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Full Perturbation Analysis of the LMI-Based Continuous-Time Linear Quadratic Regulator Problem

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Abstract: *The Linear Quadratic Regulator (LQR) problem is used to obtain a control law that can ensure stability of the closed-loop system, desired performance and minimizes a predefined cost function. The considered control problem can be implicitly realized using the solutions Q, Y of a system of Linear Matrix Inequalities (LMIs). In this paper we perform linear perturbation analysis for the discrete-time LMI based linear quadratic regulator problem. The problem of performing sensitivity analysis of the perturbed matrix inequalities is considered similarly to perturbed matrix equations, after introducing a slightly perturbed suitable right hand part. The proposed approach allows to obtain tight linear perturbation bounds for the LMIs' solutions to the linear quadratic regulator problem. In the paper numerical examples are also presented.*

Keywords: *Perturbation analysis, Linear Quadratic Regulator Problem, LMI based synthesis, Linear systems.*

1. Introduction

There are many control problems where the design constraints have a simple reformulation in terms of Linear Matrix Inequalities (LMIs). This is because the LMIs are direct byproducts of Lyapunov based criteria, and that Lyapunov techniques plays a central role in the analysis and control of linear systems [1, 2]. The Linear Quadratic Regulator (LQR) problem is a good illustration of what is mentioned above.

LMI approach is effective because it is applicable to all plants without restrictions on infinite or pure imaginary invariant zeros. LMI design is practical, interesting and useful thanks to the availability of efficient convex optimization algorithms [3] and software [4] plus the MATLAB package Yalmip and SeDuMi solver [5].

The LQR problem is known to be investigated using solutions of a Riccati equation [6]. Perturbation analysis of Riccati equations is performed in [7].

The aim of this paper is to propose an approach to perform linear perturbation analysis of the LMI based LQR problem via introducing a suitable right hand part in the considered matrix inequalities. After the considered problem is solved the obtained results can be applied in the following ways. First it is possible to estimate the errors in the calculated solution of the LQR problem, which are due to rounding errors and parametric disturbances in the considered data. Second it is possible to study the robust stability and robust performance of the closed loop system with uncertainties in the plant and in the controller. The uncertainties in the controller appear because of the sensitivity of the linear quadratic regulator problem.

Further the following notation is used: $R^{m\times n}$ – the space of real $m\times n$ matrices; $R^n = R^{n\times 1}$; I_n – the identity $n\times n$ matrix; e_n – the unit $m\times 1$ vector; M^{T} – the transpose of M; $M^{\perp}$ – the pseudo-inverse of M; $\|M\|_2 = \sigma_{\max}(M)$ – the spectral norm of M, where $\sigma_{\max}(M)$ is the maximum singular value of M; $\mathrm{vec}(M)\in R^{mn}$ – the column-wise vector representation of $M\in R^{m\times n}$; $\Pi_{m,n}\in R^{mn\times mn}$ – the vec-permutation matrix, such that $\mathrm{vec}(M^{\mathrm{T}})= \Pi_{m,n}\,\mathrm{vec}(M)$; $M \otimes P$ – the Kroneker product of the matrices M and P. The notation ":=" stands for "equal by definition".

The rest of the paper is structured as follows. Section 2 presents the problem set up and objective. In Section 3 the performed linear sensitivity analysis of the LMI-based continuous linear quadratic regulator problem is presented. In Section 4 some numerical examples are given before we conclude in Section 5 with some final remarks.

2. Problem setup and objective

In this paper we consider the linear continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t)\in R^n$ and $u(t)\in R^m$ are the system state and input vectors respectively, and A, B are constant matrices of compatible size.

Linear quadratic regulator problem means for a given initial state $x(0)$ to find a control law, which minimizes the cost function $\int_0^\infty \left(x^{\mathrm{T}}Q_p x + u^{\mathrm{T}}R_p u\right)dt$. It is also necessary to find a quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$, $P > 0$, such that $\frac{d}{dt}V(x) < -x^{\mathrm{T}}\left[Q_p + K^{\mathrm{T}}R_pK\right]x.$

In order to solve the LQR problem and to ensure closed-loop stability and specified performance it is necessary to design a state-feedback control $u=Kx$.

We consider an LMI approach to solve the linear quadratic regulator problem, as stated in [1]:

$$x^{\mathrm{T}}\left[(A+BK)^{\mathrm{T}}P + P(A+BK)\right]x < -x^{\mathrm{T}}\left(Q_p + K^{\mathrm{T}}R_pK\right)x,\ p > 0. \tag{2}$$

From Schur complement argument [8] the above inequality is equivalent to

$$(3)\qquad \begin{bmatrix} (A+BK)^{\mathrm{T}}P+P(A+BK) & K^{\mathrm{T}} & I \\ K & -R_p^{-1} & 0 \\ I & 0 & -Q_p^{-1} \end{bmatrix} < 0.$$

We pre- and post- multiply expression (3) by diag$[P^{-1}, I, I]$. Later we introduce new variables $Q = P^{-1}$, $Q > 0$ and $Y = KP^{-1}$ to obtain the following system of LMIs:

$$(4)\qquad \begin{bmatrix} AQ+QA^{\mathrm{T}}+BY+Y^{\mathrm{T}}B^{\mathrm{T}} & Y^{\mathrm{T}} & Q \\ Y & -R_p^{-1} & 0 \\ Q & 0 & -Q_p^{-1} \end{bmatrix} < 0,$$

$$Q > 0.$$

The main objective of the paper is to perform a linear sensitivity analysis of the LMI system (4), needed to solve the continuous-time linear quadratic requlator problems. Further in the paper we will use the following notation: $R_P^{-1} = R_{ip}$, $Q_p^{-1} = Q_{ip}$, $\Delta R_p^{-1} = \Delta R_{ip}$, $\Delta Q_p^{-1} = \Delta Q_{ip}$.

Suppose that the matrices A, B, R_{ip}, Q_{ip} are subject to perturbations ΔA, ΔB, ΔR_{ip}, ΔQ_{ip} and assume that they do not change the sign of the LMI system (4). The sensitivity analysis of the continuous-time LMI based linear quadratic regulator problem is aimed at determining perturbation bounds of the LMI system (4) as functions of the perturbations in the data A, B, R_{ip}, Q_{ip}.

3. Linear perturbation analysis

In this section perturbation analysis of the LMI (4) for the continuous-time system (1) is performed

$$(5)\qquad \begin{bmatrix} ABQY^{\mathrm{T}}+ABQY & (Y+\Delta Y)^{\mathrm{T}} & (Q+\Delta Q) \\ (Y+\Delta Y) & -(R_{ip}+\Delta R_{ip}) & 0 \\ (Q+\Delta Q) & 0 & -(Q_{ip}+\Delta Q_{ip}) \end{bmatrix} < 0,$$

where $$ABQY^{\mathrm{T}} = (Q+\Delta Q)(A+\Delta A)^{\mathrm{T}} + (Y+\Delta Y)^{\mathrm{T}}(B+\Delta B)^{\mathrm{T}},$$

$$ABQY = (Q+\Delta Q)(A+\Delta A) + (Y+\Delta Y)(B+\Delta B).$$

It is necessary that we study the effect of the perturbations ΔA, ΔB, ΔR_{ip}, ΔQ_{ip}, on the perturbed LMI solutions $Q^* + \Delta Q$ and $Y^* + \Delta Y$, where Q^*, Y^* and ΔQ, ΔY are the nominal solution of the inequality (4) and the perturbations, respectively. The essence of our approach is to perform perturbation analysis of the inequality (4) similarly to a perturbed matrix equation, after introducing a slightly perturbed suitable right hand part. In this way LMI (5) is obtained:

$$(6)\qquad \begin{bmatrix} ABQY^{*\mathrm{T}}+ABQY^{*} & (Y^*+\Delta Y)^{\mathrm{T}} & (Q^*+\Delta Q) \\ (Y^*+\Delta Y) & -(R_{ip}+\Delta R_{ip}) & 0 \\ (Q^*+\Delta Q) & 0 & -(Q_{ip}+\Delta Q_{ip}) \end{bmatrix} = L^*+\Delta L_1 < 0.$$

Here $\quad ABQY^{*\mathrm{T}} = (Q^* + \Delta Q)(A + \Delta A)^{\mathrm{T}} + (Y^* + \Delta Y)^{\mathrm{T}}(B + \Delta B)^{\mathrm{T}},$

$$ABQY^* = (Q^* + \Delta Q)(A + \Delta A) + (Y^* + \Delta Y)(B + \Delta B),$$

and L^* is calculated using the following nominal LMI:

$$\begin{bmatrix} AQ + BY^* + Q^*A^{\mathrm{T}} + Y^{*\mathrm{T}}B^{\mathrm{T}} & Y^{*\mathrm{T}} & Q^* \\ Y^* & -R_{ip} & 0 \\ Q^* & 0 & -Q_{ip} \end{bmatrix} = L^* < 0. \tag{7}$$

The matrix ΔL_1 includes information regarding data and closed-loop performance perturbations, the rounding errors and the sensitivity of the interior point method that is used to solve the LMIs.

If we use the relation (7) the perturbed equation (6) can be written in the following way:

$$\Delta_Q + \Omega_Q = \Delta L_1, \tag{8}$$

where

$$\Delta_Q = \begin{bmatrix} A\,\Delta Q + \Delta Q\,A^{\mathrm{T}} & 0 & \Delta Q \\ 0 & 0 & 0 \\ \Delta Q & 0 & 0 \end{bmatrix},$$

$$\Omega_Q = \begin{bmatrix} \Delta AQ^* + B\Delta Y + \Delta BY^* + Q^*\Delta A^{\mathrm{T}} + \Delta Y^{\mathrm{T}}B^{\mathrm{T}} + Y^{*\mathrm{T}}\Delta B^{\mathrm{T}} & \Delta Y^{\mathrm{T}} & 0 \\ \Delta Y & -\Delta R_{ip} & 0 \\ 0 & 0 & -\Delta Q_{ip} \end{bmatrix}.$$

Since we perform linear sensitivity analysis here the terms of second and higher order are neglected. Then the vectorized form of relation (8) can be obtained

$$\mathrm{vec}(\Delta_Q) + \mathrm{vec}(\Omega_Q) = \mathrm{vec}(\Delta L_1), \tag{9}$$

where

$$\mathrm{vec}(\Delta_Q) = [I \otimes A + A \otimes I, 0, I, 0, 0, 0, I, 0, 0]^{\mathrm{T}}\,\mathrm{vec}(\Delta Q) := T\Delta q,$$

$$\mathrm{vec}(\Omega_Q) =$$

$$= \begin{bmatrix} (Q^*\otimes I) + (I \otimes Q^*)\Pi_{n^2} & (I \otimes B) + (B \otimes I)\Pi_{n\times m} & (Y^*\otimes I) + (I \otimes Y^{*\mathrm{T}})\Pi_{m^2} & 0 & 0 \\ 0 & \Pi_{m\times m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} \times$$

$$\times\begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Y) \\ \mathrm{vec}(\Delta B) \\ \mathrm{vec}(\Delta R_{ip}) \\ \mathrm{vec}(\Delta Q_{ip}) \end{bmatrix} = \begin{bmatrix} T_{t1} & T_{t2} & T_{t3} & T_{t4} & T_{t5} \end{bmatrix} \Delta_{a\,y\,bR_{ip}Q_{ip}} := T_t \Delta_{a\,y\,bR_{ip}Q_{ip}}.$$

The derivations made allow us to obtain the expression

(10) $$T\Delta q + T_{t1}\mathrm{vec}(\Delta A) + T_{t2}\,\mathrm{vec}(\Delta Y) + T_{t3}\,\mathrm{vec}\,(\Delta B) + T_{t4}\mathrm{vec}\,(\Delta R_{ip}) + \\ + T_{t5}\,\mathrm{vec}(\Delta Q_{ip}) = \mathrm{vec}\,(\Delta L_1).$$

At the end the relative perturbation bound for the solution Q^* of the LMI (4) is obtained

(11) $$\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2} \le \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(T_{ayb1}\frac{\|\mathrm{vec}(\Delta A)\|_2}{\|\mathrm{vec}(A)\|_2} + T_{ayb2}\frac{\|\mathrm{vec}(\Delta Y)\|_2}{\|\mathrm{vec}(Y^*)\|_2} + \right. \\ \left. + T_{ayb3}\frac{\|\mathrm{vec}(\Delta B)\|_2}{\|\mathrm{vec}(B)\|_2}\right) \le \frac{1}{\|\mathrm{vec}(Q^*)\|_2}\left(T_{ayb4}\frac{\|\mathrm{vec}(\Delta R_{ip})\|_2}{\|\mathrm{vec}(R_{ip})\|_2} + \right. \\ \left. + T_{ayb5}\frac{\|\mathrm{vec}(\Delta Q_{ip})\|_2}{\|\mathrm{vec}(Q_{ip})\|_2} + L_1\frac{\|\mathrm{vec}(\Delta L_1)\|_2}{\|\mathrm{vec}(L^*)\|_2}\right),$$

where

$$\frac{T_{ayb1}}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|T_{t1}\|_2\|\mathrm{vec}(A)\|_2}{\|\mathrm{vec}(Q^*)\|_2}, \quad \frac{T_{ayb2}}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|T_{t2}\|_2\|\mathrm{vec}(Y^*)\|_2}{\|\mathrm{vec}(Q^*)\|_2},$$

$$\frac{T_{ayb3}}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|T_{t3}\|_2\|\mathrm{vec}(B)\|_2}{\|\mathrm{vec}(Q^*)\|_2}, \quad \frac{L_1}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|\mathrm{vec}(L^*)\|_2}{\|\mathrm{vec}(Q^*)\|_2},$$

$$\frac{T_{ayb4}}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|T_{t4}\|_2\|\mathrm{vec}(R_{ip})\|_2}{\|\mathrm{vec}(Q^*)\|_2},$$

$$\frac{T_{ayb5}}{\|\mathrm{vec}(Q^*)\|_2} := \frac{\|T^{\perp}\|_2\|T_{t5}\|_2\|\mathrm{vec}(Q_{ip})\|_2}{\|\mathrm{vec}(Q^*)\|_2}$$

are the individual relative condition numbers of the LMI (4) with respect to the perturbations ΔA, ΔB, ΔR_{ip}, ΔQ_{ip}, and ΔY.

Using a similar procedure the relative perturbation bounds for the solution Y^* of the LMI (4) may be obtained. We use the following expression:

(12) $$\Delta_Y + \Omega_Y = \Delta L_2\,,$$

where

$$\Delta_Y = \begin{bmatrix} B\Delta Y + \Delta Y^{\mathrm{T}} B^{\mathrm{T}} & \Delta Y^{\mathrm{T}} & 0 \\ \Delta Y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Omega_Y = \begin{bmatrix} A\Delta Q + \Delta A Q^* + \Delta B Y^* + \Delta Q A^{\mathrm{T}} + Q^* \Delta A^{\mathrm{T}} + Y^{*\mathrm{T}} \Delta B^{\mathrm{T}} & 0 & \Delta Q \\ 0 & -\Delta R_{ip} & 0 \\ \Delta Q & 0 & -\Delta Q_{ip} \end{bmatrix}.$$

Since we perform linear sensitivity analysis here the terms of second and higher order are neglected. Then the vectorized form of relation (12) can be obtained as

$$\mathrm{vec}(\Delta_Y) + \mathrm{vec}(\Omega_Y) = \mathrm{vec}(\Delta L_2), \tag{13}$$

where

$$\mathrm{vec}(\Delta_Y) = [(I \otimes B) + (B \otimes I)\Pi_{n\times m}, \Pi_{m\times m}, 0, I, 0, 0, 0, 0, 0]^{\mathrm{T}} \mathrm{vec}(\Delta Y) := W\,\Delta y,$$

$$\mathrm{vec}(\Omega_Y) =$$

$$= \begin{bmatrix} (Q^* \otimes I) + (I \otimes Q^*)\Pi_{n^2} & (I \otimes A) + (A \otimes I) & (Y^* \otimes I) + (I \otimes Y^{*\mathrm{T}})\Pi_{m^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \end{bmatrix} \times$$

$$\times \begin{bmatrix} \mathrm{vec}(\Delta A) \\ \mathrm{vec}(\Delta Q) \\ \mathrm{vec}(\Delta B) \\ \mathrm{vec}(\Delta R_{ip}) \\ \mathrm{vec}(\Delta Q_{ip}) \end{bmatrix} = \begin{bmatrix} W_{t1} & W_{t2} & W_{t3} & W_{t4} & W_{t5} \end{bmatrix} \Delta_{aqbR_pQ\,ip} := W_t \Delta_{aqbR_pQ_{ip}}.$$

The derivations made allow us to obtain the expression

$$W\Delta y + W_{t1}\,\mathrm{vec}\,(\Delta A) + W_{t2}\,\mathrm{vec}\,(\Delta Q) + W_{t3}\,\mathrm{vec}\,(\Delta B) + \\ + W_{t4}\,\mathrm{vec}\,(\Delta R_{ip}) + W_{t5}\,\mathrm{vec}\,(\Delta Q_{ip}) = \mathrm{vec}\,(\Delta L_2). \tag{14}$$

At the end the relative perturbation bound for the solution Y^* of the LMI (4) is obtained:

(15) $$\frac{\|\Delta y\|_2}{\|\operatorname{vec}(Y^*)\|_2} \le \frac{1}{\|\operatorname{vec}(Y^*)\|_2}\left(W_{ayb1}\frac{\|\operatorname{vec}(\Delta A)\|_2}{\|\operatorname{vec}(A)\|_2} + W_{ayb2}\frac{\|\operatorname{vec}(\Delta Q)\|_2}{\|\operatorname{vec}(Q^*)\|_2} + \right.$$

$$\left. + W_{ayb3}\frac{\|\operatorname{vec}(\Delta B)\|_2}{\|\operatorname{vec}(B)\|_2}\right) \le \frac{1}{\|\operatorname{vec}(Y^*)\|_2}\left(W_{ayb4}\frac{\|\operatorname{vec}(\Delta R_{ip})\|_2}{\|\operatorname{vec}(R_{ip})\|_2} + \right.$$

$$\left. + W_{ayb5}\frac{\|\operatorname{vec}(\Delta Q_{ip})\|_2}{\|\operatorname{vec}(Q_{ip})\|_2} + L_2\frac{\|\operatorname{vec}(\Delta L_2)\|_2}{\|\operatorname{vec}(L^*)\|_2}\right),$$

where

$$\frac{W_{aqb1}}{\|\operatorname{vec}(Y^*)\|_2} := \frac{\|W^{\perp}\|_2 \|W_{t1}\|_2 \|\operatorname{vec}(A)\|_2}{\|\operatorname{vec}(Y^*)\|_2},$$

$$\frac{W_{aqb2}}{\|\operatorname{vec}(W^*)\|_2} := \frac{\|W^{\perp}\|_2 \|W_{t2}\|_2 \|\operatorname{vec}(Q^*)\|_2}{\|\operatorname{vec}(Y^*)\|_2},$$

$$\frac{W_{aqb3}}{\|\operatorname{vec}(Y^*)\|_2} := \frac{\|W^{\perp}\|_2 \|W_{t3}\|_2 \|\operatorname{vec}(B)\|_2}{\|\operatorname{vec}(Y^*)\|_2},$$

$$\frac{L_2}{\|\operatorname{vec}(Y^*)\|_2} := \frac{\|W^{\perp}\|_2 \|\operatorname{vec}(L^*)\|_2}{\|\operatorname{vec}(Y^*)\|_2},$$

$$\frac{W_{aqb4}}{\|\operatorname{vec}(Y^*)\|_2} := \frac{\|W^{\perp}\|_2 \|W_{t4}\|_2 \|\operatorname{vec}(R_{ip})\|_2}{\|\operatorname{vec}(Y^*)\|_2},$$

$$\frac{W_{aqb5}}{\|\operatorname{vec}(Y^*)\|_2} := \frac{\|W^{\perp}\|_2 \|W_{t5}\|_2 \|\operatorname{vec}(Q_{ip})\|_2}{\|\operatorname{vec}(W^*)\|_2}$$

are the individual relative condition numbers of the LMI (2) with respect to the perturbations ΔA, ΔB, ΔR_{ip}, ΔQ_{ip}, and ΔQ.

4. Numerical examples

4.1. Example 1 [10]

Consider the following continuous-time system (1), where

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\mathrm{pm} & -\mathrm{pc}/m & -\mathrm{pk}/m & 1/m \end{bmatrix},$$

$$Q_p = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ R_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

and $m = 3,\ c = 1,\ k = 2,\ \mathrm{pm} = 0.4,\ \mathrm{pc} = 0.2,\ \mathrm{pk} = 0.3.$

The perturbations in the system matrices are chosen as:

$$\Delta A = A \times 10^{-i},\ \Delta B = B \times 10^{-i},$$
$$\Delta R_{ip} = R_{ip} \times 10^{-i},\ \Delta Q_{ip} = Q_{ip} \times 10^{-i},$$
$$\Delta L_1 = L^* \times 10^{-i},\ \Delta L_2 = L^* \times 10^{-i},$$
$$\Delta Q = Q^* \times 10^{-i},\ \Delta Y = Y^* \times 10^{-i}.$$

The perturbed solutions $Q^* + \Delta Q$ and $Y^* + \Delta Y$ of LMI (6) are computed using the method derived in [9] and applying the software [4]. Based on the proposed approach the relative perturbation bounds for the solutions $Q^*$and Y^* of the LMI system (4) are obtained by the linear bounds (11) and (15), respectively.

The results obtained for different size of perturbations are shown in Table 1.

Table 1

i	$\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2}$	Bound (11)	$\frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2}$	Bound (15)
8	5.1250×10^{-8}	1.1304×10^{-7}	3.8134×10^{-8}	6.7272×10^{-8}
7	5.1250×10^{-7}	1.1304×10^{-6}	3.8134×10^{-7}	6.7272×10^{-7}
6	5.1250×10^{-6}	1.1304×10^{-5}	3.8134×10^{-6}	6.7272×10^{-6}
5	5.1250×10^{-5}	1.1304×10^{-4}	3.8134×10^{-5}	6.7272×10^{-5}
4	5.1250×10^{-4}	1.1304×10^{-3}	3.8134×10^{-4}	6.7272×10^{-4}

Based on the proposed solution approach we obtain the perturbation bounds (11) and (15), which are close to the real relative perturbation bounds $\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2}$ and $\frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2}$, thus they are good in sense that they are tight.

4.2. Example 2

Consider the continuous-time system (1), where

$$A^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{mm}^2\mathrm{el}^2 g}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} & \frac{((M+\mathrm{mm})\times\mathrm{mm}\times\mathrm{el}\times g)}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} \\ 1 & 0 & \frac{-(J+\mathrm{mm}\times\mathrm{el}^2)\mathrm{fc}}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} & \frac{-\mathrm{mm}\times\mathrm{el}\times\mathrm{fc}}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} \\ 0 & 1 & \frac{-\mathrm{mm}\times\mathrm{el}\times\mathrm{fp}}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} & \frac{-(M+\mathrm{mm})\times\mathrm{fp}}{((M+\mathrm{mm})\times J + M\times\mathrm{mm}\times\mathrm{el}^2)} \end{bmatrix},$$

$$B=\begin{bmatrix} 0 \\ 0 \\ \dfrac{(J+\mathrm{mm}\times \mathrm{el}^2)}{((M+\mathrm{mm})\times J+M\times \mathrm{mm}\times \mathrm{el}^2)} \\ \dfrac{\mathrm{mm}\times \mathrm{el}}{((M+\mathrm{mm})\times J+M\times \mathrm{mm}\times \mathrm{el}^2)} \end{bmatrix},\ Q_p=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},\ R_p=1.$$

Here

$$M=0.768;\ \mathrm{mm}=0.104;\ \mathrm{el}=0.1735;\ J=2.829e^{-3};$$
$$g=9.81;\ \mathrm{fc}=0.5;\mathrm{fp}=0.1035e^{-3}.$$

The perturbations in the system matrices of the discrete-time system are chosen as

$$\Delta A=A\times 10^{-i},\Delta B=B\times 10^{-i},$$
$$\Delta R_{ip}=R_{ip}\times 10^{-i},\Delta Q_{ip}=Q_{ip}\times 10^{-i},$$
$$\Delta L_1=L^*\times 10^{-i},\Delta L_2=L^*\times 10^{-i},$$
$$\Delta Q=Q^*\times 10^{-i},\Delta Y=Y^*\times 10^{-i}.$$

The perturbed solutions $Q^*+\Delta Q$ and $Y^*+\Delta Y$ of LMI (6) are computed using the method derived in [9] and applying the software [4]. Based on the proposed approach the relative perturbation bounds for the solutions Q^* and Y^* of the LMI system (4) are obtained by the linear bounds (11) and (15), respectively.

The results obtained for different size of perturbations are shown in the Table 2.

Table 2

i	$\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2}$	Bound(11)	$\frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2}$	Bound(15)
8	3.0160×10^{-8}	2.2967×10^{-7}	2.1338×10^{-8}	1.3296×10^{-7}
7	3.0160×10^{-7}	2.2967×10^{-6}	2.1338×10^{-7}	1.3296×10^{-6}
6	3.0160×10^{-6}	2.2967×10^{-5}	2.1338×10^{-6}	1.3296×10^{-5}
5	3.0160×10^{-5}	2.2967×10^{-4}	2.1338×10^{-5}	1.3296×10^{-4}
4	3.0160×10^{-4}	2.2967×10^{-3}	2.1338×10^{-4}	1.3296×10^{-3}

Based on the proposed solution approach we obtain the perturbation bounds (11) and (15), which are close to the real relative perturbation bounds $\frac{\|\Delta q\|_2}{\|\mathrm{vec}(Q^*)\|_2}$ and $\frac{\|\Delta y\|_2}{\|\mathrm{vec}(Y^*)\|_2}$, thus they are good in sense that they are tight.

5. Conclusion

In this paper the linear perturbation analysis of the continuous-time LMI based linear quadratic regulator problem has been investigated. Tight linear perturbation bounds have been obtained for the matrix inequalities determining the problem solution. Based on derived theoretical results we have presented numerical examples to explicitly reveal the applicability and performance of the proposed solution approach to analyze the sensitivity of the LMI based linear quadratic regulator problem.

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