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Some Results on Weights of Vectors Having 2-Repeated Bursts

Bhu Dev Sharma, Barkha Rohtagi

JIIT (Deemed University), A-10, Sector 62, Noida – 201301, India
E-mails: bhudev_sharma@yahoo.com barkha.rohtagi@gmail.com

Abstract: *Beside random errors, burst error studies have attracted quite some attention. Most communications now being open to atmospheric and other disturbing effects, the error patterns are mostly in the form of bursts. In fact usually the messages are long and the strings of bursts may be short repeating in a vector itself. The idea of repeated bursts, introduced by Berardi, Dass and Verma has opened this area of study.*

In this paper we obtain results on weights of all vectors having 2-repeated bursts of the same size. Another section is devoted to the study of vectors having 2 repeated bursts with weight constraints. The study can help developing more efficient codes with these vectors as error patterns.

Keywords: *Two-repeated bursts, weight constraints, efficient codes.*

1. Introduction

In many instances of communication, as is common knowledge, errors do not occur independently but are in many a ways clustered. This led to the study of burst-error-correcting codes, introduced by Fire [4] and Regier [7], and later nicely treated by Peterson & Weldon [6]. Easy implementation and efficient functioning are added advantages with burst-correcting codes. Stone [9], and Birdwell and Wolf [1] considered multiple bursts. Chien and Tang [2] also considered a different type of burst known as "*CT burst*".

Yet another kind of error pattern called "2-repeated bursts" has been introduced by Dass, Verma and Berardi [3]. This is an extension of the idea of open- loop burst given by Fire. They obtained results regarding the number of parity-check digits required for codes correcting such errors. While some results have been obtained on bounds for 2-repeated burst error correcting codes with specified distance and parity-check digits but the important area of the weight of 2-repeated burst error correcting codes is still untouched.

The study of bursts in terms of weights was initiated by Sharma and Dass [9]. Extending their work Krishnamurthy [5] gave some combinatorial results regarding the weight of burst error correcting codes.

In this paper, we obtain results regarding the weight of all vectors having 2-repeated bursts of length b each. The paper has been organized as follows: In section 2 basic definitions, related to our study are stated with some examples. Some results on weights of 2-repeated bursts are derived in Section 3. In section 4, combinatorial results on weights of 2-repeated bursts are obtained.

In the following, we shall consider the space of n-tuples whose nonzero components are taken from the field of q code characters with elements 0, 1, 2, …, $q-1$.The weight of a vector is considered in Hamming sense as the number of non-zero entries.

2. Preliminaries

We give two definition of a burst, defined by Fire, as taken in [7].

Definition 2.1. A burst of length b is a vector all of whose nonzero components are confined to some b consecutive components, the first and the last of which is nonzero.

A vector may have not just one cluster of errors, but more than one. Lumping them into one burst, amounts to neglecting the nature of communication and unnecessarily considering longer burst which may have a part, which is not of cluster in-between. For example in a very busy communication channel, sometimes, bursts repeat themselves. Dass, Verma and Berardi [3] introduced the idea of repeated bursts. In particular they defined *"2-repeated burst"*.

A 2-repeated burst of length b may be defined as follows:

Definition 2.2. A 2-repeated burst of length b is a vector of length n whose only nonzero components are confined to two distinct sets of $b$consecutive components, the first and the last component of each set being nonzero.

Example: (0001204100300) is a 2-repeated burst of length 4 over GF(5).

Weight structure being of quite some interest, in the next section, we present some results on weights of 2-repeated bursts.

3. Results on weights of 2-repeated bursts

Let W_{2b} denotes the total weight of all vectors having 2-repeated bursts of length b in the space of all n-tuples. Before obtaining W_{2b} in terms of n and b we derive two results in the lemmas below, on counting the 2-repeated bursts.

Lemma 3.1. The total number of 2-repeated bursts, each of length b>1, in the space of all n-tuples over GF(q) is

$$[q^{(b-2)}(q-1)^2]^2\,\frac{(n-2b+1)(n-2b+2)}{2}\,. \tag{1}$$

P r o o f: Let us consider a vector of length n having 2-repeated bursts of length b each. Its only nonzero components are confined in two distinct sets of b consecutive components, the first and the last components of each set being nonzero.

To make two repeated bursts of length b each, the first burst can start from ith position, where i varies from 1 to $n - 2b + 1$. The second burst can then start from a position after the first one ends.

Let us first consider the vector having 2-repeated bursts, in which the first burst starts from the first position, their number is $(q - 1)^2 q^{b-2}$, then the second burst will have $n - 2b + 1$ starting positions and their number will also be $(q - 1)^2 q^{b-2}$. Thus the total number of 2-repeated bursts in which the first burst starts from first position, is given by

$$[(q-1)^2 q^{b-2}]^2 (n-2b+1) .$$

Next considering vector with 2-repeated bursts, in which the first burst starts from second position, the starting positions of second having reduced by 1, their number shall be

$$[(q-1)^2 q^{b-2}]^2 (n-2b) .$$

A little consideration will show that the process of constructing 2-repeated bursts will end when the second burst has just one starting position, the number then being $[(q-1)^2 q^{b-2}]^2 .1$.

Summing these all the total number of n-vectors having 2-repeated bursts of length b each will be

$$[q^{(b-2)} (q-1)^2]^2 \sum_{1}^{n-2b+1} i = [q^{(b-2)} (q-1)^2]^2 \frac{(n-2b+1)(n-2b+2)}{2} .$$

This proves the result.

Next we impose weight restriction on 2-repeated bursts and count their numbers. The results are given in lemma below.

Lemma 3.2. The total number of vectors having 2-repeated bursts of length $b>1$ with weight w $(4 \le w \le 2b)$ in the space of all n-tuples is

(2) $$\binom{2b-4}{w-4} (q-1)^w \frac{(n-2b+1)(n-2b+2)}{2} .$$

P r o o f: Let us consider a vector having 2-repeated bursts of length b each. Its only nonzero components are confined to two distinct sets of consecutive components, the first and the last component of each set being nonzero, each of these, the first and the last, components may be any of the $q-1$ nonzero field elements. As we are considering 2-repeated bursts of length b, in a vector of length n, having weight w,this will have non-zero positions as follows:

i. First and the last position of first burst;

ii. First and the last position of second burst;

iii. Some w_1 amongst the $b-2$ in-between positions of first burst and then $w - w_1 - 4$ in the in-between $b - 2$ positions of the second burst, where w_1 varies from 0 to $w - 4$;

iv. Other positions have the value 0.

Thus in combinatorial ways, analyzing as before, in the counting factor $[(q-1)^2 q^{b-2}]^2$ replacing one factor q^{b-2} by $\binom{b-2}{w_1}(q-1)^{w_1}$ and the other by $\binom{b-2}{w-4-w_1}(q-1)^{w-4-w_1}$ and summing their product for w_1 in limits $0 \le w_1 \le w-4$, each 2-repeated burst will give its number by

(3) $$(q-1)^2(q-1)^2(q-1)^{w-4}\sum_{w_1=0}^{w-4}\binom{b-2}{w_1}\binom{b-2}{w-4-w_1} =$$

$$= (q-1)^w \sum_{w_1=0}^{w-4}\binom{b-2}{w_1}\binom{b-2}{w-4-w_1}.$$

To find a close expression for

$$\sum_{w_1=0}^{w-4}\binom{b-2}{w_1}\binom{b-2}{w-4-w_1},$$

we consider the following identity

(4) $$(1+x)^{2b-4} = (1+x)^{b-2}(1+x)^{b-2}.$$

Equating coefficients of x^{w-4} from both sides, we get

$$\binom{2b-4}{w-4} = \sum_{w_1=0}^{w-4}\binom{b-2}{w_1}\binom{b-2}{w-4-w_1}.$$

Using this identity, the total number of 2-repeated bursts of length b and weight w, with sum of their starting position $\frac{(n-2b+1)(n-2b+2)}{2}$ is

$$\binom{2b-4}{w-4}(q-1)^w \frac{(n-2b+1)(n-2b+2)}{2}.$$

This proves Lemma 3.1.

Note. Let us sum up the result of Lemma 3.2, where w takes all possible values, that is, $4 \le w \le 2b$. In this way, the number of all vectors that have 2-bursts of length b each is

$$\frac{(n-2b+1)(n-2b+2)}{2}\sum_{w=4}^{w=2b}\binom{2b-4}{w-4}(q-1)^w =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}\sum_{i=0}^{i=2b-4}\binom{2b-4}{w-4}(q-1)^{i+4} =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 q^{2b-4}.$$

This is same as the result of Lemma 3.1.

Now we return to finding an expression for W_{2b}, the total weight of all vectors having 2-repeated bursts of length b in the space of all n-tuples.

Theorem 3.1. For $n \geq b$

(5) $$W_2 = \frac{n(n-1)}{2}(q-1)^2$$

and

(6) $$W_{2b} = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 q^{2b-5}[2b(q-1)+4]\ .$$

Proof: The value of W_2 follows simply by considering all vectors having any two non-zero entries out of n. Their number clearly is given by

$$\binom{n}{2}(q-1)^2 = \frac{n(n-1)}{2}(q-1)^2 .$$

This gives the value of W_2 as stated.

Next, for $b > 1$, using the Lemma 3.2, the total weight of all vectors having 2-repeated bursts of length b each is given by

$$\sum_{w=4}^{2b} w\binom{2b-4}{w-4}\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^w =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 \sum_{i=0}^{2b-4}(i+4)\binom{2b-4}{i}(q-1)^i =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 \frac{1}{(q-1)^3}\frac{d}{dq}[(q-1)^4\{1+(q-1)\}^{2b-4}] =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)\frac{d}{dq}[(q-1)^4 q^{2b-4}] =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)[(2b-4)(q-1)^4 q^{2b-5} + q^{2b-4}.4(q-1)^3] =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 q^{2b-5}[(2b-4)(q-1)+4q] =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 q^{2b-5}[2b(q-1)+4].$$

This completes the proof of the Theorem 3.1.

Further, in coding theory, an important criterion is to look for minimum weight in a group of vectors. Our following theorem is a result in that direction.

Theorem 3.2. The minimum weight of a vector having 2-repeated burst of length b>1 in the space of all n-tuples is at most

(7) $$2b - \frac{2(b-2)}{q}.$$

Proof: From Lemma 3.1, it is clear that the number of 2-repeated bursts of length b in the space of all n-tuples with symbols taken from the field of q elements is

$$[q^{(b-2)}(q-1)^2]^2 \frac{(n-2b+1)(n-2b+2)}{2}.$$

Also from Theorem 3.1, their total weight is

$$\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 q^{2b-5}[2b(q-1)+4].$$

Since the minimum weight element can at most be equal to the average weight, an upper bound on minimum weight of a 2-repeated burst of length b is given by

$$\frac{2.(n-2b+1)(n-2b+2)(q-1)^4 q^{2b-5}[2b(q-1)+4]}{[q^{(b-2)}(q-1)^2]^2(n-2b+1)(n-2b+2).2} =$$

$$= 2b - \frac{2(b-2)}{q}.$$

This proves the result.

In this section, we considered a problem which allows greater fine tuning of error patterns for efficient coding. This is obtained by considering 2-repeated bursts with weight constraints. It is not uncommon to find error patterns in the form of bursts, receiving some positions correctly. Such error vectors have only a limited number of non-zero positions spread up over the burst length. Thus we come to a problem more general than the one handled above. We study vectors with two repeated bursts having weight constraint on them.

4. Combinatorial results on weights of vectors having 2-repeated bursts with weight constraint

Let $W_{2b,w}$ denote the total weight of those vectors having 2-repeated bursts of length b each, which are of weight w or less in the space of all n-tuples over $\mathrm{GF}(q)$ .Before obtaining main results, we state a simple result in lemma below.

Lemma 4.1. Let $[1+x]^{(n,r)}$ denote the incomplete binomial expansion

$$1+\binom{n}{1}x+...+\binom{n}{r}x^r \text{ of } (1+x)^n$$

up to the term x^r, $r \le n$, in the ascending powers of x.

Then $\frac{d}{dx}[1+x]^{(n,r)} = n[1+x]^{(n-1.r-1)}$, where $\frac{d}{dx}$ for the derivative with respect to x [6].

Theorem 4.1. In the space of all n-tuples over $\mathrm{GF}(q)$, for $n \ge 2b \ge w > 1$

(8)
$$W_{2b,w} = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4\Big[2[1+(q-1)]^{(2b-4,w-4)} + \\ + (q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)}\Big].$$

P r o o f: We know, from Lemma 3.2, that the total number of vectors having 2-repeated bursts of length $b>1$ each, with weight w in the space of all n-tuples is

$$\binom{2b-4}{w-4}(q-1)^w \frac{(n-2b+1)(n-2b+2)}{2}.$$

Therefore, $W_{2b,w}$ the total weight of 2-repeated bursts of length b each with weight w or less, where $4 \le w \le 2b,$ is given by

$$W_{2b,w} = \sum_{i=4}^{w} i \binom{2b-4}{w-4} (q-1)^i \, \frac{(n-2b+1)(n-2b+2)}{2} =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)\left[4(q-1)^3 + 5\binom{2b-4}{1}(q-1)^4 + \cdots + \right.$$

$$\left. + w\binom{2b-4}{w-4}(q-1)^{w-1} \right] = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)\frac{d}{dq}\left[(q-1)^4 \quad + \right.$$

$$\left. + \binom{2b-4}{1}(q-1)^5 + \ldots + \binom{2b-4}{w-4}(q-1)^w \right] = \frac{(n-2b+1)(n-2b+2)}{2}(q-$$

$$-1)\frac{d}{dq}\left[(q-1)^4 \left\{ 1 + \binom{2b-4}{1}(q-1) + \ldots + \binom{2b-4}{w-4}(q-1)^{w-4} \right\} \right] =$$

$$= \frac{(n-2b+1)(n-2b+2)}{2}(q-1)\left[(q-1)^4 [1+(q-1)]^{(2b-4,w-4)} \right],$$

and using Lemma 4.1,

$$W_{2b,w} = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)\left[(q-1)^4 (2b-4)[1+(q-1)]^{(2b-5,w-5)} + \right.$$

$$\left. + 2(q-1)^3[1+(q-1)]^{(2b-4,w-4)} \right] = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 \left[2[1+ \right.$$

$$\left. + (q-1)]^{(2b-4,w-4)} + (q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)} \right].$$

This proves Theorem 4.1.

Next we give a recurrence relation for weights in this very general case.

Theorem 4.2. A recurrence relation between $W_{2b,w}$ and $W_{2b-1,w-1}$ is given by

(9) $$\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 \, \frac{d}{dq}\left[\frac{2.W_{2b,w}}{(n-2b+1)(n-2b+2)(q-1)^4} \right] =$$

$$= (2b-4)W_{2b-1,w-1} + \frac{(n-2b+1)(n-2b+2)}{2} \, (q-1)^4 (2b-4)[1+(q-1)]^{(2b-5,w-5)} .$$

P r o o f: From Theorem 4.1, we have

$$W_{2b,w} = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4 \left[2[1+(q-1)]^{(2b-4,w-4)} + \right.$$

$$\left. + (q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)} \right].$$

Therefore,

(10) $$W_{2b-1,w-1} = \frac{(n-2b+2)(n-2b+3)}{2}(q-1)^4 \left[2[1+(q-1)]^{(2b-5,w-5)} + \right.$$

$$\left. + (q-1)(2b-5)[1+(q-1)]^{(2b-6,w-6)} \right]$$

and

$$\text{(11)} \qquad \frac{2.W_{2b,w}}{(n-2b+1)(n-2b+2)(q-1)^4} = \Big[2[1+(q-1)]^{(2b-4,w-4)} +$$

$$+ (q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)}\Big].$$

Differentiating with respect to q and then using Lemma 4.1, we get

$$\frac{d}{dq}\frac{2.W_{2b,w}}{(n-2b+1)(n-2b+2)(q-1)^4} =$$

$$= \begin{bmatrix} 2(2b-4)[1+(q-1)]^{(2b-5,w-5)} + (q-1)(2b-4)(2b-5)[1+(q-1)]^{(2b-6,w-6)}] \\ + (2b-4)[1+(q-1)]^{(2b-5,w-5)} \end{bmatrix} =$$

$$= (2b-4)\begin{bmatrix} 2[1+(q-1)]^{(2b-5,w-5)} + (2b-5)(q-1)[1+(q-1)]^{(2b-6,w-6)} \\ + [1+(q-1)]^{(2b-5,w-5)} \end{bmatrix}.$$

The result now follows by using the value of $W_{2b-1,w-1}$.

Finally, we have the result on upper bound on the minimum weight vector in the class of vectors considered in this section.

Theorem 4.3. The minimum weight of a 2-repeated burst of length b with weight w or less in the space of all n-tuples over GF(q) is at most

$$\text{(12)} \qquad 2 + \frac{(q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)}}{[1+(q-1)]^{(2b-4,w-4)}}.$$

Proof: Using the Lemma 3.2, the total number of 2-repeated bursts of length b with weight w in the space of all n-tuples over GF(q) is

$$\frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4[1+(q-1)]^{(2b-4,w-4)}.$$

From Theorem 4.1, the total weight is

$$W_{2b,w} = \frac{(n-2b+1)(n-2b+2)}{2}(q-1)^4\Big[2[1+(q-1)]^{(2b-4,w-4)} +$$

$$+ (q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)}\Big].$$

Since the minimum weight element is at the most equal to the average weight, the minimum weight of a 2-repeated burst of length b with weight w or less is at most

$$\frac{(n-2b+1)(n-2b+2)(q-1)^4[2[1+(q-1)]^{(2b-4,w-4)}}{2.(n-2b+1)(n-2b+2)(q-1)[1+(q-1)]^{(2b-4,w-4)}} +$$

$$+ \frac{(q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)]}.2}{2.(n-2b+1)(n-2b+2)(q-1)[1+(q-1)]^{(2b-4,w-4)}} =$$

$$= 2 + \frac{(q-1)(2b-4)[1+(q-1)]^{(2b-5,w-5)}}{[1+(q-1)]^{(2b-4,w-4)}}.$$

This proves the result.

5. Concluding remarks

Multiple bursts present interesting class of error patterns over an alphabet of size q. Results for binary case can be derived immediately. Also, here we have considered vectors having two bursts of equal lengths, with or without weight constraints. Studies generalizing these considerations have also attracted our attention that will be reported separately. With these bursts as error patterns, codes capable of correcting these patterns will improve the communicate rate. Constructing such codes will be a part of continued study.

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