_1)\|_2}{\|\mathrm{vec}(\hat{C}_1)\|_2}+Z_{AYB5}\frac{|\Delta\gamma_{\mathrm{opt}}|}{|\gamma_{\mathrm{opt}}|}+M_2\frac{\|\mathrm{vec}(\Delta M_2)\|_2}{\|\mathrm{vec}(M^*)\|_2}\right)\end{aligned}, \quad (18)$$

here

$$\frac{Z_{AQB1}}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|Z_{t1}\|_2\|\mathrm{vec}(\hat{A}_r)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2},\quad \frac{Z_{AQB2}}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|Z_{t2}\|_2\|\mathrm{vec}(Q_1^*)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2},$$

$$\frac{Z_{AQB3}}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|Z_{t3}\|_2\|\mathrm{vec}(\hat{B}_1)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2},\quad \frac{M_2}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|\mathrm{vec}(M^*)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2},$$

$$\frac{Z_{AQB4}}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|Z_{t4}\|_2\|\mathrm{vec}(\hat{C}_1)\|_2}{\|\mathrm{vec}(Y_1^*)\|_2},\quad \frac{Z_{AQB5}}{\|\mathrm{vec}(Y_1^*)\|_2}:=\frac{\|Z^{\perp}\|_2\|Z_{t5}\|_2|\gamma_{\mathrm{opt}}|}{\|\mathrm{vec}(Y_1^*)\|_2}$$

are the the estimates of the individual relative condition numbers of LMI (7) with respect to the perturbations $\Delta\hat{A}_r, \Delta\hat{B}_1, \Delta\hat{C}_1, \Delta Q_1$ and $\Delta\gamma$.

4. Numerical example [10]

Consider the continuous-time index one descriptor system (1) given in Weierstrass normal form, i.e.,

$$\hat{E}=\begin{bmatrix}1 & 0 & \vdots & 0 & 0\\ 0 & 1 & \vdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \vdots & 0 & 0\\ 0 & 0 & \vdots & 0 & 0\end{bmatrix},\quad \hat{A}=\begin{bmatrix}-1 & 0 & \vdots & 0 & 0\\ 0 & -1 & \vdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \vdots & 0 & 0\\ 0 & 0 & \vdots & 0 & 0\end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \\ 0 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} 1 & 0 & \vdots & 1 & 0 \end{bmatrix}.$$

Since we would like to calculate the linear bounds, the perturbations in the system matrices are chosen in such a way as to eliminate the second and higher order terms in the derivation procedure, i.e.:

$$\Delta\hat{A}_r = A_r \times 10^{-i}, \Delta\hat{B}_1 = \hat{B}_1 \times 10^{-i},$$

$$\Delta\hat{C}_1 = \hat{C}_1 \times 10^{-i}, \ \Delta\gamma_{\text{opt}} = \gamma_{\text{opt}} \times 10^{-i},$$

$$\Delta M_1 = M_1^* \times 10^{-i}, \ \Delta M_2 = M_2^* \times 10^{-i},$$

$$\Delta Q_1 = Q_1^* \times 10^{-i}, \ \Delta Y_1 = Y_1^* \times 10^{-i} \text{ for } i = 8,7...4.$$

The perturbed solutions $Q_1^* + \Delta Q_1$ and $Y_1^* + \Delta Y_1$ are calculated applying the method presented in [9] and using the software [4]. Carrying out the proposed approach the linear relative perturbation bounds for the solutions Q_1^* and Y_1^* of the LMI system (7) are calculated using expressions (14) and (18), respectively.

We have considered different size of perturbations, calculated the linear perturbation bounds and the obtained results are shown in the following table

Table 1

I	$\frac{\|\Delta q_1\|_2}{\|\text{vec}(Q_1^*)\|_2}$	Bound (14)	$\frac{\|\Delta y_1\|_2}{\|\text{vec}(Y_1^*)\|_2}$	Bound (18)
8	6.1354×10^{-8}	1.2384×10^{-7}	4.9164×10^{-8}	0.7562×10^{-7}
7	6.1354×10^{-7}	1.2384×10^{-6}	4.9164×10^{-7}	0.7562×10^{-6}
6	6.1354×10^{-6}	1.2384×10^{-5}	4.9164×10^{-6}	0.7562×10^{-5}
5	6.1354×10^{-5}	1.2384×10^{-4}	4.9164×10^{-5}	0.7562×10^{-4}
4	6.1354×10^{-4}	1.2384×10^{-3}	4.9164×10^{-4}	0.7562×10^{-3}

Based on the proposed solution approach to perform sensitivity analysis of the continuous-time LMI based H_∞ quadratic stability problem for descriptor systems, we obtain the perturbation bounds (14) and (18). These bounds are close to the real relative perturbation bounds $\frac{\|\Delta q_1\|_2}{\|\text{vec}(Q_1^*)\|_2}$ and $\frac{\|\Delta y_1\|_2}{\|\text{vec}(Y_1^*)\|_2}$, this means that they are good in sense that they are tight.

5. Conclusion

In this paper we propose an approach to compute the linear perturbation bounds of the continuous-time LMI based H_∞ quadratic stability problem for descriptor systems. We also show how the estimates of the individual condition numbers of the LMIs considered can be calculated. Tight linear perturbation bounds are obtained for the matrix inequalities, determining the problem solution. Based on mathematical derivations we have obtained theoretical results, that applied on a numerical example show the importance of the proposed solution approach to analyze the sensitivity of the LMI based H_∞ quadratic stability problem for descriptor systems.

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