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Inclusion of the Intuitionistic Fuzzy Sets Based on Some Weak Intuitionistic Fuzzy Implication

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Abstract: *The definition of the Parametric of Inclusion Degree (PID) of the intuitionistic fuzzy sets on the basis of the weak intuitionistic fuzzy implication is presented in the paper. The axioms, which the inclusion measure must fulfill, noticed in recent literature, are given. The theorems present which of the axioms are met and which are not met by the newly introduced PID. Since the PID fulfills the basic axioms, although some in a "soft" way, it can be an alternative to other more "hard" measures of inclusion.*

Keywords: *Intuitionistic fuzzy sets, set inclusion, intuitionistic fuzzy implication.*

1. Brief introduction to Intuitionistic Fuzzy Sets (IFS)

In 1965 L. A. Zadeh defined the fuzzy set as

$$Z = \{(x, \mu_Z(x)) : x \in U\},$$

where $\mu_Z(x)$ is the value of function μ_Z: $U \to [0, 1]$ on a universe U, called the membership function. The family of all fuzzy sets on the universe U we denote as FS(U).

In 1983 K. Atanassov defined some generalization of Zadeh's fuzzy sets.

Definition 1. Intuitionistic fuzzy set A on a universe $U \neq \varnothing$ is understood as

$$A = \{ (x, \mu_A(x), \nu_A(x)) : x \in U \},$$

where μ_A and ν_A are functions from U to a closed interval [0, 1], and for every $x \in U$ $\mu_A(x) + \nu_A(x) \leq 1$ holds.

The values $\mu_A(x)$ and $\nu_A(x)$ are, respectively, the degree of membership and the degree of non-membership of element x to the set A. We would understand $A(x)$ as a couple $\langle \mu_A(x), \nu_A(x) \rangle$.

Unlike in classical fuzzy sets, the values of $\mu_A(x)$ and $\nu_A(x)$ are independent on each other (omitting $\mu_A(x) + \nu_A(x) \leq 1$). The family of all intuitionistic fuzzy sets on U we denote as IFS(U).

As classic fuzzy sets are associated with fuzzy logic, the IFSs are associated with the so called Intuitionistic Fuzzy Logic (IFL). In this logic the truth-value of the propositional variable p is given by an ordered couple $\langle a, b \rangle$, where a, b, $a+b \in [0, 1]$.

This pair will be called an Intuitionistic Fuzzy Value (IFV), and the set $\{\langle a, b \rangle \in [0, 1]^2 : a + b \leq 1\}$ will be denoted by L. The numbers a and b are interpreted as the validity- and non-validity-degree of the variable p. The truth-value of the variable p we denote by $V(p)$.

The variable with a truth-value *true* in classical logic is denoted by $\underline{1}$ and the variable *false* by $\underline{0}$. For this variables holds also $V(\underline{1}) = \langle 1, 0 \rangle$ and $V(\underline{0}) = \langle 0, 1 \rangle$.

Moreover, some kind of "fuzzy" truth-value is distinguished.

We call the variable x an Intuitionistic Fuzzy Tautology (IFT) when for $V(x) = \langle a, b \rangle$ holds: $a \geq b$ and, similarly, an Intuitionistic Fuzzy co-Tautology (IFcT), if $a \leq b$ holds. The IFT is therefore a variable "at least as true as false" or "rather true".

For every x we can define the value of negation of x in the typical form $V(\neg x) = \langle b, a \rangle$.

Consequently, the set $A^c = \{(x, \nu_A(x), \mu_A(x)) : x \in U\}$ is the standard complement of A.

In intuitionistic fuzzy propositional calculus, Intuitionistic Fuzzy Implication is an important operator.

Definition 2. The weak intuitionistic fuzzy implication[1] is a mapping $\Rightarrow$ fulfilling for any variable p, p_1, p_2, q, q_1, q_2 the properties:

(i1) if $V(p_1) \preceq V(p_2)$ then $V(p_1 \Rightarrow q) \succeq V(p_2 \Rightarrow q)$,

(i2) if $V(q_1) \preceq V(q_2)$ then $V(p \Rightarrow q_1) \preceq V(p \Rightarrow q_2)$,

(i3) $\underline{0} \Rightarrow q$ is an IFT,

(i4) $p \Rightarrow \underline{1}$ is an IFT,

(i5) $\underline{1} \Rightarrow \underline{0}$ is an IFcT,

where $\preceq$ is a fuzzy ordering relation in the set of an intuitionistic fuzzy variable.

[1] Definition 2 is given based on the definition of fuzzy implication (Baczyński, Jayaram [3, p. 2]).

For the variable p and q with $V(p) = \langle a, b\rangle$ and $V(q) = \langle c, d\rangle$ we denote $V(p) \preceq V(q)$ if and only if $a \le c$ and $b \ge d$.

We call this implication *weak implication* because properties (i3), (i4) and (i5) are also given in the typical, *strong form* $V(\underline{0} \Rightarrow q) = V(p \Rightarrow \underline{1}) = \langle 1, 0\rangle$ and $V(\underline{1} \Rightarrow \underline{0}) = \langle 0, 1\rangle$.

There are many postulated properties in literature, which the fuzzy implication must meet. Significantly fewer properties are given exactly for the intuitionistic fuzzy implication. They are usually extensions of the fuzzy implications axioms. The main ones are given below.

2. Strict and graded inclusion in intuitionistic fuzzy environment

In theory, both classical sets and their generalizations, as well as suggestions to use the concept of inclusion, is very important. In the case of vague sets the inclusion is not determined unambiguously. It is related to the various definitions of multi-valued implications, different concepts of cardinalities, or even with an ambiguous definition of inclusion. In intuitionistic fuzzy environment the problems are similar.

In this paper the conditions that the measure of inclusion must fulfill, are presented in a brief literature overview. A parametric inclusion degree based on the weak intuitionistic fuzzy implication is introduced.

In the paper *Fuzzy Sets,* L. A. Zadeh gives a basic definition of the inclusion of two fuzzy sets. Denoted by $\subseteq_{\mathrm{FS}}$ inclusion in Zadeh's sense, this definition can be written as follows.

Definition 3. Let $Y, Z \in \mathrm{FS}(U)$. It is $Y \subseteq_{\mathrm{FS}} Z$ iff $\forall x \in U$: $\mu_Y(x) \le \mu_Z(x)$.

This definition allows the stating of a kind of absolute, strict inclusion. Therefore in this case, the varying degrees of inclusion relationships are not highlighted. So other, different definitions of the inclusion, with an inclusion degree in the interval [0, 1] are considered. Formally, this degree is a value of the mapping I_{FS}: $\mathrm{FS}(U)\times\mathrm{FS}(U) \to [0, 1]$, and must meet certain conditions.

An extension of this idea is the definition of the IFSs inclusion.

Definition 4. Let $A, B \in \mathrm{IFS}(U)$. It is $A \subseteq_{\mathrm{IFS}} B$ iff $\forall x \in U$: $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$.

It is easy to notice that the inclusion stated in this way, is also "rigid". It allows us to respond negatively to both the inclusion of sets $A=\{(x, 1, 0)\colon x \in U\}$ and $B=\{(x, 0, 1)\colon x \in U\}$, and sets with identical membership function and non-membership function different only at one point $x_0 \in U$, in which $\nu_A(x_0) < \nu_B(x_0)$.

Due to the doubts depicted in the example above, different definitions of inclusions and inclusions degree are also introduced for IFSs.

Using the operators □ and ◊ given by Atanassov as

$$\Box A = \{(x, \mu_A(x), 1-\mu_A(x))\colon x \in U\},$$

and

$$\Diamond A = \{(x, 1-\nu_A(x), \nu_A(x))\colon x \in U\},$$

we can consider also a weaker inclusion in the form

$$A \subseteq_{\Box} B \text{ iff } \forall x \in U\text{: } \mu_A(x) \le \mu_B(x),$$

or

$$A \subseteq_{\Diamond} B \text{ iff } \forall x \in U\text{: } \nu_A(x) \ge \nu_B(x).$$

We denote that these two inclusions are really the crisp inclusions of fuzzy sets, because $\Box A$, $\Diamond A \in \mathrm{FS}(U)$.

Another generalization is the proposal of the designate of the inclusion degree using the mapping I_{IFS}: $\mathrm{IFS}(U) \times \mathrm{IFS}(U) \to [0, 1]$, with real values or $\mathrm{ID}_{\mathrm{IFS}}$: $\mathrm{IFS}(U) \times \mathrm{IFS}(U) \to L$ with intuitionistic fuzzy values.

While defining the degree of inclusions it is most often considered what set of axioms must be adopted for function I_{IFS} and what – for $\mathrm{ID}_{\mathrm{IFS}}$ so that their values could be considered as inclusion degrees. The axioms proposed are usually an extension of the axioms known for fuzzy sets. In various sources there is no compliance, even with respect to their canon. Usually the axioms proposed are a consequence of the approach given by S i n h a and D o u g h e r t y [14], and also, independently, by K i t a i n i k [10]. The analysis of the work of Sinha, Dougherty and Kitainik is contained in [6].

C o r n e l i s and K e r r e [7] extend the approach given also in [6]. According to them an intuitionistic fuzzy ***inclusion measure*** **Inc** is an $\mathrm{IFS}(U) \times \mathrm{IFS}(U) \to L$ mapping fulfilling:

(CK 1) $\mathrm{Inc}(A, B) = \mathrm{Inc}(B^c, A^c)$;

(CK 2) $\mathrm{Inc}(A, B \cap C) = \mathrm{INF}(\mathrm{Inc}(A, B), \mathrm{Inc}(A, C))$,

where $B \cap C$ denotes the classical intersection with $(B \cap C)(x) = \mathrm{INF}(B(x), C(x)) =$

$$= \mathrm{INF}(\langle \mu_B(x), \nu_B(x)\rangle, \langle \mu_C(x), \nu_C(x)\rangle) \stackrel{\text{def}}{=} \langle \min(\mu_B(x), \mu_C(x)), \max(\nu_B(x), \nu_C(x))\rangle;$$

(CK 3) $\mathrm{Inc}(A, B) = \mathrm{Inc}(P(A), P(B))$,

where P is a $\mathrm{IFS}(U) \to \mathrm{IFS}(U)$ mapping defined for $x \in U$ as $P(A)(x) = A(p(x))$, and p is a permutation of U;

(CK 4a) $\mathrm{Inc}(A, B) = \langle 1, 0\rangle$ iff $A \subseteq_{\mathrm{IFS}} B$;

(CK 4b) $\mathrm{Inc}(A, B) = \langle 0, 1\rangle$ iff $(\exists x \in U\text{: } (A(x) = \langle 1, 0\rangle \text{ and } B(x) = \langle 0, 1\rangle)$;

(CK 4c) if $A, B \in \mathrm{FS}(U)$ then $\mathrm{Inc}(A, B) \in D$,

where $D = \{\langle a, b\rangle \in L\text{: } a+b=1\}$.

The properties (CK 1-CK 4) are called contrapositivity, distributivity, symmetry and faithfulness, respectively. The property (CK 4c) is called a heritage condition.

V l a c h o s and S e r g i a d i s [15] and later X i e, H a n and M i [16] state that the ***subsethood measure*** or the ***inclusion measure*** **Inc:** $\mathrm{IFS}(U) \times \mathrm{IFS}(U) \to [0, 1]$ must fulfill the following properties:

(VS 1) $\mathrm{Inc}(A, B) = 1$ iff $A \subseteq_{\mathrm{IFS}} B$;

(VS 2) if $A^c \subseteq_{\mathrm{IFS}} A$ then $\mathrm{Inc}(A, A^c) = 0$ iff $\mathrm{A} = U_{\mathrm{IFS}}$,

where $U_{\mathrm{IFS}} = \{(x, 1, 0)\text{: } x \in U\}$;

(VS 3a) if $A \subseteq_{\mathrm{IFS}} B \subseteq_{\mathrm{IFS}} C$ then $\mathrm{Inc}(C, A) \le \mathrm{Inc}(B, A)$;

(VS 3b) if $A \subseteq_{IFS} B$ then $\mathrm{Inc}(C, A) \le \mathrm{Inc}(C, B)$.

L u o and Y u [12] proposed the ***inclusion degree function*** in the form

$$\mathrm{Inc}: \mathrm{IFS}(U)\times \mathrm{IFS}(U) \to [0, 1] \text{ satisfying:}$$

(LY 1) if $A \subseteq_{IFS} B$ then $\mathrm{Inc}(A, B)=1$;

(LY 2) $\mathrm{Inc}(U_{IFS}, \varnothing_{IFS}) = 0$,

where $\varnothing_{IFS} = \{(x, 0, 1): x\in U\}$;

(LY 3) if $A \subseteq_{IFS} B \subseteq_{IFS} C$ then $\mathrm{Inc}(C, A) \le \min(\mathrm{Inc}(B, A), \mathrm{Inc}(C, B))$.

Q u i a n, L i a n g and D a n g [13], considering the inclusion (not directly IFSs) have established, that the ***degree for the set A included in set B*** must be a number from the closed interval [0, 1] fulfilling properties (LY 1) and (VS 3b) only.

In a recent publication G r z e g o r z e w s k i [9] gives a new approach to the inclusion degree defining the possible and necessary inclusion degree of IFSs, but he gives the axioms only for FS, called mapping $\mathrm{Inc}: \mathrm{FS}(U)\times\mathrm{FS}(U) \to [0, 1]$ an ***inclusion indicator*** (or ***subsethood measure***). The indicator must fulfill the axioms equivalent to (CK 4a), (CK 4b), (CK 1) and (CK 2), and the fifth axiom: $\mathrm{Inc}(A, B) = \mathrm{Inc}(\Phi(A), \Phi(B))$, where Φ: $\mathrm{FS}(U) \to \mathrm{FS}(U)$ is a mapping defined by $\mu_{\Phi(A)}(x) = \mu_A(\varphi(x))$ with φ denoting a function φ: $U\to U$.

Let us note that this axiom is "strange" if φ can be any function. Considering, for example, $U = \{t, s\}$ and $A = \{0/s, 1/t\}$ and $B = \{0/s , 0/t\}$ if (CK 4a) must be satisfied, then $\mathrm{Inc}(A, B) \neq 1$ is obtained. However, taking for example a fixed function $\varphi(x) = s$, we obtain $\Phi(A) = \{0/t, 0/s\} = \Phi(B)$ and further $\mathrm{Inc}(\Phi(A), \Phi(B))= =1\neq \mathrm{Inc}(A, B)$. Maybe some additional conditions must be imposed on function φ, probably as in (CK 3). Later in this paper this axiom will not be considered.

The inclusion degree must be the measure of informing about to what extent, how much, the set A is contained in B. Otherwise, this degree can be understood as a measure related to the classical definition of the inclusion of A in B, which is as follows:

$$A \subseteq B \quad \text{iff} \quad \forall x\in U: (x\in A \Rightarrow_{class} x\in B),$$

where $\Rightarrow_{class}$ denoted the classical implication.

The quantifier "for all" means that the implication must be true for all elements of U. If the values of the implications are the IF values , then all of them would have the value $\langle 1, 0\rangle$. This is a very sharp assumption. That is why in the fuzzy environment we usually define the inclusion of sets not in the form *yes*/*no*, but *yes, to some degree*.

If we expand the classical definition recognizing the quantifier "for all" as the conjunction of all implications for $x\in U$, we will get the basic method for calculating the inclusion degree in the form (based on idea of B a n d l e r and K o h o u t [4]):

$$\mathrm{Inc}(A, B) = \operatorname*{INF}_{x\in U} V(A(x) \Rightarrow B(x)),$$

where

$$\underset{x\in U}{\mathrm{INF}}\ (\langle a(x),\, b(x)\rangle) \stackrel{\mathrm{def}}{=} \langle \inf_{x\in U} a(x),\ \sup_{x\in U} b(x)\rangle.$$

We note that an inclusion degree of the IFSs could be defined also in other forms, based on the cardinality of the appropriate sets or on the basis of their distance.

3. The parametric inclusion degree

Lemma 1. The parametric intuitionistic logical connective $\rightarrow_\gamma$ with a truth-value:

$$V(x\rightarrow_\gamma y) = \left\langle \frac{b+c+\gamma}{2\gamma+1},\ \frac{a+d+\gamma-1}{2\gamma+1} \right\rangle,$$

where $V(x) = \langle a, b\rangle$, $V(y) = \langle c, d\rangle$, and $\gamma \in \Re$, $\gamma \geq 1$, is a Weak Intuitionistic Fuzzy Implication, fulfilling Definition 2 (see: [8]).

Definition 5. Let A, $B \in$ IFS(U). We call the value

$$\mathrm{PID}(A, B) = \underset{x\in U}{\mathrm{INF}}\ V(A(x) \rightarrow_\gamma B(x)) \in L$$

a Parametric Inclusion Degree (PID) of A into B.

Remarks

Remark 1. If $A = \varnothing_{\mathrm{IFS}}$ and $B = U_{\mathrm{IFS}}$, then $\mathrm{PID}(A, B)= \left\langle \frac{\gamma+2}{2\gamma+1},\ \frac{\gamma-1}{2\gamma+1} \right\rangle \in D$ is an IFT and $\mathrm{PID}(B, A)= \left\langle \frac{\gamma}{2\gamma+1},\ \frac{\gamma+1}{2\gamma+1} \right\rangle \in D$ is an IFcT.

Remark 2. $\mathrm{PID}(A, B)= \langle 1, 0\rangle$ iff $\gamma=1$ and $A = \varnothing_{\mathrm{IFS}}$ and $B = U_{\mathrm{IFS}}$.

Remark 3. $\mathrm{PID}(A, A)$ is an IFT, but $\mathrm{PID}(A, A) \neq \langle 1, 0\rangle$.

Remark 4. There does not exist any $x\in U$ for which $\mathrm{PID}(A, B) = \langle 0, 1\rangle$.

The following theorems provide fulfillment of the axioms given in the references by PID. We note that the axioms (VS) and (LY) relate to the measures of the inclusion with values from the interval [0, 1]. In the next theorems we consider the adequate ($\cong$) axioms with values in L.

Theorem 1. The Parametric Inclusion Degree fulfills the axioms:

a) (PID 1)≡(CK 1): $\mathrm{PID}(A, B) = \mathrm{PID}(B^c, A^c)$;

b) (PID 2)≡(CK 2): $\mathrm{PID}(A, B\cap C) = \mathrm{INF}(\mathrm{PID}(A, B), \mathrm{PID}(A, C))$;

c) (PID 3)≡(CK 3): $\mathrm{PID}(A, B) = \mathrm{PID}(P(A), P(B))$,

where P is a IFS(U) $\rightarrow$ IFS(U) mapping defined for $x\in U$ as $P(A)(x) = A(p(x))$, where p is a permutation of U;

d) (PID 4)≡(CK 4c):

if $A, B\in$ FS(U), then $\mathrm{PID}(A, B)\in D =\{\langle a, b\rangle\in L\colon a+b=1\}$;

e) (PID 5)≅(VS 3a): if $A \subseteq_{\mathrm{IFS}} B \subseteq_{\mathrm{IFS}} C$ then $\mathrm{PID}(C, A) \preceq \mathrm{PID}(B, A)$;

f) (PID 6)≅(VS 3b): if $A \subseteq_{\text{IFS}} B$ then $\text{PID}(C, A) \preceq \text{PID}(C, B)$;

g) (PID 7)≅(LY 3):

if $A \subseteq_{\text{IFS}} B \subseteq_{\text{IFS}} C$, then $\text{PID}(C, A) \preceq \text{INF}(\text{PID}(B, A), \text{PID}(C, B))$.

P r o o f:

a) $$\text{PID}(A, B) = \underset{x\in U}{\text{INF}}\, V(A(x) \to_\gamma B(x)) = \underset{x\in U}{\text{INF}} \left\langle \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \right\rangle = \underset{x\in U}{\text{INF}}\, V(B^{c}(x) \to_\gamma A^{c}(x)) = PID(B^{c}, A^{c}).$$

b) $$\text{PID}(A, B\cap C) = \underset{x\in U}{\text{INF}}\, V(A(x) \to_\gamma (B\cap C)(x)) =$$

$$= \underset{x\in U}{\text{INF}}\, V(\langle \mu_A(x), \nu_A(x)\rangle \to_\gamma \text{INF}(\langle \mu_B(x), \nu_B(x)\rangle, \langle \mu_C(x), \nu_C(x)\rangle)) =$$

$$= \underset{x\in U}{\text{INF}} \left\langle \frac{\nu_A(x)+\min(\mu_B(x),\mu_C(x))+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\max(\nu_B(x),\nu_C(x))+\gamma-1}{2\gamma+1} \right\rangle =$$

$$= \underset{x\in U}{\text{INF}} \left\langle \min\left(\frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1}, \frac{\nu_A(x)+\mu_C(x)+\gamma}{2\gamma+1} \right), \max\left(\frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1}, \frac{\mu_A(x)+\nu_C(x)+\gamma-1}{2\gamma+1} \right) \right\rangle =$$

$$= \underset{x\in U}{\text{INF}}\, \text{INF} \left(\left\langle \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \right\rangle, \left\langle \frac{\nu_A(x)+\mu_C(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\nu_C(x)+\gamma-1}{2\gamma+1} \right\rangle \right) =$$

$$= \text{INF} \left(\underset{x\in U}{\text{INF}} \left\langle \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \right\rangle, \underset{x\in U}{\text{INF}} \left\langle \frac{\nu_A(x)+\mu_C(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+\nu_C(x)+\gamma-1}{2\gamma+1} \right\rangle \right) =$$

$$= \text{INF}(\text{PID}(A, B), \text{PID}(A, C)).$$

c) $$\text{PID}(A, B) = \underset{y\in U}{\text{INF}}\, V(A(y) \to_\gamma B(y)) = \underset{p(x)\in U}{\text{INF}}\, V(A(p(x)) \to_\gamma B(p(x))) =$$

$$= \underset{x\in U}{\text{INF}}\, V(A(p(x)) \to_\gamma B(p(x))) = \text{PID}(P(A), P(B)).$$

d) If $A, B \in \text{FS}(U)$ then $A(x) = \langle \mu_A(x), 1-\mu_A(x)\rangle$ and $B(x) = \langle \mu_B(x), 1-\mu_B(x)\rangle$.

Therefore

$$\text{PID}(A, B) = \underset{x\in U}{\text{INF}} \left\langle \frac{1-\mu_A(x)+\mu_B(x)+\gamma}{2\gamma+1}, \frac{\mu_A(x)+1-\mu_B(x)+\gamma-1}{2\gamma+1} \right\rangle =$$

$$= \left\langle \inf_{x\in U} \frac{\gamma+1-\mu_A(x)+\mu_B(x)}{2\gamma+1},\ \sup_{x\in U} \frac{\gamma+\mu_A(x)-\mu_B(x)}{2\gamma+1} \right\rangle \overset{(*)}{=}$$

$$= \left\langle 1-\sup_{x\in U}\left\{1-\frac{\gamma+1-\mu_A(x)+\mu_B(x)}{2\gamma+1}\right\},\ \sup_{x\in U} \frac{\gamma+\mu_A(x)-\mu_B(x)}{2\gamma+1} \right\rangle =$$

$$= \left\langle 1-\sup_{x\in U}\left\{\frac{\gamma+\mu_A(x)-\mu_B(x)}{2\gamma+1}\right\},\ \sup_{x\in U} \frac{\gamma+\mu_A(x)-\mu_B(x)}{2\gamma+1} \right\rangle \in D.$$

The equality (*) holds, because for $f(x)\in[0, 1]$ the property $\inf_{x\in U}\{f(x)\} = 1 - \sup_{x\in U}\{1-f(x)\}$ is valid.

e) $A \subseteq_{\text{IFS}} B \subseteq_{\text{IFS}} C$ means $\forall x\in U$: $\mu_A(x)\le \mu_B(x)\le \mu_C(x)$ and $\forall x\in U$: $\nu_A(x)\ge \nu_B(x)\ge \nu_C(x)$.

By definition $\text{PID}(C, A) = \underset{x\in U}{\text{INF}} \left\langle \frac{\nu_C(x)+\mu_A(x)+\gamma}{2\gamma+1},\ \frac{\mu_C(x)+\nu_A(x)+\gamma-1}{2\gamma+1} \right\rangle$

and $\text{PID}(B, A) = \underset{x\in U}{\text{INF}} \left\langle \frac{\nu_B(x)+\mu_A(x)+\gamma}{2\gamma+1},\ \frac{\mu_B(x)+\nu_A(x)+\gamma-1}{2\gamma+1} \right\rangle$.

Because $\forall x\in U$: $\nu_B(x)\ge \nu_C(x)$, then

$\inf_{x\in U} \frac{\nu_C(x)+\mu_A(x)+\gamma}{2\gamma+1} \le \inf_{x\in U} \frac{\nu_B(x)+\mu_A(x)+\gamma}{2\gamma+1}$ and because $\forall x\in U$: $\mu_B(x)\le \mu_C(x)$, then

$\sup_{x\in U} \frac{\mu_C(x)+\nu_A(x)+\gamma-1}{2\gamma+1} \ge \sup_{x\in U} \frac{\mu_B(x)+\nu_A(x)+\gamma-1}{2\gamma+1}$, therefore

$$\text{PID}(C, A) \preceq \text{PID}(B, A).$$

f) Analogous to e).

g) Readily apparent from e) and f) ■

Theorem 2. The Parametric Inclusion Degree does not fulfill the axioms:

a) (CK 4a)≅(VS 1), but holds

(PID 8a): if $\text{PID}(A, B) = \langle 1, 0\rangle$, then $A \subseteq_{\text{IFS}} B$;

(PID 8b): if $A \subseteq_{\text{IFS}} B$, then $\text{PID}(A, B)$ is an IFT;

b) (CK 4b), but holds

(PID 9a): if $\text{PID}(A, B) = \langle 0, 1\rangle$, then ($\exists x\in U$: ($A(x) = \langle 1, 0\rangle$ and $B(x) = \langle 0, 1\rangle$);

(PID 9b): if $\exists x\in U$: ($A(x) = \langle 1, 0\rangle$ and $B(x) = \langle 0, 1\rangle$), then $\text{PID}(A, B)$ is an IFcT;

c) (VS 2);

d) (LY 1);

e) (LY 2).

P r o o f:

a) $\text{PID}(A, B) = \langle 1, 0\rangle$ iff $\underset{x\in U}{\text{INF}} \left\langle \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1},\ \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \right\rangle =$

$=\langle 1, 0\rangle$

iff $\forall x \in U$: $\dfrac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1} = 1$ and $\dfrac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} = 0$

iff $\forall x \in U$: $\mu_B(x) + \nu_A(x) = 1 + \gamma$ and $\mu_A(x) + \nu_B(x) = 1 - \gamma$.

Because $\gamma \geq 1$, then $1 + \gamma \geq 2$ and $1 - \gamma \leq 0$. If PID(A, B) = $\langle 1, 0\rangle$, then $\gamma=1$ and $\forall x \in U$: $\mu_B(x) = \nu_A(x) = 1$ and $\mu_A(x) = \nu_B(x) = 0$.

So it is obtained $\forall x \in U$: ($A(x) = \langle 0, 1\rangle$ and $B(x) = \langle 1, 0\rangle$), therefore $A \subseteq_{\text{IFS}} B$.

In the other direction, let $A \subseteq_{\text{IFS}} B$, this means $\forall x \in U$: $\mu_A(x) \leq \mu_B(x)$ and $\nu_B(x) \leq \nu_A(x)$.

The inequality $\inf\limits_{x\in U} \dfrac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1} \geq \sup\limits_{x\in U} \dfrac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1}$ is equivalent to $\inf\limits_{x\in U} (\nu_A(x)+\mu_B(x)) \geq \sup\limits_{x\in U} (\mu_A(x)+\nu_B(x)-1)$, and this holds, because $\forall x \in U$:$\nu_A(x)+\mu_B(x)\geq 0$ and $\mu_A(x) + \nu_B(x) -1 \leq 0$.

In the second case, if it were otherwise, there would exist $x \in U$ such that $\mu_A(x)+\nu_B(x)-1>0$, i. e., $\mu_A(x)+\nu_B(x)>1$.

But, from the assumption $\nu_A(x)+\mu_B(x) \geq \mu_A(x)+\nu_B(x)$, therefore the sum $\nu_A(x)+\mu_B(x)+\mu_A(x)+\nu_B(x)>2$, which is impossible. As a result PID(A, B) is an IFT.

b) Based on Remark 4, property (PID 9a) is (formally) fulfilled.

The property (PID 9b) is fulfilled because if $\exists x \in U$:$A(x)=\langle 1, 0\rangle$ and $B(x)=\langle 0, 1\rangle$, then

$$\inf_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1} = \frac{\gamma}{2\gamma+1} \quad\text{and}\quad \sup_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} = \frac{1+\gamma}{2\gamma+1},$$

and because $\dfrac{\gamma}{2\gamma+1} \leq \dfrac{1+\gamma}{2\gamma+1}$, so PID($A$, B) = $\left\langle \dfrac{\gamma}{2\gamma+1}, \dfrac{1+\gamma}{2\gamma+1} \right\rangle \in D$ is an IFcT.

c) Readily apparent from Definition 5 and Remark 4.

d) Readily apparent from Theorem 2 a) and (PID 8b).

e) Readily apparent from Definition 5 and Remark 4 ■

Remarks

Remark 5. If $A \subseteq_{\text{IFS}} B$, then PID($A$, B) $\succeq$ PID(B, A).

Proof: Let $A \subseteq_{\text{IFS}} B$. By definition

$$\text{PID}(A, B) = \left\langle \inf_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1},\ \sup_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \right\rangle,$$

$$\text{PID}(B, A) = \left\langle \inf_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma}{2\gamma+1},\ \sup_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma-1}{2\gamma+1} \right\rangle.$$

Since $\forall x \in U$: $\mu_A(x) \leq \mu_B(x)$ and $\nu_B(x) \leq \nu_A(x)$, and therefore

$$\inf_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1} = \frac{\nu_A(x_0)+\mu_B(x_0)+\gamma}{2\gamma+1} \geq \frac{\mu_A(x_0)+\nu_B(x_0)+\gamma}{2\gamma+1} \geq \inf_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma}{2\gamma+1}$$

and similarly,

$$\sup_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1} \leq \sup_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma-1}{2\gamma+1}.$$

Finally, PID(A, B) $\succeq$ PID(B, A) ■

Remark 6. The property (PID 5) can be extended to:

(PID 5a): if $A \subseteq_{\text{IFS}} B$, then PID($B$, C) $\preceq$ PID(A, C).

Remark 7. The property (PID 9b) can be extended to the form

(PID 9c): if $\exists x_0 \in U$: score$_A(x_0)$ – score$_B(x_0)$ ≥ 1, then PID(A,B) is an IFcT, where score$_A(x)$:= $\mu_A(x) - \nu_A(x)$.

Namely it is

$$\inf_{x\in U} \frac{\nu_A(x)+\mu_B(x)+\gamma}{2\gamma+1} \leq \frac{\nu_A(x_0)+\mu_B(x_0)+\gamma}{2\gamma+1} \overset{(*)}{\leq} \frac{\mu_A(x_0)+\nu_B(x_0)+\gamma-1}{2\gamma+1} \leq$$

$$\leq \sup_{x\in U} \frac{\mu_A(x)+\nu_B(x)+\gamma-1}{2\gamma+1},$$

and inequality (*) holds therefore from the assumption, $\exists x_0 \in U$: $\mu_A(x_0) - \nu_A(x_0) - \mu_B(x_0) + \nu_B(x_0) \geq 1$, which is equivalent to $\nu_A(x_0) + \mu_B(x_0) \leq \mu_A(x_0) + \nu_B(x_0) - 1$.

4. Conclusion

The definition of the parametric degree of inclusion of intuitionistic fuzzy sets on the basis of weak intuitionistic implication is presented in the paper. The axioms, which must fulfill the inclusion measure, noticed in recent literature, are given. The theorems indicate which of the axioms are met and which are not met, by the newly introduced inclusion degree PID. Since PID fulfills the basic axioms, although some of them in a “soft” way, it can be an alternative to other, more “hard” measures of inclusion. Some properties may be difficult to accept – for example, the contained in Remark 3, property PID(A, A) $\neq \langle 1, 0\rangle$. We note, however, that the properties of PID allow its use in some applications, for example, to determine the optimal solution based on the level of inclusion of the IFS describing, the solution and IFSs describing, the best or worst solution.

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