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Upper Bounds for the Solution of the Parameter Dependent Lyapunov Equation

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Abstract: *This research work considers the problem of scalar and matrix solution bounds derivation for one class of parameter dependent Lyapunov equations (PDLEs). It is assumed, that the coefficient matrix is a matrix polytope, where the uncertain vector is defined on the unit simplex. It is shown that this problem can be efficiently solved by making use of some previously obtained results, concerning the exact conditions for positive definiteness of homogeneous matrix polynomials (HMPs). The main contribution consists in the definition of two upper bounds – for the trace and the maximum eigenvalue of the solution of a PDLE. The applicability of these results is illustrated by a numerical example.*

Keywords: *Lyapunov equation, uncertain systems, matrix polytope, robust stability.*

1. Introduction

The problem of valid bounds definition for the solution of the algebraic Lyapunov equation has a long history. The current interest is due to both theoretical and practical reasons. In some cases the direct solution of this equation is impossible, due to its high order and in other ones, it is sufficient to have at disposal only some estimates for it. The main difficulty arises from the fact, that the available upper bounds are valid under some very conservative restrictions on the coefficient matrix. Due to this, valid solution bounds are possible only for some special subsets of positive (negative) stable coefficient matrices [1, 4, 8].

Robustness of a linear dynamic system, subjected to structured real parametric uncertainty, belonging to a compact vector set (e.g., the unit simplex), has been recognised as a key issue in the analysis of control systems, since it is not realistic to expect that the exact model of any system is available. As a consequence, several attempts to get bounds for the solution of the PDLE were made. Additional difficulties arise from the simple fact, that the coefficient matrix is not exactly known any more.

This research work is the first attempt to get always valid upper bounds for the PDLE in the case when the coefficient matrix is a matrix polytope and the uncertainty vector is defined on the unit simplex. It is shown that this problem can be solved by making use of some known results concerning the positive definiteness of HMPs. This results in the definition of two always valid upper trace and maximum eigenvalue solution bounds, which are parameter independent. Up to our best knowledge, these are the first suggested bounds for the solution of the considered class of PDLEs. Their applicability is illustrated by a numerical example.

2. Preliminaries and problem formulation

The notation $A > (\geq) 0$ indicates, that A is a positive (semi-) definite matrix, $A = [a_{ij}] \in \mathbf{R}_n$ and $a = (a_i) \in \mathbf{R}^N$ denote a real $n\times n$ matrix and an $N\times 1$ vector with entries a_{ij} and a_i, respectively. The sum of N nonnegative scalars α_i is $|\alpha|$ and $A - B\ [\leq]\ 0$ means, that all entries of matrix $A - B$ are non-positive. Define also, the vector sets $\mathbf{X}_n \equiv \{x \in \mathbf{R}^n : x^{\mathrm{T}} x = 1\}$ and $\boldsymbol{\omega}_N \equiv \{\alpha = (\alpha_i) \in R^N : |\alpha| = 1\}$.

Consider a HMP of an arbitrary degree k with $\chi(k) = \dfrac{(k+N-1)!}{k!(N-1)!}$, $0! = 1$, symmetric coefficients:

$$\Pi(\alpha, k) = \sum_{|k|=k} \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N} P_{k_1 k_2 \dots k_N} \in \mathbf{R}_n,\ \alpha \in \boldsymbol{\omega}_N, \tag{1}$$

which can be equivalently represented as a HMP of even degree 2τ with $\chi(2\tau)$ coefficients:

$$\Pi(\alpha, k) = |\alpha|^d\, \Pi(\alpha, k) = \Pi(\alpha, 2\tau) = \sum_{i,j=1, i\leq j}^{\chi(2\tau)} \overline{\alpha}_i \overline{\alpha}_j \Pi_{ij},\ d + k = 2\tau. \tag{2}$$

Here, $\overline{\alpha}_i = \alpha_1^{\tau_1} \alpha_2^{\tau_2} \dots \alpha_N^{\tau_N}, |\tau| = \tau, i = 1, 2, \dots, \chi(\tau)$, denotes the i-th monomial of degree τ, and Π_{ij} is the coefficient, corresponding to monomial $\overline{\alpha}_i \overline{\alpha}_j$ of degree 2τ. Let $\overline{\alpha}_v = (\overline{\alpha}_i)^{\mathrm{T}} \in \mathbf{R}^{\chi(\tau)}$ be the vector containing all $\chi(\tau)$ monomials of degree τ and consider the homogeneous scalar polynomial (HSP):

$$p(\alpha, 2\tau, x) = x^{\mathrm{T}} \Pi(\alpha, 2\tau) x = \sum_{i,j=1, i\leq j}^{\chi(2\tau)} \overline{\alpha}_i \overline{\alpha}_j\, p_{ij}(x) = \overline{\alpha}_v^{\mathrm{T}} C(x) \overline{\alpha}_v, \tag{3}$$

$$C(x) = [c_{ij}(x)] \in \mathbf{R}_{\chi(\tau)},\ x \in \mathbf{X}_n,$$

Where $p_{ij}(x) = x^{\mathrm{T}} \prod_{ij} x$. The symmetric matrix $C(x)$ is said to be a Coefficient Matrix (CM) for the HSP in (3). Define also, the HSP:

(4) $$p(\alpha, 2\tau) = \sum_{i,j=1, i\le j}^{\chi(2\tau)} \overline{\alpha}_i \overline{\alpha}_j p_{ij} = \overline{a}_v^{\mathrm{T}} C \overline{a}_v,$$

$$p_{ij} \le p_{ij}(x), \forall x \in \mathbf{X}_n \Rightarrow C = [c_{ij}][\le] C(x), \ x \in \mathbf{X}_n.$$

Having in mind that $\alpha \in \boldsymbol{\omega}_N$, one gets

(5) $$p(\alpha, 2\tau) \le p(\alpha, 2\tau, x) \quad \forall \alpha, x.$$

Consider the PDLE

(6) $$A^{\mathrm{T}}(\alpha)P(\alpha) + P(\alpha)A(\alpha) = Q(\alpha) = \Pi_Q(\alpha, d_Q) > 0,$$

$$A(\alpha) = \sum_1^N \alpha_i A_i \ \ \alpha \in \boldsymbol{\omega}_N,$$

where all matrices $A_i \in \mathbf{R}_n$ are positive stable and the right-hand side is an arbitrary given positive definite HMP. Let the matrix polytope $A(\alpha)$ be positive stable on the whole uncertainty set, then $P(\alpha)$ is the unique positive definite solution to (6). It is desired to find upper bounds for the solution, its trace and maximum eigenvalue.

As it was already said, the problem of deriving valid upper bounds for Lyapunov equation faces serious difficulties, even in the case, when the coefficient and the right-hand side matrices A and Q are fixed. Upper bounds have been obtained for only two special cases:

(i) $A^{\mathrm{T}} + A > 0$ [1, 4] and

(ii) $A^{\mathrm{T}}(A^{\mathrm{T}}A)^{\frac{1}{2}} + (A^{\mathrm{T}}A)^{\frac{1}{2}}A > 0$ [6, 9].

It has been proved that (i) always implies (ii), i.e., requirement (ii) is less conservative with respect to validity. As it can be expected, the difficulties in the case considered here are even more, due to the parameter dependence in (6).

3. Main result

Several important previous results are required.

Theorem 1. The HMP in (1) is positive definite, if and only if there exists some sufficiently large integer d, such that all $\chi(2\tau)$ matrix coefficients of the HMP (2) are positive definite.

The necessity part of Theorem 1 is proved in [5, 14] and it generalizes the famous Polya's Theorem [2], concerning positive HSPs in $\alpha \in \boldsymbol{\omega}_N$, for the case of matrix valued functions. Theorem 1 represents an asymptotically exact condition and it provides a systematic way to decide whether a given HMP is positive definite. Unfortunately, this result is very conservative, due to the obvious fact, that (2) can be positive definite, even if some coefficients are not strictly positive definite. In an attempt to overcome this shortcoming (conservativeness), by means of considering inequalities between some entries of the uncertainty vector, an alternative approach has been suggested and consecutively improved and generalized in [10, 12, 13], and it can be briefly summarized as the following list of statements.

Theorem 2. For any given vector α , there exist two sets of HSPs, such that

$$p_f(\alpha, 2\tau, x) = x^{\mathrm{T}} \prod_f(\alpha, 2\tau)x =$$

$$= \overline{\alpha}_v^{\mathrm{T}} C_f(x)\overline{\alpha}_v \le p(\alpha, 2\tau, x), C_f(x) = [c_{ij,f}(x)],\ f = 1, ..., t.$$

(7) $$p_f(\alpha, 2\tau) = \overline{\alpha}_v^{\mathrm{T}} C_f \overline{\alpha}_v \le p_f(\alpha, 2\tau, x),$$

$$C_f = [c_{ij,f}][\le]C(x),\ c_{ij,f} = \lambda_{\min}(\Pi_{ij,f})\ \forall x,\ f = 1, ..., t.$$

A HMP in (1) is positive definite, if and only if there exists an integer τ, such that $C_f > 0,\ f = 1, ..., t.$

Remark 1. If there exists some integer *d*, such that all $\chi(2\tau)$ matrix coefficients of the HMP (2) are positive definite, then all *t* matrices C_f are diagonal and positive definite.

The definition of the polynomials and the respective CMs in (7) is discussed in details in [7, 10, 12, 13] and that is why it is omitted here. More attention is paid to the problem how Theorems 1 and 2 can be applied to get valid upper bounds for the solution of the PDLE (6).

Lemma 1. [11] $A(\alpha)$ is positive stable on the uncertainty set, if and only if there exists HMP $\Pi(\alpha, d_R) = R(\alpha)$, $d_R \le 0.5n(n-1)+1$ such that $\Pi_{\mathrm{RA}}(\alpha, d_R+1) = A^{\mathrm{T}}(\alpha)R(\alpha) + R(\alpha)A(\alpha) > 0.$

Lemma 2. Let the positive scalar μ be chosen to satisfy the matrix inequality

(8) $$\mu\Pi(\alpha, d_R+1) - Q(\alpha) > 0\ \ \forall\alpha.$$

Then $R_{\mathrm{U}}(\alpha) = \mu R(\alpha)$ is an upper matrix bound for the solution of the PDLE.

Proof: Having in mind (6) and (8), it is easy to get the following PDLE

$$A^{\mathrm{T}}(\alpha)[R_{\mathrm{U}}(\alpha) - P(\alpha)] + [R_{\mathrm{U}}(\alpha) - P(\alpha)]A(\alpha) \ge 0\ \ \forall\alpha.$$

Since $A(\alpha)$ is positive stable for all α, matrix $R_{\mathrm{U}}(\alpha) - P(\alpha)$ must be positive semi-definite by necessity, in accordance with Lyapunov Stability Theorem, i.e., $P(\alpha) \le R_{\mathrm{U}}(\alpha)\ \forall\alpha$ ■

Denote $x^{\mathrm{T}}[\Pi_{\mathrm{RA}}(\alpha, d_R+1) - Q(\alpha)]x = p_{\mathrm{RA}}(\alpha, d_R+1, x) - p(\alpha, d_Q, x).$ Then the problem for determining an upper matrix bound for the solution of (6) can be equivalently stated as follows: given a HMP $R(\alpha)$, such that $\Pi_{\mathrm{RA}}(\alpha, d_R+1) > 0$, determine integers d_1, d_2 and a positive scalar μ , such that

$$\mu|\alpha|^{d_1} p_{\mathrm{RA}}(\alpha, d_R+1, x) = \mu p_{\mathrm{RA}}(\alpha, d_1+d_R+1, x) \ge$$

$$\ge |\alpha|^{d_2} p_Q(\alpha, d_Q, x) = p_Q(\alpha, d_1+d_Q)\ \ \forall\alpha, x.$$

The integers d_1, d_2 can always be chosen to satisfy the equalities $d_1 + d_R + 1 = d_2 + d_Q = 2\tau$, which helps to put the problem in a quadratic with respect to the vector of monomials of degree τ compact matrix form:

(9) $$p_{\mathrm{REQ}}(\alpha, 2\tau, x, \mu) = \mu p_{\mathrm{RE}}(\alpha, 2\tau, x) - p_Q(\alpha, 2\tau, x) =$$
$$= \overline{\alpha}_v^{\mathrm{T}} [\mu C_{\mathrm{RA}}(x) - C_Q(x)] \overline{\alpha} \geq 0 \ \ \forall \alpha, x,$$

where the entries of the two CMs are denoted as $c_{\mathrm{RA},ij}(x) = x^{\mathrm{T}} \Pi_{ij} x,$ $c_{Q,ij}(x) = x^{\mathrm{T}} Q_{ij} x,$ respectively.

The application of Theorem 1 for the problem solution will be illustrated now.

Lemma 3. There exists some integer τ, such that for $\mu = \max \lambda_{\max}(Q_{ij} \Pi_{ij}^{-1}),\ i, j = 1, \ldots, \chi(2\tau),\ i \leq j,$ one has $P(\alpha) \leq R_U(\alpha)\ \forall \alpha,$ where Π_{ij}, Q_{ij} denotes the (*ij*)-th matrix coefficient of $|\alpha|^{d_1} \Pi_{RA}(\alpha, d_R + 1)$ and $|\alpha|^{d_2} Q(\alpha)$, respectively and $d_1 + d_R + 1 = d_2 + d_Q = 2\tau$.

Proof: There exists some integer τ, such that all entries of the CM $c_{\mathrm{RE},ij}(x) > 0\ \forall x \Leftrightarrow \Pi_{ij} > 0,$ in accordance with Theorem 1. The above choice for μ guarantees that the CM in (9) is non-negative for all *x*, which sufficiently satisfies inequality (9), since $\overline{\alpha}_v$ is a non-negative vector for all $\alpha \in \boldsymbol{\omega}_N$.

The asymptotically exact condition of Theorem 2 can also be used to derive upper bounds. Before that, a well known result is recalled.

Theorem 3 [3]. Let $M = [m_{ij}]$ be an arbitrary square matrix and denote $|M| = [|m_{ij}|],\ \rho(M) = \max|\lambda(M)|$ (spectral radius). In this case

$$\rho(M) \leq \rho(|M|) \leq \rho(N), \quad N[\geq]|M|.$$

Corollary 1. Let $M(x) = [m_{ij}(x)],\ m_{ij}(x) = x^{\mathrm{T}} M_{ij} x,\ M_{ij} = M_{ij}^{\mathrm{T}}.$ For any $x \in \mathbf{X}_n$

$$\lambda_{\max}[M(x)] \leq \rho[M(x)] \leq \rho(M^+),\ M^+ = [m_{ij}^+], m_{ij}^+ = \max\{\lambda_{\max}(M_{ij}),\ -\lambda_{\min}(M_{ij})\}.$$

Proof: It follows easy from Theorem 3, since $|M(x)|[\leq] M^+$ for all $x \in \mathbf{X}_n$ ■

Consider the set of HSPs in (7) and the HSP in (8). Denote $C_Q(x) C_f^{-1} = C_{Q,f}(x),\ f = 1, \ldots, t,$ and $C_{Q\max} = [c_{Q\max,ij}], c_{Q\max,ij} = \lambda_{\max}(Q_{ij}).$

Lemma 4. Let $\Pi_{\mathrm{RA}}(\alpha,\ d_R + 1) = A^{\mathrm{T}}(\alpha) R(\alpha) + R(\alpha) A(\alpha) > 0,$ $\Pi(\alpha,\ d_R) = R(\alpha).$ There exists some integer τ, such that for

$$\mu = \min(\mu_1, \mu_2), \mu_1 = \max \rho[(C_{Q,f}^+],\ \ \mu_2 = \max \lambda_{\max}(C_{Q\max} C_f^{-1}),\ f = 1, \ldots, t,$$

one has $P(\alpha) \leq R_{\mathrm{U}}(\alpha) = \mu R(\alpha)\ \forall \alpha.$

P r o o f: In accordance with Theorem 2, there exists some τ, such that all CMs in (7) are positive definite and having in mind the HSP in (9), for any given uncertain vector, there exists some HSP which is a lower bound for it, i.e.,

$$\mu p_{\mathrm{RE}}(\alpha, 2\tau, x) \geq \mu p_f(\alpha, 2\tau, x) \ \geq \mu p_f(\alpha, 2\tau) = \mu \overline{\alpha}_v{}^{\mathrm{T}} C_f \overline{\alpha}_v \ \ \forall \alpha, x, f = 1, ..., t,$$

which means that if μ is chosen to satisfy $\overline{\alpha}_v^{\mathrm{T}}[\mu C_f - C_Q(x)]\overline{\alpha} \geq 0 \ \forall \alpha, x, f$, then the inequality (9) will be sufficiently satisfied, as well. Let $\mu = \mu_1$. It follows from Corollary 1, that all matrices $\mu C_f - C_Q(x)$ are positive semi-definite for all x, or $P(\alpha) \leq R_{\mathrm{U}}(\alpha) = \mu_1 R(\alpha), \forall \alpha$. Since $\overline{\alpha}_v^{\mathrm{T}} C_Q(x)\overline{\alpha} \leq \overline{\alpha}_v^{\mathrm{T}} C_{Q\max}\overline{\alpha}_v \ \forall \alpha, x,$ then for the choice $\mu = \mu_2$, one has $\mu C_f - C_{Q\max} \geq 0 \ \forall f$, or $P(\alpha) \leq R_{\mathrm{U}}(\alpha) = \mu_2 R(\alpha) \ \forall \alpha$. This proves the matrix bound for the solution of the PDLE (6) ■

Before presenting some scalar solution bounds, the following result is needed. For any positive integer d and vector $\alpha \in \boldsymbol{\omega}_N$, $|\alpha|^d = \sum_{|d|=d} \theta_{d_1 \dots d_{d_N}} \alpha_1^{d_1} \dots \alpha_N^{d_N} = 1$. Having in mind this fact, any HSP $p(\alpha, d), \quad \alpha \in \boldsymbol{\omega}_N$, can be represented and bounded from above as follows

$$p(\alpha, d) = \sum_{|d|=d} \alpha_1^{d_1} \dots \alpha_N^{d_N} c_{d_1 \dots d_N} =$$

$$\sum_{|d|=d} (\theta_{d_1 \dots d_{d_N}} \alpha_1^{d_1} \dots \alpha_N^{d_N})(\theta_{d_1 \dots d_N})^{-1} c_{d_1 \dots d_N} \leq \max(\theta_{d_1 \dots d_N})^{-1} c_{d_1 \dots d_N}. \tag{10}$$

Corollary 2. Let $P(\alpha) \leq R_{\mathrm{U}}(\alpha) = \mu R(\alpha) = \mu \sum_{|d|=d_R} \alpha_1^{d_1} \dots \alpha_N^{d_N} R_{d_1 \dots d_N} \ \ \forall \alpha.$ Then the following upper trace and maximum eigenvalue bounds are valid:

$$\mathrm{tr}[P(\alpha)] \leq \mu \max\{(\theta_{d_1 \dots d_N})^{-1} \mathrm{tr}(R_{d_1 \dots d_N})\}, \tag{11}$$

$$\lambda_{\max}[P(\alpha)] \leq \mu \max\{(\theta_{d_1 \dots d_N})^{-1} \lambda_{\max}[R_{d_1 \dots d_N}]\} \ \ \forall \alpha.$$

P r o o f: It can be easily obtained, having in mind the estimate (10) and taking into account that for any symmetric matrix sum

$$S = \sum_{s=1}^{m} S_s \Rightarrow \mathrm{tr}(S) = \sum_{s=1}^{m} \mathrm{tr}(S_s), \lambda_{\max}(S) \leq \sum_{s=1}^{m} \lambda_{\max}(S_i) \ ■$$

The obtained bounds (11) will be illustrated next.

4. Numerical example

Consider the polytopes $A(\alpha)$ and $R(\alpha)=\sum_{i=1}^{3}R_i$, described by their vertices:

$$A_1=\begin{bmatrix}0.6895 & -4.137 & -4.728\\ 9.8500 & 3.152 & 0.197\\ -13.1999 & 13.987 & 15.169\end{bmatrix},$$

$$A_2=\begin{bmatrix}0.3276 & -0.3432 & -0.4836\\ 2.0280 & 0.5616 & -0.2184\\ 1.9968 & 1.4976 & 1.2480\end{bmatrix}, A_3=\begin{bmatrix}0.7056 & -2.856 & -2.352\\ 4.8496 & 1.344 & 0.224\\ 5.600 & 8.232 & 7.168\end{bmatrix},$$

$$R_1=\begin{bmatrix}0.7880 & 0.1970 & 0.2955\\ 0.1970 & 0.5910 & 0.3940\\ 0.2955 & 0.3940 & 0.7880\end{bmatrix},$$

$$R_2=\begin{bmatrix}0.1404 & 0.0312 & 0.0156\\ 0.0312 & 0.0624 & 0.0312\\ 0.0156 & 0.0312 & 0.0780\end{bmatrix}, R_3=\begin{bmatrix}0.3864 & 0.0560 & 0.1120\\ 0.0560 & 0.3920 & 0.2240\\ 0.1120 & 0.2240 & 0.3360\end{bmatrix}.$$

The minimal eigenvalues of the six coefficient matrices of the HMP

$$A^{\mathrm{T}}(\alpha)R(\alpha)+R(\alpha)A(\alpha) \text{ are:}$$
$$\lambda_{11}=0.1403, \lambda_{22}=0.00172,$$
$$\lambda_{33}=0.05448, \lambda_{12}=0.02293, \lambda_{13}=-0.001035, \lambda_{23}=-0.01998.$$

Therefore, for d = 0, robust stability of the polytope cannot be concluded, according to Theorem 1.

Step 1. Application of Theorem 2 leads to the following results. The two coefficient matrices in (7) are:

$$C_1=\begin{bmatrix}\lambda_{11} & 0 & 0.5\lambda_{13}\\ 0 & \lambda_{22} & 0.5\lambda_{23}\\ 0.5\lambda_{13} & 0.5\lambda_{23} & \lambda_{33,1}\end{bmatrix}, C_2=\begin{bmatrix}\lambda_{11} & 0 & 0.5\lambda_{13}\\ 0 & \lambda_{22,2} & 0.5\lambda_{23}\\ 0.5\lambda_{13} & 0.5\lambda_{23} & \lambda_{33}\end{bmatrix},$$

where $\lambda_{33,1}=0.63, \lambda_{22,2}=0.0025$, i.e.,

$$C_1 = \begin{bmatrix} 0.1403 & 0 & -0.0005175 \\ 0 & 0.00172 & -0.00999 \\ -0.0005175 & -0.00999 & 0.63 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.1403 & 0 & -0.0005175 \\ 0 & 0.0025 & -0.00999 \\ -0.0005175 & -0.00999 & 0.05448 \end{bmatrix}.$$

Simple computations show that both matrices are positive definite, which means that $A^{\mathrm{T}}(\alpha)R(\alpha) + R(\alpha)A(\alpha)$ is a positive HMP for all $\alpha \in \boldsymbol{\omega}_N$, or equivalently, $A(\alpha)$ is a positive stable polytope on the whole uncertainty set.

Step 2. Determination of the parameter $\mu = \min(\mu_1, \mu_2)$.

Let $Q(\alpha) = Q$ in (6) be an arbitrary fixed positive definite matrix. By making use of the representation $x^{\mathrm{T}}Qx = |\alpha|^2 x^{\mathrm{T}}Qx = \alpha^{\mathrm{T}}C_Q(x)\alpha,\ C_Q(x) = x^{\mathrm{T}}Qx[c_{ij}], c_{ij} = 1$, the problem is put in the required form (9). It is easy to see that in this case $\mu = \mu_1 = \mu_2$. Computation of the maximum eigenvalues of the matrices $C_{Q\max}C_f^{-1},\ f = 1, 2$, shows that

$$P(\alpha) \le R_{\mathrm{U}}(\alpha) = \mu R(\alpha), \forall \alpha,\quad \mu = \lambda_{\max}(Q)\max(669.65, 2124.2) = 2124.2\lambda_{\max}(Q).$$

Step 3. The traces and the maximum eigenvalues of the three vertices of $R(\alpha)$ are:

$$\mathrm{tr}(R_1) = 2.167,\ \mathrm{tr}(R_2) = 0.2808,\ \mathrm{tr}(R_3) = 1.1144,$$
$$\lambda_{\max}(R_1) = 1.328,\ \lambda_{\max}(R_2) = 0.1596,\ \lambda_{\max}(R_3) = 0.6431.$$

Now, one can easily compute the upper scalar bounds for the solution trace and maximum eigenvalue from (11) as follows:

$$\mathrm{tr}[P(\alpha)] \le 4603.14\lambda_{\max}(Q),\quad \lambda_{\max}[P(\alpha)] \le 2820.94\lambda_{\max}(Q)\quad \forall \alpha,$$

and any fixed positive right-hand side matrix Q with maximum eigenvalue $\lambda_{\max}(Q)$.

Consider now, a PDLE (6) with a parameter dependent right-hand side matrix given by $Q(\alpha) = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)Q_1 + 2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)Q_2$, where the two matrix coefficients are:

$$Q_1 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & -1 \\ 1 & -1 & 6 \end{bmatrix} > 0,\ Q_2 = \begin{bmatrix} -1.333 & -0.6667 & -0.6667 \\ 0.6667 & -1.333 & 0.6667 \\ -0.6667 & 0.6667 & -2 \end{bmatrix} > 0.$$

By using Theorem 2, the positive definiteness of this HMP on the uncertainty set is concluded. The entries of matrix $C_Q(x)$ are $c_{Q,ij}(x) = x^{\mathrm{T}}Q_1x,\ i = j = 1, 2, 3,$

and $c_{Q,ij}(x) = x^{\mathrm{T}} Q_2 x,\ i, j = 1, 2, 3,\ i < j.$ The following computational results have been obtained:

$$P(\alpha) \le R_{\mathrm{U}}(\alpha) = \mu R(\alpha)\ \ \forall \alpha,\ \ \mu = \min(\mu_1, \mu_2) = \min(9438.5, 9983) = 9438.5.$$

After the important parameter μ is determined, the computation of the respective upper scalar bounds can be easily obtained.

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