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On the Sensitivity of the Matrix Equations $X \pm A^* X^{-1} A = Q$ [1]

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***Abstract**: We analyse and compare the effectiveness and the accuracy of the existing methods for estimating the sensitivity of the solution to the nonlinear matrix equations* $X \pm A^* X^{-1} A = Q$. *Object of the analysis are the perturbation bounds concerning equations* $X \pm A^* X^{-1} A = Q$, *as well as the methods related to the estimation of the sensitivity of the solution to the equations* $X^s \pm A^* X^{-t} A = Q$ *for the particular case, when* $s = 1$, $t = 1$. *We examine the behaviour and the reliability of the bounds, proposed in nine sources, through experiments with nine non-trivial numerical examples in both real and complex cases.*

***Keywords**: Perturbation bounds, non-linear matrix equation, sensitivity.*

1. Introduction and notations

Consider the non-linear matrix equations

(1) $$X+A^*X^{-1}A=Q,$$

and

(2) $$X-A^*X^{-1}A=Q,$$

where A and Q ($Q > 0$) are $n \times n$ complex or real matrices, and A^* stands for the complex conjugate transpose of A in the complex case and for the transpose of A in the real case. The relatively wide field of application of the equations determines the interest of the researchers in the existence of a positive definite solution and its sensitivity in terms of conditioning and perturbation bounds. In 2001 Shu-Fang Xu [13] using his earlier conclusion for well conditioning, obtains a perturbation bound for the maximal solution of the matrix equation (1) and a computable bound

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for approximate solutions. Subsequently, Ran and Reurings [11] give perturbation bounds for the more general matrix equation $X + A^* \mathcal{F}(X) A = Q$, where $\mathcal{F}(X)$ is a map from $\mathbb{P}(m)$ into $\mathbb{P}(m)$ or $-\mathbb{P}(m)$, $\mathbb{P}(m)$ is the set of all $m\times m$ positive semidefinite matrices; Ji-Guand Sun and Shu-Fang Xu [12] propose perturbation bounds and explicit expression of the condition number for (1); Konstantinov et al. [7], Hasanov and Ivanov [4], and Hasanov [2] evaluate perturbation bounds for equation $X \pm A^*X^{-1}A = Q$; Hasanov and Ivanov [3], Ivanov [6], Boneva et al. [1] derive perturbation bounds for equations $X \pm A^*X^{-n}A = Q$; Jing Li and Yuhai Zhang [9] obtain explicit expression of the condition number for the unique positive definite solution, as well as evaluate perturbation bound and backward error of an approximate solution of equation $X - A^*X^{-p}A = Q$; Xiaoyan Yin et al. [14] propose perturbation estimates for the matrix equation $X + A^*X^{-t}A = Q$. Usually the authors present their estimates as “new” [6, 4, 14, 2], “sharper” [12, 9], “improved” and they illustrate their arguments by numerical examples. As it is pointed out in [8] a perturbation bound should be rigorous in the sense that its domain of applicability should be defined and therefore known in advance, it should be sharp or exact in some sense and if the bound is too pessimistic for some cases, this should be clearly stated.

In [10] an experimental analysis of the effectiveness and the reliability of the existing perturbation bounds for the real and the complex generalized matrix equation $X^S + \sigma A^*X^{-t}A = Q$, $\sigma = \pm 1$, when s or t, or both are greater than 1, is given. The experimental analysis is based on four examples, applied for eight different cases of this equation: $\sigma = +1$, $s = 1$, $t = -2$; $s = 2$, $t = -2$; $s = 2$, $t = 3$; $s = 3$, $t = 2$ and $\sigma = -1$, $s = 1$, $t = -1/2$; $s = 2$, $t = -3$; $s = 1$, $t = 3/4$; $s = 1$, $t = 1/3$. The models are taken from the two preferred by the authors examples for effectiveness analysis of perturbation bounds for this equation. Among the bounds considered in the paper, the bound superior with respect to closeness to the estimated quantity and comprehensive application is pointed out. The observed bounds properties hold true for problems belonging to the class of the experimental models used.

As noted, historically equations (1) and (2) enjoyed considerable interest by the researchers and there are several perturbation bounds proposed in the literature now. In this paper, we intend to classify existing perturbation bounds for equations (1) and (2) and to indicate areas of features and applications of the different bounds. By means of numerical experiments, we present a comparison analysis of the perturbation bounds for the matrix equations (1) and (2) proposed in [2-4, 6, 7, 9, 12-14] in sense of applicability and effectiveness. For the experiments we choose the examples proposed by the authors themselves for projection the properties of their own bounds. As the examples, except for the proposed in [7], relate to the field of real numbers and matrices, for the purpose of completeness, in the experiments we also include a modification of the examples for the case of complex perturbations in the coefficient matrices of equations (1) and (2). The computations are carried out on a PC with 2.61 GHz Pentium Dual-Core and machine epsilon $\varepsilon = 2.2\times10^{-16}$ using MATLAB (MATLAB is a trade mark of MathWorks). Section 2 is devoted to the brief description of the perturbation bounds, object of our analysis.

For sake of convenience, the original notations are used. In Section 3, the numerical examples are described and the results are listed.

Throughout the paper we use the following notations: $\mathbb{R}$, and $\mathbb{C}$ – the sets of real and complex numbers, respectively; I – the identity matrix; I_n – the identity matrix of order n; $\mathrm{vec}(A) \in \mathbb{C}^{n^2}$ – the column-wise vector representation of the matrix $A \in \mathbb{C}^{n\times n}$, where $\mathbb{C}^n = \mathbb{C}^{n\times 1}$; $A \otimes B = [A(k,l)B]$ – the Kronecker product of the matrices $A = [A(k,l)]$ and B; $\mathrm{P}_{n^2} \in \mathbb{R}^{n^2\times n^2}$ – the so called vec-permutation matrix, such that $\mathrm{vec}(Y^{\mathrm{T}}) = \mathrm{P}_{n^2}\mathrm{vec}(Y)$ for each $Y \in \mathbb{C}^{n\times n}$; $\|\cdot\|$ – a unitary invariant norm; $\|\cdot\|_2$ – the Euclidean vector or the spectral matrix norm; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius norm; $\alpha = \|A\|_2$, $\zeta = \|X^{-1}\|_2$ and $E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$. The notation ":=" stands for "equal by definition".

2. Perturbation bounds

Consider the perturbed non-linear matrix equations

(3) $$\tilde{X} + \tilde{A}^*\tilde{X}^{-1}\tilde{A} = \tilde{Q},$$

and

(4) $$\tilde{X} - \tilde{A}^*\tilde{X}^{-1}\tilde{A} = \tilde{Q},$$

where $\tilde{A} = A + \delta A$ and $\tilde{Q} = Q + \delta Q$, $\tilde{Q} > 0$ are the matrix coefficients in (1) and (2) perturbed with small perturbations δA and δQ, respectively. The perturbations in the coefficients lead to a perturbation in the solution $\tilde{X} = X + \delta X$, if it exists.

We consider the perturbation bounds for equations (1) and (2) proposed by Hasanov [2], Hasanov and Ivanov [3] and [4], Ivanov [6], Konstantinov et al. [7], Jing Li and Yuhai Zhang [9], Sun and Xu [12], Shu-Fang Xu [13] and Yin, Liu and Fang [14].

2.1. The bound of Shu-Fang Xu [13]

Theorem 2.1.1 (Theorem 3.1 from [13]). Let A, $\tilde{A}$, Q, $\tilde{Q} \in C^{n\times n}$ with Q and $\tilde{Q}$ Hermitian positive definite. If

(5) $$\|A\|_2\|Q^{-1}\|_2 < \frac{1}{2},\ \|\tilde{A} - A\|_2 < \frac{1}{2}\left(\frac{1}{2} - \|A\|_2\|Q^{-1}\|_2\right)\|Q^{-1}\|_2^{-1},$$

$$\|\tilde{Q} - Q\|_2 \le \left(\frac{1}{2} - \|A\|_2\|Q^{-1}\|_2\right)\|Q^{-1}\|_2^{-1},$$

then the maximal solutions X and $\widetilde{X}$ of the matrix equations (1) and (3) exist and satisfy that

(6) $$\frac{\left\|\widetilde{X}-X\right\|_2}{\|X\|_2} \le \frac{1}{\frac{1}{2}-\|A\|_2\left\|Q^{-1}\right\|_2}\left(\frac{\left\|\widetilde{A}-A\right\|_2}{\|A\|_2}+\frac{\left\|\widetilde{Q}-Q\right\|_2}{\|Q\|_2}\right) := \mathrm{est}_{\mathrm{xu01a}}.$$

The bound $\mathrm{est}_{\mathrm{xu01a}}$ is an elegant perturbation bound which calculation does not require knowledge of the solution to the perturbed (3) or the unperturbed (1) equations.

Theorem 2.1.2 (Theorem 3.3 from [13]). Let $\widetilde{X}$ approximate the minimal solution X of the matrix equation (1). If $\|A\|_2\left\|Q^{-1}\right\|_2 < \frac{1}{2}$ and $\left\|\widetilde{X}^{-1}\right\|_2 \le 2\left\|\left(\widetilde{X}+A^*\widetilde{X}^{-1}A\right)^{-1}\right\|_2$, and the residual $R\left(\widetilde{X}\right) := \widetilde{X}+A^*\widetilde{X}^{-1}A-Q$ satisfies that $\left\|R\left(\widetilde{X}\right)\right\|_2 \le \left(\frac{1}{2}-\|A\|_2\left\|Q^{-1}\right\|_2\right)\left\|Q^{-1}\right\|_2^{-1}$, then

(7) $$\frac{\left\|\widetilde{X}-X\right\|_2}{\|X\|_2} \le \frac{1}{\frac{1}{2}-\|A\|_2\left\|Q^{-1}\right\|_2}\frac{\left\|R\left(\widetilde{X}\right)\right\|_2}{\|Q\|_2} := \mathrm{est}_{\mathrm{xu01b}}.$$

2.2. The bound of Ji-Guand Sun and Shu-Fang Xu [12]

Let X be the maximal solution of (1), $\mathbf{L}$ be the linear operator $\mathbf{L}Z=Z-B^*ZB$, $\mathbf{W}$ be the operator $\mathbf{W}Z=\mathbf{L}^{-1}(B^*Z+Z^*B)$, where $B=X^{-1}A$. Denote $\beta=\|B\|_2$, $w=\|\mathbf{W}\|$, $l=\left\|\mathbf{L}^{-1}\right\|^{-1}$, $\varepsilon=\frac{1}{l}\|\delta Q\|+w\|\delta A\|+\frac{\zeta}{l}\|\delta A\|^2$, $d=\frac{\zeta}{l}\left[(\alpha+\|\delta A\|)\zeta+\beta\right]\|\delta A\|$.

Theorem 2.2.1 (Theorem 2.1 from [12]). If $d<\min\left\{1,\frac{(1-\beta)(\alpha\zeta+\beta)}{l}\right\}$ and

$$\varepsilon<\min\left\{\frac{l(1-d)^2}{\zeta\left[l+2\beta^2+ld+2\sqrt{(ld+\beta^2)(l+\beta^2)}\right]},\frac{(1-d)\left[(1-\beta)(\alpha\zeta+\beta)-ld\right]}{\zeta\left[(1+\beta)(\alpha\zeta+\beta)+ld\right]}\right\},$$

then the perturbed matrix equation (3) has the maximal solution $\widetilde{X}$, and moreover,

(8) $$\left\|\widetilde{X}-X\right\| \le \frac{2l\varepsilon}{l(1+\zeta\varepsilon-d)+\sqrt{l^2(1+\zeta\varepsilon-d)^2-4\zeta l\varepsilon\left(l+\beta^2\right)}} := \mathrm{est}_{\mathrm{sunx03}}.$$

2.3. The bounds of Konstantinov et al. [7]

The bounds apply to both equations (1) and (2). Rewrite equations (1) and (2) in the equivalent form

(9) $$X-\sigma A^*X^{-1}A=Q,\ \sigma=\pm 1.$$

Denote $L = I_{n^2} + \sigma\left(X^{-1}A\right)^{\mathrm{T}} \otimes \left(A^* X^{-1}\right)$, $M_1 := -L^{-1} = M_{1r} + iM_{1i}$,

$M_1^0 = \begin{bmatrix} M_{1r} & -M_{1i} \\ M_{1i} & M_{1r} \end{bmatrix}$, $M_{21} := -\sigma L^{-1}\left(I_n \otimes \left(A^* X^{-1}\right)\right) = M_{21r} + iM_{21i}$,

$M_{22} := -\sigma L^{-1}\left(\left(\left(X^{-1}A\right)^{\mathrm{T}} \otimes I_n\right)\mathrm{P}_{n^2}\right) = M_{22r} + iM_{22i}$,

$M_2^0 = \begin{bmatrix} M_{21r} + M_{22r} & M_{22i} - M_{21i} \\ M_{21i} + M_{22i} & M_{21r} - M_{22r} \end{bmatrix}$, $M^0 = \left[m_{kl}^0\right]$, $m_{kl}^0 = \left\| M_k^{0\,\mathrm{T}} M_l^0 \right\|_2$

for k, l = 1, 2, $d = \begin{bmatrix} d_1 & d_2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \|\delta A\|_{\mathrm{F}} & \|\delta Q\|_{\mathrm{F}} \end{bmatrix}^{\mathrm{T}}$.

- Local bound

The local bound is

(10) $$\|\delta X\|_{\mathrm{F}} < \min\left\{ \left\| \left[M_1^0, M_2^0\right] \right\|_2 \|d\|_2, \sqrt{d^T M^0 d} \right\} + O\left(\|d\|^2\right) := \mathrm{est}_{\mathrm{konpap03a}},$$

- Non local bound

Let a_0:=$\mathrm{est}_{\mathrm{konpap03a}} + \left\|M_1^0\right\|_2 \zeta d_2^2$, $a_1 := a_{11} d_2 + a_{12} d_2^2$,

$a_2 := a_{20} + a_{21} d_2 + a_{22} d_2^2$,

$a_{11} := \zeta\left(\left\| L^{-1}\left(I_n \otimes \left(A^* X^{-1}\right)\right) \right\|_2 + \left\| L^{-1}\left(\left(X^{-1}A\right)^{\mathrm{T}} \otimes I_n \right) P_{n^2} \right\| \right)$, $a_{12} := \left\|M_0^1\right\|_2 \zeta^2$,

$a_{20} := \zeta^3 \left\| L^{-1}\left(A^{\mathrm{T}} \otimes A^*\right) \right\|_2$, $a_{21} := \zeta^3 \left\| L^{-1}\left(A^{\mathrm{T}} \otimes I_n\right) P_{n^2} + L^{-1}\left(I_n \otimes A^*\right) \right\|_2$,

$a_{22} := \left\|M_1^0\right\|_2 \zeta^3$.

If $d \in \Omega$, where

(11) $$\Omega := \left\{ d \in \mathbb{R}_+^2 : a_1 - \zeta a_0 + 2\sqrt{a_0\left(a_2 + \zeta\left(1 - a_1\right)\right)} \le 1 \right\},$$

then the non-local perturbation bound $\|\delta X\|_{\mathrm{F}} \le \mathrm{est}_{\mathrm{konpap03b}}$ is valid for equation (9), where $\mathrm{est}_{\mathrm{konpap03b}}$ is determined by

(12) $$\mathrm{est}_{\mathrm{konpap03b}} := \frac{2a_0}{1 - a_1 + \zeta a_0 + \sqrt{\left(1 - a_1 + \zeta a_0\right)^2 - 4a_0\left(a_2 + \zeta\left(1 - a_1\right)\right)}}.$$

2.4. The bound of H a s a n o v and I v a n o v [3]

Consider the unperturbed and the perturbed matrix equations

(13) $$X - \sigma A^* X^{-t} A = Q,$$

(14) $$\widetilde{X} - \sigma \widetilde{A}^* \widetilde{X}^{-t} \widetilde{A} = \widetilde{Q}.$$

Theorem 2.4.1 (Theorem 6 from [3]). For $\sigma = \pm 1$ let X and $\widetilde{X}$ be positive definite solutions of the matrix equations (13) and (14), respectively. If

$$\varepsilon = 1 - \sum_{k=1}^{t} \left\| \widetilde{X}^{-k} \widetilde{A} \right\|_2 \left\| X^{k-(t+1)} A \right\|_2 > 0, \text{ then}$$

(15) $$\|\delta X\|_2 \le \frac{1}{\varepsilon}\left[\|\delta Q\|_2 + \left(\left\|\widetilde{X}^{-t}\widetilde{A}\right\|_2 + \left\|X^{-t}A\right\|_2\right)\|\delta A\|_2\right] := \mathrm{est}_{\mathrm{hasi04a}}.$$

Theorem 2.4.2 (Theorem 7 from [3]). For $\sigma = \pm 1$ let X and $\widetilde{X}$ be positive definite solutions of the matrix equations (13) and (14), respectively. If

$$\eta = 1 - \sum_{k=1}^{t}\left\|\widetilde{X}^{-k}A\right\|_2\left\|X^{k-(t+1)}A\right\|_2 > 0,$$ then

(16) $$\|\delta X\|_2 \le \frac{1}{\eta}\left[\|\delta Q\|_2 + 2\left\|\widetilde{X}^{-t}\widetilde{A}\right\|_2\|\delta A\|_2 + \left\|\widetilde{X}^{-t}\right\|_2\|\delta A\|_2^2\right] := \mathrm{est}_{\mathrm{hasi04b}}.$$

Corollary 2.4.1 (Corollary 8 from [3]). For $\sigma = -1$ let X and $\widetilde{X}$ be positive definite solutions of matrix equations (13) and (14), respectively. If $\eta = 1 - \|A\|_2\sum_{k=1}^{t}\left\|\widetilde{Q}^{-1}\right\|_2^k\left\|X^{k-(t+1)}A\right\|_2 > 0$, then

(17) $$\|\delta X\|_2 \le \frac{1}{\eta}\left[\|\delta Q\|_2 + \left\|\widetilde{Q}^{-1}\right\|_2^t\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2\right] := \mathrm{est}_{\mathrm{hasi04c}}.$$

Corollary 2.4.2 (Corollary 9 from [3]). For $\sigma = -1$ let X and $\widetilde{X}$ be positive definite solutions of matrix equations (13) and (14), respectively. If $\eta = 1 - \|A\|_2^2\sum_{k=1}^{t}\left\|\widetilde{Q}^{-1}\right\|_2^k\left\|Q^{-1}\right\|_2^{t+1-k} > 0$, then

(18) $$\|\delta X\|_2 \le \frac{1}{\eta}\left[\|\delta Q\|_2 + \left\|\widetilde{Q}^{-1}\right\|_2^t\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2\right] := \mathrm{est}_{\mathrm{hasi04d}}.$$

Theorem 2.4.3 (Theorem 10 from [3]). Let $A, \widetilde{A}, Q, \widetilde{Q} \in \mathbb{C}^{n\times n}$ and $Q, \widetilde{Q}$ be positive definite matrices and $\sigma = -1$. If $\theta = \frac{\sqrt{t}}{t} - -\|A\|_2\sqrt{\left\|Q^{-1}\right\|_2^{t+1}} > 0$,

$$\left\|\widetilde{Q} - Q\right\|_2 \le \frac{1}{\left\|Q^{-1}\right\|_2}\left(1 - \sqrt[t+1]{(1-\theta)^2}\right), \qquad \left\|\widetilde{A} - A\right\|_2 < \frac{\left(t - \sqrt{t}\right)\sqrt{\left\|Q^{-1}\right\|_2^{t+1}}}{t\left\|Q^{-1}\right\|_2^{t+1}}\theta,$$ then

equations (13), (14) with $\sigma = -1$ have unique positive definite solutions X and $\widetilde{X}$, respectively. For these solutions it is satisfied that

(19) $$\|\delta X\|_2 \le \frac{1}{\eta}\left[\|\delta Q\|_2 + \left(2\|A\|_2 + \|\delta A\|_2\right)\left\|\widetilde{Q}^{-1}\right\|_2^t\|\delta A\|_2\right] := \mathrm{est}_{\mathrm{hasi04e}}.$$

Theorem 2.4.4 (Theorem 11 from [3]). Let X be a positive definite solution of the matrix equation (12) with $\sigma = -1$ and let $\widetilde{X}$ approximate X, so that $\widetilde{X} - \widetilde{A}^*\widetilde{X}^{-t}\widetilde{A} > 0$.

If $\nu = 1 - \|A\|_2\sum_{k=1}^{t}\left\|Q^{-1}\right\|_2^{t+1-k}\left\|\widetilde{X}^{-k}A\right\|_2 > 0$, then

(20) $$\|\delta X\|_2 \le \frac{1}{\nu}\left\|\widetilde{X} - A^*\widetilde{X}^{-t}A - Q\right\|_2 := \mathrm{est}_{\mathrm{hasi04f}}.$$

2.5. The bound of I v a n o v [6]

Denote the linear operator $LZ := Z - \sigma\sum_{k=0}^{t-1} B_k^* Z B_{t-k-1}$, $B_k = X^{-(k+1)}A$ and the operator $WZ := L^{-1}\left(\left(X^{-t}A\right)^* Z + Z^*\left(X^{-t}A\right)\right)$. Denote $w = \|\mathbf{W}\|$, $l = \left\|\mathbf{L}^{-1}\right\|^{-1}$, $\varepsilon = \frac{1}{l}\|\delta Q\| + w\|\delta A\| + \frac{\zeta^t}{l}\|\delta A\|^2$, $d = \frac{\zeta^{t+1}}{l}\|\delta A\|\left(2\alpha + \|\delta A\|\right)$,

$$\nu_* = \frac{2l\varepsilon}{l(1 + t\zeta\varepsilon - td) + \sqrt{l^2(1 + t\zeta\varepsilon - td)^2 - 4l\zeta\varepsilon\left(tl + t^2\alpha^2\zeta^{t+1}\right)}}.$$

Theorem 2.5.1 (Theorem 2.1 from [6]). Let X be a positive definite solution of (13) with $\sigma = -1$. Assume that

(i) the operator L is invertible,

(ii) $d < \frac{1}{t}$,

(iii) $\varepsilon \le \frac{1}{t\zeta}\frac{l(1-td)^2}{l + ltd + 2t\alpha^2\zeta^{t+1} + 2\sqrt{\left(l + t\alpha^2\zeta^{t+1}\right)\left(ltd + t\alpha^2\zeta^{t+1}\right)}}$.

Then the perturbed equation (14) with $\sigma = -1$ has a positive definite solution $\widetilde{X}$ and

(21) $$\|\delta X\| \le \nu_* := \mathrm{est}_{\mathrm{iva05a}}.$$

Theorem 2.5.2 (Theorem 2.3 from [6]). If $\theta = \sqrt{\frac{t^t}{(t+1)^{t+1}}} - \|A\|_2\sqrt{\left\|Q^{-1}\right\|_2^{t+1}} > 0$, then there exists a unique positive definite solution X of (13) with $\sigma = -1$ and the solution has the property $\left\|X^{-1}\right\|_2 < \frac{t+1}{t}\left\|Q^{-1}\right\|_2$. If

(i) the operator L is invertible,

(ii) $d < \min\left(\frac{1}{t}; \frac{\sqrt{(t+1)^{t+1}} - \sqrt{t^t}}{\sqrt{\left[(t+1)\left\|Q^{-1}\right\|_2\right]^{t+1}}}\frac{\zeta^{t+1}\alpha}{l}\theta\right)$ for $\zeta = \left\|X^{-1}\right\|_2$;

(iii) $\varepsilon \le \frac{1}{t\zeta}\frac{l(1-td)^2}{l + ltd + 2t\alpha^2\zeta^{t+1} + 2\sqrt{\left(l + t\alpha^2\zeta^{t+1}\right)\left(ltd + t\alpha^2\zeta^{t+1}\right)}}$;

(iv) $\frac{\zeta}{1-\zeta\nu_*} < \frac{t+1}{t}\left\|Q^{-1}\right\|_2$ for $\zeta = \left\|X^{-1}\right\|_2$;

(v) $\quad \|\delta Q\| < \min\left[\frac{1}{\|Q^{-1}\|_2}\left(1-\sqrt[t+1]{(1-\theta)^2}\right); \frac{t+1}{t}\frac{1-\zeta\nu_*}{\zeta} - \frac{1}{\|Q^{-1}\|_2}\right]$,

then there exists a unique positive definite solution $\widetilde{X} = X + \delta X$ of the perturbed equation (14) with $\sigma = -1$. For the solution it is fulfilled that $\left\|\widetilde{X}^{-1}\right\|_2 < \frac{t+1}{t}\left\|\widetilde{Q}^{-1}\right\|_2$; and

(22) $$\left\|X - \widetilde{X}\right\|_F \le \nu_* := \mathrm{est}_{\mathrm{iva05b}}.$$

Theorem 2.5.3 (Remark 2.2 from [6]). From Theorem 2.1 the authors get a first-order perturbation bound for the solution *X:*

(23) $$\left\|\widetilde{X} - X\right\| \le \varepsilon + O\left(\|\delta A, \delta Q\|^2\right) := \mathrm{est}_{\mathrm{iva05c}} \text{ as } \delta A \to 0,\ \delta Q \to 0.$$

Theorem 2.5.4 (Theorem 2.4 from [6]). Let X be a positive definite solution of (13) with $\sigma = 1$. Assume that

(i) the operator L is invertible,

(ii) $d < \frac{1}{t}$, and

(iii) $\varepsilon \le \frac{1}{t\zeta}\frac{l(1-td)^2}{l + ltd + 2t\alpha^2\zeta^{t+1} + 2\sqrt{\left(l + t\alpha^2\zeta^{t+1}\right)\left(ltd + t\alpha^2\zeta^{t+1}\right)}}$.

Then the perturbed equation (14) with $\sigma = 1$ has a positive definite solution $\widetilde{X}$ and

(24) $$\|\delta X\| \le \nu_* := \mathrm{est}_{\mathrm{iva05d}}.$$

Theorem 2.5.5 (Theorem 2.5 from [6]). If $\theta = \frac{\sqrt{t}}{t} - \|A\|_2\sqrt{\left\|Q^{-1}\right\|_2^{t+1}} > 0$, then there exists a unique positive definite solution X of (13) with $\sigma = 1$ and the solution has the property $X \ge Q$. If

(i) the operator L is invertible,

(ii) $d < \min\left(\frac{1}{t}; \frac{\left(t-\sqrt{t}\right)\sqrt{\left\|Q^{-1}\right\|_2^{t+1}}}{t\left\|Q^{-1}\right\|_2^{t+1}}\frac{\zeta^{t+1}\alpha}{l}\theta\right)$, $\zeta = \left\|X^{-1}\right\|_2$;

(iii) $\varepsilon \le \frac{1}{t\zeta}\frac{l(1-td)^2}{l + ltd + 2t\alpha^2\zeta^{t+1} + 2\sqrt{\left(l + t\alpha^2\zeta^{t+1}\right)\left(ltd + t\alpha^2\zeta^{t+1}\right)}}$;

(iv) $\|\delta Q\|_{\mathrm{F}} \le \frac{1}{\|Q^{-1}\|_2}\left(1-\sqrt[t+1]{(1-\theta)^2}\right)$, then there exists a unique solution $\widetilde{X} = X + \delta X$ of the perturbed equation (14) with $\sigma = 1$. For the solution it is fulfilled that $\widetilde{X} \geq \widetilde{Q}$; and

(25)
$$\left\| X - \widetilde{X} \right\| \leq \nu_* := \mathrm{est}_{\mathrm{iva05e}}.$$

2.6. The bound of H a s a n o v and I v a n o v [4]

Theorem 2.6.1 (Theorem 2.1 from [4]). Let A, $\widetilde{A}$, Q, $\widetilde{Q} \in \mathbb{C}^{n\times n}$ be coefficient matrices for matrix equations (1) and (3). Let

$$b = 1 - \left\| X^{-1}A \right\|_2^2 + \left\| X^{-1} \right\|_2 \left\| \delta Q \right\|,\ c = \left\| \delta Q \right\| + 2\left\| X^{-1}A \right\|_2 \left\| \delta A \right\| + \left\| X^{-1} \right\|_2 \left\| \delta A \right\|^2.$$

where X is the maximal positive definite solution of equation (1). If

(26)
$$\left\| X^{-1}A \right\|_2 < 1 \text{ and } 2\left\| \delta A \right\| + \left\| \delta Q \right\| \leq \frac{\left(1 - \left\| X^{-1}A \right\|_2\right)^2}{\left\| X^{-1} \right\|_2},$$

then $D = b^2 - 4c\left\| X^{-1} \right\|_2 \geq 0$, the perturbed matrix equation (3) has the maximal positive definite solution $\widetilde{X}$ and

(27)
$$\left\| \delta X \right\| \leq \frac{b - \sqrt{D}}{2\left\| X^{-1} \right\|_2} := \mathrm{est}_{\mathrm{hasi06a}}.$$

Theorem 2.6.2 (Theorem 3.1 from [4]). Let A, $\widetilde{A}$, Q, $\widetilde{Q} \in \mathbb{C}^{n\times n}$ be coefficient matrices for matrix equations (2) and (4). Let $b = 1 - \left\| X^{-1}A \right\|_2^2 + \left\| X^{-1} \right\|_2 \left\| \delta Q \right\|$, $c = \left\| \delta Q \right\| + 2\left\| X^{-1}A \right\|_2 \left\| \delta A \right\| + \left\| X^{-1} \right\|_2 \left\| \delta A \right\|^2$. where X is a unique positive definite solution of equation (2). If

(28)
$$\left\| X^{-1}A \right\|_2 < 1 \text{ and } 2\left\| \delta A \right\| + \left\| \delta Q \right\| \leq \frac{\left(1 - \left\| X^{-1}A \right\|_2\right)^2}{\left\| X^{-1} \right\|_2},$$

then $D = b^2 - 4c\left\| X^{-1} \right\|_2 \geq 0$, and the positive definite solutions X and $\widetilde{X}$ of the respective equations (2) and (3) satisfy

(29)
$$\left\| \delta X \right\| \leq \frac{b - \sqrt{D}}{2\left\| X^{-1} \right\|_2} := \mathrm{est}_{\mathrm{hasi06b}}.$$

2.7. The bound of J i n g L i and Y u h a i Z h a n g [9]

Consider the matrix equation (13) with $\sigma = 1$. Let $\overline{k} = \lambda_{\max}\left(A^*A\right)$, $\underline{k} = \lambda_{\min}\left(A^*A\right)$, $\overline{q} = \lambda_{\max}(Q)$ and $\underline{q} = \lambda_{\min}(Q)$, and the pair (α, β) be the solution of the system $\alpha = \underline{q} + \underline{k}\beta^{-t}$, $\beta = \overline{q} + \overline{k}\alpha^{-t}$.

Theorem 2.7 (Theorem 3.1 from [9]). Assume that A, $\widetilde{A}$, Q, $\widetilde{Q} \in \mathbb{C}^{n\times n}$ with Q and $\widetilde{Q}$ being positive definite. Let $\varepsilon = \left(\beta + t\underline{q} - t\beta\right)\alpha^{t+1} + \beta^2\alpha^{t-1}\|\delta Q\|_2$, $\zeta = \dfrac{\varepsilon^2}{4\beta^3\alpha^t} - \alpha^t\|\delta Q\|_2$. If $\|\delta A\|_2 \le \sqrt{\|A\|_2^2 + \zeta} - \|A\|_2$, then

(30) $$\frac{\|\widetilde{X} - X\|_2}{\|X\|_2} \le \xi\|\delta A\|_2 + \varpi\|\delta Q\|_2 := \mathrm{est}_{\mathrm{liz09}},$$

where $\xi = \dfrac{2\left(2\|A\|_2 + \|\delta A\|_2\right)\beta}{\varepsilon + \sqrt{\varepsilon^2 - 4\alpha^t\beta^3\left(\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2 + \alpha^t\|\delta Q\|_2\right)}}$ and

$$\varpi = \frac{2\alpha^t\beta}{\varepsilon + \sqrt{\varepsilon^2 - 4\alpha^t\beta^3\left(\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2 + \alpha^t\|\delta Q\|_2\right)}}.$$

2.8. The bound of Yin, Liu, Fang [14]

Consider the unperturbed and the perturbed matrix equations

(31) $$X^s + A^* X^{-t} A = Q$$

and

(32) $$\widetilde{X}^s + \widetilde{A}^* \widetilde{X}^{-t} \widetilde{A} = \widetilde{Q}.$$

Let $\overline{k} = \lambda_{\max}\left(A^*A\right)$, $\underline{k} = \lambda_{\min}\left(A^*A\right)$, $\overline{q} = \lambda_{\max}(Q)$ and $\underline{q} = \lambda_{\min}(Q)$, $x_* = \left[\lambda_{\min}(Q)\dfrac{t}{t+s}\right]^{\frac{t}{s}}$, $\widetilde{x}_* = \left[\lambda_{\min}(\widetilde{Q})\dfrac{t}{t+s}\right]^{\frac{t}{s}}$ and the pair (α, β) be the solution of the system $\alpha = \underline{q} + \underline{k}\beta^{-t}$, $\beta = \overline{q} + \overline{k}\alpha^{-t}$.

Theorem 2.8.1 (Theorem 3.1 from [14]). Let

(i) $$\theta = \sqrt{\frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}} - \|A\|_2\sqrt{\|Q^{-1}\|_2^{\frac{s+t}{s}}} > 0,$$

(ii) $$\|\delta Q\|_2 \le \frac{1}{\|Q^{-1}\|_2}\left[1 - (1-\theta)^{\frac{2s}{s+t}}\right],$$

(iii) $$\|\delta A\|_2 < \frac{\sqrt{\left(\dfrac{s+t}{s}\right)^{\frac{s+t}{s}}} - \sqrt{\left(\dfrac{t}{s}\right)^{\frac{t}{s}}}}{\sqrt{\left(\dfrac{s+t}{s}\right)^{\frac{s+t}{s}}\|Q^{-1}\|_2^{\frac{s+t}{s}}}}\theta.$$

Then equations (30), (31) have unique special Hermitian positive definite solutions X_L and $\widetilde{X}_L$, respectively. Moreover,

(33) $$\|\delta X\|_{\rm F} \le \frac{\|T_1^{-1}\|_2}{1-\|T_1^{-1}\|_2\|T_2\|_2\|\widetilde{A}\|_2^2}\left(\|\delta Q\|_{\rm F} + 2\|X_L^{-t}A\|_2\|\delta A\|_{\rm F} + \|X_L^{-t}\|_2\|\delta A\|_{\rm F}^2\right) :=$$

$$:= \mathrm{est}_{\mathrm{yinlf09a}}.$$

where

$$T_1 = \sum_{i=1}^{s} \widetilde{X}_L^{s-i} \otimes X_L^{i-1},\ T_2 = \sum_{i=1}^{t} X_L^{-t-1+i} \otimes \widetilde{X}_L^{-i}.$$

Theorem 2.8.2 (Remark 3.1 from [14]). Let

$$\xi = \max\left\{\frac{1}{s\widetilde{x}_*^{s-1} - t\|\widetilde{A}\|^2/\widetilde{x}_*^{t+1}}, \frac{1}{sx_*^{s-1} - t\|\widetilde{A}\|^2/x_*^{t+1}}\right\}.$$

Substituting $\frac{\|T_1^{-1}\|_2}{1-\|T_1^{-1}\|_2\|T_2\|_2\|\widetilde{A}\|_2^2} \le \begin{cases} \dfrac{1}{s\widetilde{x}_*^{s-1} - t\|\widetilde{A}\|_2^2/\widetilde{x}_*^{t+1}} & \text{if } x_* \ge \widetilde{x}_* \\ \dfrac{1}{sx_*^{s-1} - t\|\widetilde{A}\|_2^2/x_*^{t+1}} & \text{if } x_* < \widetilde{x}_* \end{cases}$ into (33), the authors obtain a weak estimate

(34) $$\|\delta X\|_{\rm F} \le \xi\left(\|\delta Q\|_{\rm F} + 2\|X_L^{-t}A\|_2\|\delta A\|_{\rm F} + \|X_L^{-t}\|_2\|\delta A\|_{\rm F}^2\right) := \mathrm{est}_{\mathrm{yinlf09b}}.$$

Moreover, since $X_L \in [\beta_1 I, \beta_2 I]$, then $\|X_L^{-t}\|_2 \le \beta_1^{-t}$, and $\|X_L^{-t}A\|_2 \le \beta_1^{-t}\|A\|_2$, the authors obtain another weak estimate

(35) $$\|\delta X\|_{\rm F} \le \xi\left(\|\delta Q\|_{\rm F} + \frac{2\|A\|_2\|\delta A\|_{\rm F} + \|\delta A\|_{\rm F}^2}{\beta_1^t}\right) := \mathrm{est}_{\mathrm{yinlf09c}},$$

where β_1 and β_2 are the maximal positive solutions of the equations $x^{s+t} - \lambda_{\min}(Q)x^t + \lambda_{\max}(A^*A) = 0$ and $x^{s+t} - \lambda_{\max}(Q)x^t + \lambda_{\min}(A^*A) = 0$, respectively.

Theorem 2.8.3 (Theorem 3.2 from [14]). Under the assumptions of Theorem 2.8.1, the unique special solutions X_L and $\widetilde{X}_L$ to the matrix equation (31) and (32) exist, and

(36) $$\|\delta X\|_{\rm F} \le \frac{\|L_1^{-1}\|_2}{1-\|L_1^{-1}\|_2\|A\|_2\|\widetilde{A}\|_2\|L_2\|_2}\left(\|\delta Q\|_{\rm F} + \left(\|\widetilde{A}^*\widetilde{X}_L^{-t}\|_2 + \|X_L^{-t}A\|_2\right)\|\delta A\|_{\rm F}\right) :=$$

$$:= \mathrm{est}_{\mathrm{yinlf09d}},$$

where $L_1 = \sum_{i=1}^{s} X_L^{s-i} \otimes \widetilde{X}_L^{i-1},\ L_2 = \sum_{i=1}^{t} X_L^{-t-1+i} \otimes \widetilde{X}_L^{-i}$.

Theorem 2.8.4 (Remark 3.2 from [14]). Similar to Theorem 2.8.2, the authors get a weak estimate:

(37) $$\|\delta X\|_F \le \eta\left(\|\delta Q\|_F + \left(\tilde{\beta}_1^{-t}\|\tilde{A}\|_2 + \beta_1^{-t}\|A\|_2\right)\|\delta A\|_F\right) := \mathrm{est}_{\mathrm{yinlf09e}},$$

where β_1 and $\tilde{\beta}_1$ are the maximal positive solutions of equation $x^{s+t} - \lambda_{\min}(Q)x^t + \lambda_{\max}(A^*A) = 0$ and $x^{s+t} - \lambda_{\max}(\tilde{Q})x^t + \lambda_{\min}(\tilde{A}^*\tilde{A}) = 0$, respectively and

$$\eta = \max\left\{\frac{1}{s\tilde{x}_*^{\,s-1} - t\|A\|_2\|\tilde{A}\|_2 / \tilde{x}_*^{\,t+1}}, \frac{1}{s\tilde{x}_*^{\,s-1} - t\|A\|_2\|\tilde{A}\|_2 / x_*^{t+1}}\right\}.$$

2.9. The bound of H a s a n o v [2]

Theorem 2.9.1 (Theorem 2.3 from [2]). Let A, $\tilde{A}$, Q, $\tilde{Q} \in \mathbb{C}^{n\times n}$ ($Q > 0$, $\tilde{Q} > 0$) be coefficient matrices for matrix equations (1) and (3) , P is a positive definite matrix. We assume that equation (1) has a positive definite solution and X_L is its maximal solution. Let $\alpha = \|PX_L^{-1}AP^{-1}\|_2$, $\beta = \|PX_L^{-1}P\|_2$, $b = 1-\alpha^2 + \beta\|P^{-1}\delta QP^{-1}\|$, $c = \|P^{-1}\delta QP^{-1}\| + 2\alpha\|P^{-1}\delta AP^{-1}\| + \beta\|P^{-1}\delta AP^{-1}\|^2$. If $\alpha < 1$ and $2\|P^{-1}\delta AP^{-1}\| + \|P^{-1}\delta QP^{-1}\| \le \dfrac{(1-\alpha)^2}{\beta}$, then $D = b^2 - 4c\beta \ge 0$, the perturbed matrix equation (3) has the maximal positive solution $\tilde{X}_L$ and

(38) $$\|\delta X\| \le \|P\|_2^2 \frac{b-\sqrt{D}}{2\beta} := \mathrm{est}_{\mathrm{has10a}}.$$

Theorem 2.9.2 (Theorem 2.4 from [2]). Let A, $\tilde{A}$, Q, $\tilde{Q} \in \mathbb{C}^{n\times n}$ ($Q > 0$, $\tilde{Q} > 0$) be coefficient matrices for the matrix equations (2) and (4), and P be a positive definite matrix. Let $\alpha = \|PX^{-1}AP^{-1}\|_2$, $\beta = \|PX^{-1}P\|_2$, $b = 1-\alpha^2 + \beta\|P^{-1}\delta QP^{-1}\|$,

$c = \|P^{-1}\delta QP^{-1}\| + 2\alpha\|P^{-1}\delta AP^{-1}\| + \beta\|P^{-1}\delta AP^{-1}\|^2$. If $\alpha < 1$ and

$2\|P^{-1}\delta AP^{-1}\| + \|P^{-1}\delta QP^{-1}\| \le \dfrac{(1-\alpha)^2}{\beta}$, then $D = b^2 - 4c\beta \ge 0$ and

(39) $$\|\delta X\| \le \|P\|_2^2 \frac{b-\sqrt{D}}{2\beta} := \mathrm{est}_{\mathrm{has10b}}.$$

These bounds $\mathrm{est}_{\mathrm{has10a}}$ and $\mathrm{est}_{\mathrm{has10b}}$ are very simple and easy for calculation, but their existence is associated with the choice of an appropriate positive definite matrix P. The authors propose to choose $P = Q^{1/2}$. The lack of a general rule for the choice of the matrix P is a restriction for the application of the bound.

3. Experimental results

Denote the ratio of the perturbation bounds to the estimated value as follows:

$$\text{xu01a} := \frac{\text{est}_{\text{xu01a}} \|X\|_2}{\|\widetilde{X} - X\|_2}, \ \text{xu01b} := \frac{\text{est}_{\text{xu01b}} \|X\|_2}{\|\widetilde{X} - X\|_2}, \ \text{sunx03} = \frac{\text{est}_{\text{sunx03}}}{\|\widetilde{X} - X\|_{\text{F}}},$$

$$\text{konpap03a} := \frac{\text{est}_{\text{konpap03a}}}{\|\delta X\|_{\text{F}}}, \ \text{konpap03b} := \frac{\text{est}_{\text{konpap03b}}}{\|\delta X\|_{\text{F}}}, \ \text{hasi04a} := \frac{\text{est}_{\text{hasi04a}}}{\|\delta X\|_2},$$

$$\text{hasi04b} := \frac{\text{est}_{\text{hasi04b}}}{\|\delta X\|_2}, \ \text{hasi04c} := \frac{\text{est}_{\text{hasi04c}}}{\|\delta X\|_2}, \ \text{hasi04d} := \frac{\text{est}_{\text{hasi04d}}}{\|\delta X\|_2}, \ \text{hasi04e} := \frac{\text{est}_{\text{hasi04e}}}{\|\delta X\|_2},$$

$$\text{hasi04f} = \frac{\text{est}_{\text{hasi04f}}}{\|\delta X\|_2}, \ \text{iva05a} := \frac{\text{est}_{\text{iva05a}}}{\|\delta X\|_{\text{F}}}, \ \text{iva05b} := \frac{\text{est}_{\text{iva05b}}}{\|X - \widetilde{X}\|_{\text{F}}}, \ \text{iva05c} := \frac{\text{est}_{\text{iva05c}}}{\|\widetilde{X} - X\|_{\text{F}}},$$

$$\text{iva05d} := \frac{\text{est}_{\text{iva05d}}}{\|\delta X\|_{\text{F}}}, \ \text{iva05e} := \frac{\text{est}_{\text{iva05e}}}{\|X - \widetilde{X}\|_{\text{F}}}, \ \text{hasi06a} := \frac{\text{est}_{\text{hasi06a}}}{\|\delta X\|_{\text{F}}}, \ \text{hasi06b} := \frac{\text{est}_{\text{hasi06b}}}{\|\delta X\|_{\text{F}}},$$

$$\text{liz09} := \frac{\text{est}_{\text{liz09}} \|X\|_2}{\|\widetilde{X} - X\|_2}, \ \text{yinlf09a} := \frac{\text{est}_{\text{yinlf09a}}}{\|\delta X\|_{\text{F}}}, \ \text{yinlf09b} := \frac{\text{est}_{\text{yinlf09b}}}{\|\delta X\|_{\text{F}}},$$

$$\text{yinlf09c} := \frac{\text{est}_{\text{yinlf09c}}}{\|\delta X\|_{\text{F}}}, \ \text{yinlf09d} := \frac{\text{est}_{\text{yinlf09d}}}{\|\delta X\|_{\text{F}}}, \ \text{yinlf09e} := \frac{\text{est}_{\text{yinlf09e}}}{\|\delta X\|_{\text{F}}},$$

$$\text{has10a} := \frac{\text{est}_{\text{has10a}}}{\|\delta X\|_{\text{F}}} \ \text{and} \ \text{has10b} := \frac{\text{est}_{\text{has10b}}}{\|\delta X\|_{\text{F}}}.$$

Consider the unperturbed and the perturbed matrix equations (1) and (3), respectively.

3.1. Example 1

The model from Example 5.1 in [12] is used. Consider the nonlinear matrix equation (1) with coefficient matrices $A = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where $0 < a < 1$, and a maximal solution $X_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 - a^2 \end{bmatrix}$. Take $a = 0.5 - 10^{-k}$, and suppose that the perturbations in the coefficient matrices are $\delta A = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-9}$, $\delta Q = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-8}$. The relative condition number, obtained by Sun and Xu in [12] is 1.3198. This means that the solution X_L of the equation is well-conditioned. The perturbed equation (3) is a discrete-time algebraic matrix Riccati equation [5]. But the perturbed matrix coefficient $\widetilde{Q} = Q + \delta Q$ is not symmetric and the MATLAB's discrete-time algebraic matrix Riccati equation solver **dare** can not be used for obtaining the solution $\widetilde{X}$ to the perturbed equation (3). Hence the perturbation δX in the solution and the perturbed solution $\widetilde{X}$ are unknown. We can only consider the bounds, whose computation does not need this information. The perturbation bound in the solution is estimated by the bounds est_{xu01a} (6), est_{sunx03} (8), $\mathrm{est}_{konpap03a}$ (10), $\mathrm{est}_{konpap03b}$ (12), est_{iva05a} (21), est_{iva05b} (22), est_{iva05c} (23), $\mathrm{est}_{hasi06a}$ (27), $\mathrm{est}_{yinlf09b}$ (34), $\mathrm{est}_{yinlf09c}$ (35), $\mathrm{est}_{yinlf09e}$ (37) and est_{has10a} (38). The numerical results on the absolute perturbation bounds in terms of Frobenius norm, except for the bound est_{sunx03}, which concerns the spectral norm of the perturbation in the solution, are listed in Table 3.1.1. The cases when the conditions of existence of a bound are violated are denoted by an asterisk. The cases when the matrix $P = Q^{1/2}$ used by est_{has10a} (38) is not appropriate are denoted by hyphen.

Table 3.1.1. Example 1, $a = 0.5 - 10^{-k}$, real case

k	9	7	5	3	1
est_{xu01a} (6)	*	1.3779×10^{-1}	1.3779×10^{-3}	1.3784×10^{-5}	1.4389×10^{-7}
est_{sunx03} (8)	2.5580×10^{-8}	2.5580×10^{-8}	2.5580×10^{-8}	2.5578×10^{-8}	2.5463×10^{-8}
$\mathrm{est}_{konpap03a}$ (10)	1.4815×10^{-8}	1.4815×10^{-8}	1.4814×10^{-8}	1.4806×10^{-8}	1.4009×10^{-8}
$\mathrm{est}_{konpap03b}$ (12)	1.4815×10^{-8}	1.4815×10^{-8}	1.4814×10^{-8}	1.4806×10^{-8}	1.4009×10^{-8}
est_{iva05a} (21)	2.5580×10^{-8}	2.5580×10^{-8}	2.5580×10^{-8}	2.5578×10^{-8}	2.5463×10^{-8}
est_{iva05b} (22)	*	2.5580×10^{-8}	2.5580×10^{-8}	2.5578×10^{-8}	2.5463×10^{-8}
est_{iva05c} (23)	2.5580×10^{-8}	2.5580×10^{-8}	2.5580×10^{-8}	2.5578×10^{-8}	2.5463×10^{-8}
$\mathrm{est}_{hasi06a}$ (27)	1.7754×10^{-8}	1.7754×10^{-8}	1.7754×10^{-8}	1.7754×10^{-8}	1.7754×10^{-8}
$\mathrm{est}_{yinlf09b}$ (34)	*	*	*	*	3.6988×10^{-8}
$\mathrm{est}_{yinlf09c}$ (35)	*	*	*	*	3.6294×10^{-8}
$\mathrm{est}_{yinlf09e}$ (37)	*	*	*	*	3.6988×10^{-8}
est_{has10a} (38)	5.2978×10^{-9}	5.2978×10^{-9}	5.2980×10^{-9}	5.3179×10^{-9}	7.2982×10^{-9}

The results listed in Table 3.1.1 show that bounds est_{has10a} (38) of Hasanov [2], $\mathrm{est}_{konpap03a}$ (10) and $\mathrm{est}_{konpap03b}$ (12) of Konstantinov et al. [7] give the lowest values.

The bound est_{xu01a} (6) is an a priori estimate, since for its calculation it is not necessary to know the solutions X and $\widetilde{X}$ of the unperturbed and the perturbed equation, respectively. But, as it is seen, it exhibits quite strange behaviour, although the conditions (5) of existence are fulfilled.

Let us now modify the model and take the perturbations in the matrix coefficients as $\delta A = (1+i)\begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-9}$ and $\delta Q = (1+i)\begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-8}$, where i is the imaginary unit. The results are shown in Table 3.1.2.

Table 3.1.2. Example 1, $a = 0.5 - 10^{-k}$, complex case

k	9	7	5	3	1
est_{xu01a} (6)	*	3.4539×10^{-2}	3.4540×10^{-4}	3.4608×10^{-6}	4.3174×10^{-8}
est_{sunx03} (8)	1.7804×10^{-9}	1.7804×10^{-9}	1.7804×10^{-9}	1.7803×10^{-9}	1.7723×10^{-9}
$\mathrm{est}_{konpap03a}$ (10)	1.7666×10^{-9}	1.7666×10^{-9}	1.7666×10^{-9}	1.7631×10^{-9}	1.4133×10^{-9}
$\mathrm{est}_{konpap03b}$ (12)	1.7666×10^{-9}	1.7666×10^{-9}	1.7666×10^{-9}	1.7631×10^{-9}	1.4133×10^{-9}
est_{iva05a} (21)	1.7804×10^{-9}	1.7804×10^{-9}	1.7804×10^{-9}	1.7803×10^{-9}	1.7723×10^{-9}
est_{iva05b} (22)	*	1.7804×10^{-9}	1.7804×10^{-9}	1.7803×10^{-9}	1.7723×10^{-9}
est_{iva05c} (23)	1.7804×10^{-9}	1.7804×10^{-9}	1.7804×10^{-9}	1.7803×10^{-9}	1.7723×10^{-9}
$\mathrm{est}_{hasi06a}$ (27)	2.3555×10^{-9}	2.3555×10^{-9}	2.3555×10^{-9}	2.3555×10^{-9}	2.3555×10^{-9}
$\mathrm{est}_{yinlf09b}$ (34)	*	4.4436×10^{-3}	4.4169×10^{-5}	4.4210×10^{-7}	4.9073×10^{-9}
$\mathrm{est}_{yinlf09c}$ (35)	*	4.4436×10^{-3}	4.4168×10^{-5}	4.4122×10^{-7}	3.9259×10^{-9}
$\mathrm{est}_{yinlf09e}$ (37)	*	8.8545×10^{-3}	8.7778×10^{-5}	8.2997×10^{-7}	4.9073×10^{-9}
est_{has10a} (38)	7.2856×10^{-10}	7.2856×10^{-10}	7.2857×10^{-10}	7.3001×10^{-10}	8.1886×10^{-10}

Here for the different values of the parameter k the bounds est_{xu01a}, est_{iva05a}, $\mathrm{est}_{yinlf09b}$, $\mathrm{est}_{yinlf09c}$ and $\mathrm{est}_{yinlf09e}$ are not constant in their sharpness. The bound est_{has10a},, followed by the bounds $\mathrm{est}_{konpap03a}$ and $\mathrm{est}_{konpap03b}$, is the most effective one.

Take now a = 0.99 and suppose that the perturbations in the coefficient matrices are $\delta A = \begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-k}$, $\delta Q = \begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-k}$. The relative condition number in this case is 2.892, therefore, the solution X_L remains well-conditioned. The results are listed in Table 3.1.3.

Table 3.1.3. Example 1, a = 0.99, real case

k	10	9	8	7	6
est_{xu01a} (6)	*	*	*	*	*
est_{sunx03} (8)	5.4420×10^{-10}	5.4420×10^{-9}	5.4424×10^{-8}	5.4462×10^{-7}	5.4844×10^{-6}
$\mathrm{est}_{konpap03a}$ (10)	4.2685×10^{-10}	4.2685×10^{-9}	4.2685×10^{-8}	4.2685×10^{-7}	4.2685×10^{-6}
$\mathrm{est}_{konpap03b}$ (12)	4.2688×10^{-10}	4.2708×10^{-9}	4.2914×10^{-8}	4.5230×10^{-7}	*
est_{iva05a} (21)	5.4428×10^{-10}	5.4502×10^{-9}	5.5270×10^{-8}	6.6861×10^{-7}	*
est_{iva05b} (22)	*	*	*	*	*
est_{iva05c} (23)	5.4420×10^{-10}	5.4420×10^{-9}	5.4420×10^{-8}	5.4420×10^{-7}	5.4421×10^{-6}
$\mathrm{est}_{hasi06a}$ (27)	1.8494×10^{-8}	1.8501×10^{-7}	1.8579×10^{-6}	1.9441×10^{-5}	*
$\mathrm{est}_{yinlf09b}$ (34)	*	*	*	*	*
$\mathrm{est}_{yinlf09c}$ (35)	*	*	*	*	*
$\mathrm{est}_{yinlf09e}$ (37)	*	*	*	*	*
est_{has10a} (38)	–	–	–	–	–

Let the perturbations in the matrix coefficients be chosen as $\delta A = (1+i)\begin{bmatrix} 0.9501 & 0.6068 \\ 0.2311 & 0.4860 \end{bmatrix} \times 10^{-k}$, $\delta Q = (1+i)\begin{bmatrix} -0.4326 & -0.7701 \\ 0.7701 & 0.2877 \end{bmatrix} \times 10^{-k}$. The results are shown in Table 3.1.4.

Table 3.1.4. Example 1, a = 0.99, complex case

k	10	9	8	7	6
est_{xu01a} (6)	*	*	*	*	*
est_{sunx03} (8)	3.8954×10^{-10}	3.8954×10^{-9}	3.8958×10^{-8}	3.8995×10^{-7}	3.9369×10^{-6}
$\mathrm{est}_{konpap03a}$ (10)	3.4979×10^{-10}	3.4979×10^{-9}	3.4979×10^{-8}	3.4979×10^{-7}	3.4979×10^{-6}
$\mathrm{est}_{konpap03b}$ (12)	3.4981×10^{-10}	3.4995×10^{-9}	3.5133×10^{-8}	3.6651×10^{-7}	*
est_{iva05a} (21)	3.8958×10^{-10}	3.8996×10^{-9}	3.9387×10^{-8}	4.4486×10^{-7}	*
est_{iva05b} (22)	*	*	*	*	*
est_{iva05c} (23)	3.8954×10^{-10}	3.8954×10^{-9}	3.8954×10^{-8}	3.8954×10^{-7}	3.8957×10^{-6}
$\mathrm{est}_{hasi06a}$ (27)	1.7578×10^{-8}	1.7585×10^{-7}	1.7656×10^{-6}	1.8436×10^{-5}	*
$\mathrm{est}_{yinlf09b}$ (34)	*	*	*	*	*
$\mathrm{est}_{yinlf09c}$ (35)	*	*	*	*	*
$\mathrm{est}_{yinlf09e}$ (37)	*	*	*	*	*
est_{has10a} (38)	–	–	–	–	–

As it is seen from the Tables 3.1.3 and 3.1.4, for this value of the parameter a the bound $\mathrm{est}_{konpap03a}$, followed by the bounds est_{sunx03} and est_{iva05c}, is the most effective one in terms of sharpness and relevance. It should be noted that the bounds $\mathrm{est}_{konpap03a}$ and est_{iva05c} are local bounds and they are applicable to sufficiently small perturbations in the data. In general, no one can say how small the perturbations must be in order to be considered small enough. The local bound may give results even when the perturbed equation has no solution.

3.2. Example 2 (Example 4.2 from [13])

Consider the matrix equation (1) with coefficient matrices $A = \frac{1}{10}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$, $Q = X + A^* X^{-1} A$ and X = diag(1, 2, 3, 2, 1). Let the perturbations in the coefficient matrices A and Q be $\delta A = 10^{-2j} A_0$ and $\delta Q = 10^{-2j} Q_0$, respectively, with $A_0 = \frac{1}{\|C\|} C$, $Q_0 = \frac{1}{\|C^{\mathrm{T}} + C\|}\left(C^{\mathrm{T}} + C\right)$. C is a random matrix, which is generated by MATLAB's function rand. The solution $\widetilde{X}$ of the perturbed matrix equation (3) is computed applying MATLAB's function dare (i.e. $\widetilde{X}$ = dare(0, I, $\widetilde{A}$, 0, $\widetilde{A}^{\mathrm{T}}$)). The matrix I is the identity matrix of appropriate dimension. In Table 3.2.1 are listed the ratio of the perturbation bound to the estimated value for j = 1, …, 5 averaged as the arithmetic mean of 200 randomly perturbed runs. The example compares the properties of the bounds $\mathrm{est}_{\mathrm{xu01a}}$ (6), $\mathrm{est}_{\mathrm{xu01b}}$ (7), $\mathrm{est}_{\mathrm{sunx03}}$ (8), $\mathrm{est}_{\mathrm{konpap03a}}$ (10), $\mathrm{est}_{\mathrm{konpap03b}}$ (12), $\mathrm{est}_{\mathrm{hasi04a}}$ (15), $\mathrm{est}_{\mathrm{hasi04b}}$ (16), $\mathrm{est}_{\mathrm{iva05a}}$ (21), $\mathrm{est}_{\mathrm{iva05b}}$ (22), $\mathrm{est}_{\mathrm{iva05c}}$ (23), $\mathrm{est}_{\mathrm{hasi06a}}$ (27), $\mathrm{est}_{\mathrm{yinlf09a}}$ (33), $\mathrm{est}_{\mathrm{yinlf09b}}$ (34), $\mathrm{est}_{\mathrm{yinlf09c}}$ (35), $\mathrm{est}_{\mathrm{yinlf09d}}$ (36), $\mathrm{est}_{\mathrm{yinlf09e}}$ (37) and $\mathrm{est}_{\mathrm{has10a}}$ (38). All conditions of existence of the bounds are satisfied. The bound $\mathrm{est}_{\mathrm{xu01a}}$ [13] is an order of magnitude ruder than others. Depending on the results, the following groups are formed – the bounds $\mathrm{est}_{\mathrm{sunx03}}$, $\mathrm{est}_{\mathrm{iva05a}}$, $\mathrm{est}_{\mathrm{iva05b}}$, which exceed the estimated value by 5 to 6 times, followed by $\mathrm{est}_{\mathrm{iva05c}}$, $\mathrm{est}_{\mathrm{xu01b}}$, then the group of $\mathrm{est}_{\mathrm{yinlf09c}}$, $\mathrm{est}_{\mathrm{yinlf09d}}$ and the closest to the estimated value is the group of the bounds $\mathrm{est}_{\mathrm{konpap03a}}$, $\mathrm{est}_{\mathrm{konpap03b}}$, $\mathrm{est}_{\mathrm{hasi04a}}$, $\mathrm{est}_{\mathrm{hasi04b}}$, $\mathrm{est}_{\mathrm{hasi06a}}$ and $\mathrm{est}_{\mathrm{yinlf09b}}$. The bound $\mathrm{est}_{\mathrm{has10a}}$ of H a s a n o v [2] is the sharpest.

Table 3.2.1. Example 2, real case

k	1	2	3	4	5
xu01a	51.8880	51.5730	51.5700	51.7500	51.7500
xu01b	5.1852	5.1851	5.1851	5.1851	5.1851
sunx03	6.1863	5.7670	5.7634	5.7633	5.7633
konpap03a	1.4651	1.4569	1.4569	1.4569	1.4569
konpap03b	1.4871	1.4572	1.4569	1.4569	1.4569
hasi04a	1.5194	1.5154	1.5154	1.5154	1.5154
hasi04b	1.5342	1.5156	1.5154	1.5154	1.5154
iva05a	6.3165	5.7680	5.7634	5.7633	5.7633
iva05b	6.4225	5.7689	5.7634	5.7633	5.7633
iva05c	5.8506	5.7642	5.7633	5.7633	5.7633
hasi06a	1.5461	1.5157	1.5154	1.5154	1.5154
yinlf09b	1.6042	1.5861	1.5759	1.5859	1.5859
yinlf09c	2.3139	2.3061	2.3060	2.3060	2.3060
yinlf0d	2.5558	2.5474	2.5473	2.5473	2.5473
yinlf09e	1.5900	1.5860	1.5859	1.5859	1.5859
has10a	0.9728	0.9514	0.9512	0.9512	0.9512

Let us now modify the example choosing the random matrix C as $(1+i)C$ and let us keep all other data unchanged. The results are listed in Table 3.2.2.

Table 3.2.2. Example 2, complex case

k	1	2	3	4	5
xu01a	52.5480	52.2190	52.2150	52.215	52.2150
xu01b	5.1765	5.1763	5.1769	5.1763	5.1763
sunx03	6.2496	5.8247	5.8210	5.8210	5.8210
konpap03a	1.4829	1.4649	1.4648	1.4648	1.4648
konpap03b	1.4948	1.4651	1.4648	1.4648	1.4648
hasi04a	1.5416	1.5233	1.5233	1.5233	1.5233
hasi04b	1.5342	1.5234	1.5233	1.5233	1.5233
iva05a	6.3843	5.8258	5.8211	5.8210	5.8210
iva05b	6.4971	5.8267	5.8211	5.8210	5.8210
iva05c	5.9071	5.8219	5.8210	5.8210	5.8210
hasi06a	1.5573	1.5236	1.5233	1.5233	1.5233
yinlf09b	1.6131	1.5943	1.5942	1.5941	1.5941
yinlf09c	2.3335	2.3181	2.3180	2.3180	2.3180
yinlf0de	2.5765	2.5560	2.5558	2.5558	2.5558
yinlf09e	1.5995	1.5942	1.5942	1.5941	1.5941
has10a	0.9745	0.9530	0.9528	0.9528	0.9528

In this example, the behaviour of the bounds in the complex case is the same as in the real case.

3.3. Example 3

Consider equation (1) with values of the matrix coefficient A and the solution X, given in Example 1 from [6]. Let $A = \frac{2\sqrt{3}}{45}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$, $Q = X + A^{\mathrm{T}}X^{-1}A$. Consider the two values of the unperturbed solution X=diag(0.725, 2, 3, 2, 1) and X=diag(0.25, 2, 3, 2, 1). Let the perturbed equation (3) have matrix coefficients $\widetilde{A} = A + \varepsilon(I+E)$, $\widetilde{X} = X + \varepsilon(I-E)$, $\widetilde{Q} = \widetilde{X} + \widetilde{A}^{\mathrm{T}}\widetilde{X}^{-1}\widetilde{A}$, $\varepsilon = 0.1^{2j}$. We study the perturbation bounds est_{xu01a} (6), est_{xu01b} (7), est_{sunx03} (8), $\mathrm{est}_{konpap03a}$ (10), $\mathrm{est}_{konpap03b}$ (12), $\mathrm{est}_{hasi04a}$ (15), $\mathrm{est}_{hasi04b}$ (16), est_{iva05a} (21), est_{iva05b} (22), est_{iva05c} (23), $\mathrm{est}_{hasi06a}$ (27), $\mathrm{est}_{yinlf09a}$ (33), $\mathrm{est}_{yinlf09b}$ (34), $\mathrm{est}_{yinlf09c}$ (35), $\mathrm{est}_{yinlf09d}$ (36), $\mathrm{est}_{yinlf09e}$ (37) and est_{has10a} (38). The ratio of the perturbation bounds and the estimated value for j=2, 3, 4, 5 are listed in Table 3.3.1.

After modifying the example by choosing $\varepsilon = 0.1^{2j}(1+i)$, the results obtained for the complex case are listed in Table 3.3.2.

Among the bounds considered in this example the bound est_{has10a}, of Hasanov [2] followed by $\mathrm{est}_{konpap03a}$ and $\mathrm{est}_{konpap03b}$ of Konstantinov et al. [7], gives the sharpest estimates. When the solution of equation (1) is chosen as X=diag(0.25, 2, 3, 2, 1), the bounds est_{xu01a}, est_{xu01b}, est_{iva05b}, $\mathrm{est}_{yinlf09b}$ – $\mathrm{est}_{yinlf09e}$ can not be used due to violated conditions of existence.

Table 3.3.1. Example 3, real case

X=diag(0.725, 2, 3, 2, 1)				
k	2	3	4	5
xu01a	111.6322	111.6357	111.6358	111.6358
xu01b	5.6960	5.6960	5.6960	5.6960
sunx03	7.3408	7.3149	7.3146	7.3146
konpap03a	1.6147	1.6152	1.6152	1.6152
konpap03b	1.6167	1.6152	1.6152	1.6152
hasi04a	1.6499	1.6505	1.6505	1.6505
hasi04b	1.6513	1.6505	1.6505	1.6505
iva05a	6.5843	6.5428	6.5424	6.5424
iva05b	6.5843	6.5428	6.5424	6.5424
iva05c	6.5463	6.5424	6.5424	6.5424
hasi06a	1.7956	1.7949	1.7949	1.7949
yinlf09b	2.7529	2.7515	2.7515	2.7515
yinlf09c	3.0357	3.0363	3.0363	3.0363
yinlf0de	1.7942	1.7949	1.7949	1.7949
yinlf09e	3.6486	3.6512	3.6512	3.6512
has10a	1.2104	1.2096	1.2096	1.2096
X=diag(0.25, 2, 3, 2, 1)				
k	2	3	4	5
xu01a	*	*	*	*
xu01b	*	*	*	*
sunx03	8.4182	8.1404	8.1378	8.1378
konpap03a	2.4882	2.4889	2.4889	2.4889
konpap03b	2.5093	2.4891	2.4889	2.4899
hasi04a	2.7378	2.7378	2.7358	2.7358
hasi04b	2.7395	2.7358	2.7358	2.7358
iva05a	8.0022	7.2847	7.2787	7.2786
iva05b	*	*	*	*
iva05c	7.2937	7.2788	7.2786	7.2786
hasi06a	2.7502	2.7359	2.7358	2.7358
yinlf09b	*	*	*	*
yinlf09c	*	*	*	*
yinlf0de	*	*	*	*
yinlf09e	*	*	*	*
has10a	1.9602	1.9558	1.9558	1.9558

Table 3.3.2. Example 3, complex case

X = diag(0.725, 2, 3, 2, 1)				
k	2	3	4	5
xu01a	1117264	111.7297	111.7298	111.7298
xu01b	5.6960	5.6960	5.6960	5.6960
sunx03	7.2980	7.2605	7.2601	7.2601
konpap03a	1.6048	1.6053	1.6053	1.6053
konpap03b	1.6076	1.6053	1.6053	1.6053
hasi04a	1.6398	1.6404	1.6404	1.6404
hasi04b	1.6418	1.6404	1.6404	1.6404
iva05a	6.5534	6.4942	6.4937	6.4937
iva05b	6.5001	6.4937	6.4937	6.4937
iva05c	6.5463	6.5424	6.5424	6.5424
hasi06a	1.6426	1.6405	1.6404	1.6404
yinlf09b	1.7852	1.7840	1.7840	1.7840
yinlf09c	2.7350	2.7347	2.7347	2.7347
yinlf0de	3.0199	3.0195	3.0195	3.0195
yinlf09e	1.7832	1.7840	1.7840	1.7840
has10a	1.2884	1.2869	1.2869	1.2869

Table 3.3.2 (c o n t i n u e d)

X=diag(0.25, 2, 3, 2, 1)				
k	2	3	4	5
xu01a	*	*	*	*
xu01b	*	*	*	*
sunx03	8.9435	8.5127	8.5087	8.5078
konpap03a	2.5635	2.5641	2.5641	2.5641
konpap03b	2.5948	2.5644	2.5642	2.5641
hasi04a	2.8189	2.8168	2.8168	2.8168
hasi04b	2.8502	2.6197	2.6105	2.6104
iva05a	8.0022	7.2847	7.2787	7.2786
iva05b	*	*	*	*
iva05c	7.6332	7.6106	7.6104	7.6104
hasi06a	2.7502	2.7359	2.7358	2.7358
yinlf09b	*	*	*	*
yinlf09c	*	*	*	*
yinlf0d	*	*	*	*
yinlf09e	*	*	*	*
has10a	1.8497	1.8439	1.8439	1.8439

3.4. Example 4 (Example from [7])

Consider the complex matrix equation (1) with solution $X = I_3$ and coefficients being the random matrix

$$A = \begin{bmatrix} 2.1896\times10^{-1} + i5.3462\times10^{-2} & 6.7930\times10^{-1} + i7.6982\times10^{-3} & 5.1942\times10^{-1} + i4.1749\times10^{-1} \\ 4.7045\times10^{-2} + i5.2970\times10^{-1} & 9.3469\times10^{-1} + i3.8342\times10^{-1} & 8.3097\times10^{-1} + i6.8677\times10^{-1} \\ 6.7886\times10^{-1} + i6.7115\times10^{-1} & 3.8350\times10^{-1} + i6.6842\times10^{-2} & 3.4572\times10^{-2} + i5.8898\times10^{-1} \end{bmatrix}$$

and $Q = X + A^{\mathrm{T}} X^{-1} A$. The perturbations in the perturbed matrix equation (3) are taken as $\delta A = \delta X = (1+i)10^{-k} E$ and $\delta Q = \widetilde{X} + \widetilde{A}^{*} \widetilde{X}^{-1} \widetilde{A} - Q$. The results obtained for k = 1, …, 10 are listed in Table 3.4.1.

Table 3.4.1. Example 4

k	10	9	8	7	6	5	4	3	2
xu01a	*	*	*	*	*	*	*	*	*
xu01b	*	*	*	*	*	*	*	*	*
sunx0)	*	*	*	*	*	*	*	*	*
konpap03a	7.0344	7.0344	7.0344	7.0344	7.0344	7.0344	7.0344	7.0343	7.0343
konpap03b	7.0344	7.0344	7.0344	7.0344	7.0348	7.0392	7.0831	7.5979	*
hasi04a	*	*	*	*	*	*	*	*	*
hasi04b	*	*	*	*	*	*	*	*	*
iva05a	9.8749	9.8746	9.8749	9.8751	9.8772	9.8976	10.111	15.250	*
iva05b	*	*	*	*	*	*	*	*	*
iva05c	9.8749	9.8749	9.8749	9.8749	9.8749	9.8750	9.8759	9.8847	9.9727
hasi06a	*	*	*	*	*	*	*	*	*
yinlf09b	*	*	*	*	*	*	*	*	*
yinlf09c	*	*	*	*	*	*	*	*	*
yinlf0de	*	*	*	*	*	*	*	*	*
yinlf09e	*	*	*	*	*	*	*	*	*
has10a	–	–	–	–	–	–	–	–	–

As it is seen only bounds $\mathrm{est}_{\mathrm{konpap03a}}$, $\mathrm{est}_{\mathrm{konpap03b}}$, $\mathrm{est}_{\mathrm{iva05a}}$ and $\mathrm{est}_{\mathrm{iva05c}}$ estimate the perturbations in the solution of the perturbed equation (3). The bound $\mathrm{est}_{\mathrm{has10a}}$ can not be used due to inappropriate matrix P. The conditions of existence of the other bounds are violated and these bounds do not work.

Consider the unperturbed and the perturbed matrix equations (2) and (4), respectively. Compare the effectiveness and the reliability of the bounds $\mathrm{est}_{\mathrm{konpap03a}}$

(10), $\text{est}_{\text{konpap03b}}$ (12), $\text{est}_{\text{hasi04a}}$ (15), $\text{est}_{\text{hasi04b}}$ (16), $\text{est}_{\text{hasi04c}}$ (17), $\text{est}_{\text{hasi04d}}$ (18), $\text{est}_{\text{hasi04e}}$ (19), $\text{est}_{\text{hasi04f}}$ (20), $\text{est}_{\text{iva05d}}$ (24), $\text{est}_{\text{iva05e}}$ (25), $\text{est}_{\text{hasi06b}}$ (29), $\text{est}_{\text{liz09}}$ (30) and $\text{est}_{\text{has10b}}$ (39).

3.5. Example 5

The Example is based on the model of Example 6.2 from [9]. Consider equation (2) for $A = \begin{bmatrix} 0 & 0.95 + k/40 \\ 0 & 0 \end{bmatrix}$, k = 1, 2, 3, 4, $Q = X - A^* X^{-1} A$, $X = \begin{bmatrix} 1 & 0 \\ 0 & 1.7083 \end{bmatrix}$. Let the perturbations on A and Q be $\delta A = I_2 \times 10^{-5}$, $\delta Q = \begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix} \times 10^{-5}$. The solution $\widetilde{X}$ of the perturbed equation (4) is obtained using the MATLAB's function dare – the discrete-time algebraic matrix Riccati equation solver ($\widetilde{X} = \text{dare}((\widetilde{A}^*)^{-1}\widetilde{A}, I, \widetilde{Q}, \widetilde{A}\widetilde{Q}^{-1}\widetilde{A}, 0, I)$). We compute and list in Table 3.5.1 the ratios of the estimated error and the true error. Then, we modify the example choosing the complex matrices $\delta A = (1+i) I_2 \times 10^{-5}$ and $\delta Q = (1+i)\begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix} \times 10^{-5}$ for the perturbations on A and Q. The results are listed in Table 3.5.2.

Table 3.5.1. Example 5, real case

k	1	2	3	4
konpap03a	1.7126	1.7393	1.7858	1.7908
konpap03b	1.7129	1.7397	1.7861	1.7912
hasi04a	2.4407×10^1	1.1990×10^5	*	*
hasi04b	2.4407×10^1	1.1990×10^5	*	*
hasi04c	1.2831	1.3003	1.3505	1.3448
hasi04d	1.2831	1.3003	1.3505	1.3448
hasi04e	*	*	*	*
hasi04f	*	*	*	*
iva05d	2.3899	2.4011	2.4392	2.4207
iva05e	*	*	*	*
hasi06b	2.5790×10^1	*	*	*
liz09	8.6166	1.0699×10^1	1.3908×10^1	1.7800×10^1
has10b	–	–	–	–

Table 3.5.2. Example 5, complex case

k	1	2	3	4
konpap03a	1.9890	1.9990	2.0000	1.9872
konpap03b	1.9892	1.9992	2.0002	1.9874
hasi04a	3.7577×10^1	3.6784×10^8	*	*
hasi04b	3.7578×10^1	3.7145×10^8	*	*
hasi04c	2.4486	2.7522	2.9985	3.0429
hasi04d	2.4486	2.7522	2.9985	3.0429
hasi04e	*	*	*	*
hasi04f	*	*	*	*
iva05d	2.4369	2.4491	2.4503	2.4347
iva05e	*	*	*	*
hasi06b	4.0946×10^1	*	*	*
liz09	1.6449×10^1	2.2652×10^1	3.0888×10^1	4.0273×10^1
has10b	–	–	–	–

The results for k = 1, 2, 3, 4 listed in Tables 3.5.1 and 3.5.2 show that the conditions of Theorems 2.4.3-2.5.5 in the cases $k > 1$, as well as the conditions of Theorems 2.4.1 and 2.4.2, when $k > 2$ are violated. Thus we can not use $\text{est}_{\text{hasi04e}}$, est$_{hasi04f}$, est$_{iva05e}$ as perturbation bounds. However, the bounds est$_{konpap03a}$, est$_{konpap03b}$, est$_{hasi04c}$ and est$_{hasi04d}$ give quite sharp perturbation bounds. In the real case, the bounds est$_{hasi04c}$, est$_{hasi04d}$ of H a s a n o v, I v a n o v [4] are sharper than the bounds est$_{konpap03a}$, est$_{konpap03b}$ of K o n s t a n t i n o v et al. [7]. In the complex case, the MATLAB gives a warning that the solution may be inaccurate due to poor scaling or eigenvalues near the stability boundary. This reveals that in the case of complex perturbations the solution $\widetilde{X}$ of the perturbed equation (4) is ill-conditioned. In this case the bounds est$_{konpap03a}$, est$_{konpap03b}$ are the sharpest ones.

3.6. Example 6

For this Example we use the settings of Example 1 from [3]. Consider the matrix equation (2), $A = \frac{\delta_k}{\|A\|} A_0$ and $Q = X - A^* X^{-1} A$, where $\delta_k = \frac{19}{20} - 10^{-k}$,

$$A_0 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix},$$ with solution $$X = \begin{pmatrix} 2.5 & 1 & 1 & 1 & 1 \\ 1 & 2.5 & 1 & 1 & 1 \\ 1 & 1 & 2.5 & 1 & 1 \\ 1 & 1 & 1 & 2.5 & 1 \\ 1 & 1 & 1 & 1 & 2.5 \end{pmatrix}.$$ Consider the perturbed equation (4), where $\widetilde{A} = A + e \frac{1}{\|C^{\mathrm{T}} + C\|} \left(C^{\mathrm{T}} + C \right)$ and C is a random matrix generated by the MATLAB's function randn. The perturbed solution is $\widetilde{X} = X + eE$ and $\widetilde{Q} = \widetilde{X} - \widetilde{A}^* \widetilde{X}^{-1} \widetilde{A}$. To observe the effectiveness of the bounds in the real and the complex case we choose $e = 10^{-kj}$, k = 2, 3, j = 2, 3, 4, 5, as in Example 4 from [3], and then $e = (1+i)10^{-kj}$, where i is the imaginary unit. The average of the ratios of the perturbation bound to the estimated value for 200 random runs are listed in Table 3.6.1 for the real case and in Table 3.6.2 for the complex case.

Table 3.6.1. Example 6, real case

j	2	3	4	5
		$k = 2$		
konpap03a	1.2265	1.2265	1.2265	1.2265
konpap03b	1.2269	1.2265	1.2265	1.2265
hasi04a	1.6389	1.6389	1.6389	1.6389
hasi04b	1.6389	1.6389	1.6389	1.6389
hasi04c	1.3575	1.3574	1.3574	1.3574
hasi04d	1.3576	1.3574	1.3574	1.3574
hasi04e	*	*	*	*
hasi04f	1.8047	1.8047	1.8047	1.8047
iva05d	5.4908	5.4691	5.4689	5.4689
iva05e	*	*	*	*
hasi06b	1.7562	1.7556	1.7556	1.7566
liz09	5.4788×10^1	5.0368×10^1	5.0334×10^1	5.0334×10^1
has10b	1.2610	1.2575	1.2574	1.2574

Table 3.6.1 (c o n t i n u e d)

		$k = 3$		
konpap03a	1.2336	1.2336	1.2337	1.3561
konpap03b	1.2336	1.2336	1.2337	1.3561
hasi04a	1.6550	1.6550	1.6552	1.8149
hasi04b	1.6550	1.6550	1.6552	1.8149
hasi04c	1.3649	1.3649	1.3650	1.4834
hasi04d	1.3649	1.3649	1.3650	1.4834
hasi04e	*	*	*	*
hasi04f	1.8411	1.8411	1.8413	2.0293
iva05d	5.4907	5.4905	5.4911	6.0754
iva05e	*	*	*	*
hasi06b	1.7800	1.7800	1.7801	1.9434
liz09	5.1444×10^1	5.1407×10^1	5.1412×10^1	5.5870×10^1
has10b	1.2870	1.2870	1.2869	1.3652

Among the bounds considered, the bounds $\mathrm{est}_{\mathrm{konpap03a}}$ and $\mathrm{est}_{\mathrm{konpap03b}}$ give the closest estimates of the perturbation in the solution.

Table 3.6.2. Example 6, complex case

j	2	3	4	5
		$k = 2$		
konpap03a	1.2283	1.2283	1.2283	1.2283
konpap03b	1.2287	1.2283	1.2283	1.2283
hasi04a	1.6395	1.6395	1.6395	1.6395
hasi04b	1.6395	1.6395	1.6395	1.6395
hasi04c	1.3580	1.3579	1.3579	1.3579
hasi04d	1.3581	1.3579	1.3579	1.3579
hasi04e	*	*	*	*
hasi04f	1.8047	1.8047	1.8047	1.8047
iva05d	5.4987	5.4678	5.4675	5.4675
iva05e	*	*	*	*
hasi06b	1.7613	1.7604	1.7604	1.7604
liz09	5.8153×10^1	5.0398×10^1	5.0351×10^1	5.0350×10^1
has10b	1.0834	1.0798	1.0798	1.0798
		$k = 3$		
konpap03a	1.2272	1.2278	1.2272	1.2895
konpap03b	1.2272	1.2272	1.2272	1.2895
hasi04a	1.6510	1.6510	1.6511	1.7318
hasi04b	1.6510	1.6510	1.6511	1.7318
hasi04c	1.3619	1.3619	1.3620	1.4218
hasi04d	1.3619	1.3619	1.3620	1.4218
hasi04e	*	*	*	*
hasi04f	1.8411	1.8411	1.8412	1.9294
iva05d	5.4587	5.4584	5.4587	5.7560
iva05e	*	*	*	*
hasi06b	1.7717	1.7717	1.7718	1.8563
liz09	5.1345×10^1	5.1294×10^1	5.1296×10^1	5.3551×10^1
has10b	1.1569	1.1568	1.4568	1.2062

For the complex case of Example 6, the bound $\mathrm{est}_{\mathrm{has10b}}$ is the sharpest ones.

3.7. Example 7 (Example 1 from [4])

Consider equation (2) with $A = \begin{pmatrix} 2\alpha & \alpha \\ \alpha & \alpha/10 \end{pmatrix}$, solution $X = \mathrm{diag}[1 \quad 0.99]$ and right hand $Q = X - A^{\mathrm{T}} X^{-1} A$. We take α = 0.41, 0.4, 0.39, 0.35, 0.3, 0.2. Assume perturbations on matrices A and Q: $\delta A = \begin{bmatrix} 10 & 6 \\ 2 & 4 \end{bmatrix} \times 10^{-8}$, $\delta Q = \begin{bmatrix} 4 & 7 \\ 4 & 4 \end{bmatrix} \times 10^{-8}$ for the real case and $\delta A = (1+i)\begin{bmatrix} 10 & 6 \\ 2 & 4 \end{bmatrix} \times 10^{-8}$, $\delta Q = (1+i)\begin{bmatrix} 4 & 7 \\ 4 & 4 \end{bmatrix} \times 10^{-8}$ for the complex case. The solution of the perturbed equation (4) is computed with the MATLAB's function dare. The ratio of the perturbation bound and the estimated value are listed in Table 3.7.1 – the real case and in Table 3.7.2 – the complex case.

Table 3.7.1. Example 7, real case

α	0.41	0.4	0.39	0.35	0.3	0.2
konpap03a	1.6313	1.5977	1.5647	1.4393	1.2997	1.1086
konpap03b	1.6313	1.5977	1.5647	1.4393	1.2997	1.1086
hasi04a	3.6792×10^2	3.3002×10^1	1.7200×10^1	5.8071	3.1187	1.6026
hasi04b	3.6791×10^2	3.3002×10^1	1.7200×10^1	5.8071	3.1187	1.6026
hasi04c	2.5356×10^2	2.3143×10^1	1.2232×10^1	4.3554	2.4868	1.4208
hasi04d	2.5355×10^2	2.3143×10^1	1.2232×10^1	4.3554	2.4868	1.4208
hasi04e	*	*	*	*	*	*
hasi04f	*	*	*	*	*	1.7473
iva05d	2.2605	2.2335	2.2071	2.1085	2.0035	1.8660
iva05e	*	*	*	*	*	*
hasi06b	3.7045×10^2	3.2750×10^1	1.7093×10^1	5.8010	3.1313	1.6191
liz09	*	*	*	2.327×10^2	2.3692×10^1	3.1488
has10b	–	–	–	–	–	0.9397

Table 3.7.2 Example 7, complex case

α	0.41	0.4	0.39	0.35	0.3	0.2
konpap03a	2.1751	2.1201	2.0662	1.8632	1.6396	1.4228
konpap03b	2.1751	2.1201	2.0662	1.8632	1.6396	1.4228
hasi04a	5.4325×10^2	4.8625×10^1	2.5289×10^1	8.4668	4.5011	2.2714
hasi04b	5.4325×10^2	4.8625×10^1	2.5289×10^1	8.4668	4.5011	2.2714
hasi04c	5.4129×10^2	4.8610×10^1	2.5285×10^1	8.4665	4.5011	2.2715
hasi04d)	5.4127×10^2	4.8610×10^1	2.5285×10^1	8.4665	4.5011	2.2715
hasi04e)	*	*	*	*	*	*
hasi04f	*	*	*	*	*	1.7929
iva05d	2.6587	2.5258	2.4841	2.3305	2.1686	1.9405
iva05e	*	*	*	*	*	*
hasi06b	5.5589×10^2	4.9025×10^1	2.5513×10^1	8.5610	4.5600	2.3057
liz09	*	*	*	3.9513×10^2	4.2787×10^1	5.0251
has10b	–	–	–	–	–	1.3818

For this example the conditions of Theorems 2.4.3 and 2.5.5 are not satisfied and thus the perturbation bounds $\mathrm{est}_{\mathrm{hasi04e}}$, $\mathrm{est}_{\mathrm{iva05e}}$ do not give a result. The perturbation bounds $\mathrm{est}_{\mathrm{hasi04a}}$, $\mathrm{est}_{\mathrm{hasi04b}}$, $\mathrm{est}_{\mathrm{hasi04c}}$, $\mathrm{est}_{\mathrm{hasi04d}}$, $\mathrm{est}_{\mathrm{hasi06b}}$ are not effective – they are conservative with variable accuracy in different values of the parameter α. Due to an inappropriate choice of the matrix P, the bound $\mathrm{est}_{\mathrm{has10b}}$ gives results for $\alpha = 0.2$ only. The only bounds that give acceptable results in this example are the bounds $\mathrm{est}_{\mathrm{konpap03a}}$, $\mathrm{est}_{\mathrm{konpap03b}}$ and $\mathrm{est}_{\mathrm{iva05d}}$. The bounds $\mathrm{est}_{\mathrm{konpap03a}}$ and $\mathrm{est}_{\mathrm{konpap03b}}$ are sharper.

3.8. Example 8 (Example 2 from [4])

Consider equation (2) with $A = \frac{1}{3}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, solution $X = \begin{pmatrix} 2 & 1 \\ 1 & 7 \end{pmatrix}$ and $Q = X - A^{\mathrm{T}} X^{-1} A$. Assume that the perturbations on A and Q are $\delta A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times 10^{-5}$ and $\delta Q = \begin{pmatrix} 1 & 5 \\ 5 & 4 \end{pmatrix} \times 10^{-10}$. The positive definite solution of the perturbed equation (4) is computed with function dare. The ratios of the perturbation bound to the estimated value are listed in Table 3.8.1. In Table 3.8.2, the results obtained when the perturbations in the data matrices are $\delta A = (1+i)\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times 10^{-5}$ and $\delta Q = (1+i)\begin{pmatrix} 1 & 5 \\ 5 & 4 \end{pmatrix} \times 10^{-10}$ are listed.

Table 3.8.1. Example 8, real case

konpap03a	2.9847
konpap03b	2.9849
hasi04a	3.8108
hasi04b	3.8109
hasi04c	3.3977
hasi04d	3.3979
hasi04e	*
hasi04f	1.2296
iva05d	3.8423
iva05e	*
hasi06b	3.7067
liz09	5.7355×10^1
has10b	0.9139

Table 3.8.2. Example 8, complex case

konpap03a	3.9775
konpap03b	3.9779
hasi04a	5.1253
hasi04b	5.1255
hasi04c	4.5699
hasi04d	4.5701
hasi04e	*
hasi04f	1.2383
iva05d	5.1206
iva05e	*
hasi06b	4.9398
liz09	7.7204×10^1
has10b	1.2179

For this example the most effective bound is the bound $\mathrm{est}_{\mathrm{has10b}}$, followed by the bound $\mathrm{est}_{\mathrm{hasi04f}}$. The bound $\mathrm{est}_{\mathrm{liz09}}$ is conservative. The conditions of existence of the bounds $\mathrm{est}_{\mathrm{hasi04e}}$ and $\mathrm{est}_{\mathrm{iva05e}}$ are violated. The other considered bounds exceed the estimated value by 4 to 5 times.

3.9. Example 9 (Example from [7])

Consider the complex matrix equation (2) with solution $X = I$ and matrix coefficients

$$A = \begin{bmatrix} 2.1896\times10^{-1} + i5.3462\times10^{-2} & 6.7930\times10^{-1} + i7.6982\times10^{-3} & 5.1942\times10^{-1} + i4.1749\times10^{-1} \\ 4.7045\times10^{-2} + i5.2970\times10^{-1} & 9.3469\times10^{-1} + i3.8342\times10^{-1} & 8.3097\times10^{-1} + i6.8677\times10^{-1} \\ 6.7886\times10^{-1} + i6.7115\times10^{-1} & 3.8350\times10^{-1} + i6.6842\times10^{-2} & 3.4572\times10^{-2} + i5.8898\times10^{-1} \end{bmatrix}$$

and $Q = X - A^{\mathrm{T}} X^{-1} A$. The perturbed matrix equation (4) is obtained from equation (2) using the perturbations $\delta A = \delta X = (1+i)10^{-k} E$ and $\delta Q = \widetilde{X} + \widetilde{A}^{*} \widetilde{X}^{-1} \widetilde{A} - Q$. The results obtained for k = 1, …, 10 are listed in Table 3.9.1.

Table 3.9.1. Example 9

k	10	9	8	7	6	5	4	3	2
konpap03a	5.5878	5.5878	5.5878	5.5878	5.5878	5.5878	5.5878	5.5878	5.5878
konpap03b	5.7129	5.5878	5.5878	5.5878	5.5880	5.5898	5.6080	5.8042	*
hasi04a	*	*	*	*	*	*	*	*	*
hasi04b	*	*	*	*	*	*	*	*	*
hasi04c	4.2285	4.2285	4.2285	4.2285	4.2286	4.2288	4.2309	4.2522	4.4746
hasi04d	4.2285	4.2285	4.2285	4.2285	4.2286	4.2288	4.2309	4.2519	4.4715
hasi04e	*	*	*	*	*	*	*	*	*
hasi04f	*	*	*	*	*	*	*	*	*
iva05d	8.6459	8.6459	8.6459	8.6461	8.6477	8.6636	8.8297	12.051	*
iva05e	*	*	*	*	*	*	*	*	*
hasi06b	*	*	*	*	*	*	*	*	*
liz09	*	*	*	*	*	*	*	*	*
has10b	*	*	*	*	*	*	*	*	*

For this example of a complex matrix equation (2) the sharpest bounds are $est_{hasi04c}$ and $est_{hasi04d}$. The conditions of existence from Theorems 2.4.1-2.4.4, 2.5.5, 2.6.2, 2.7 and 2.9.1 are not satisfied and thus the perturbation bounds $est_{hasi04a}$, $est_{hasi04b}$, $est_{hasi04e}$, $est_{hasi04f}$, est_{iva05e}, $est_{hasi06b}$, est_{liz09}, est_{has10b} do not give a result.

4. Concluding remarks

Analysing the behaviour of the perturbation bounds considered in the paper, we can point out as most effective the bounds $est_{konpap03a}$, $est_{konpap03b}$ of Konstantinov et al. [7] and est_{has10a}, est_{has10b} of Hasanov [2]. The dependence of the bounds $est_{konpap03a}$, $est_{konpap03b}$ on many parameters makes them difficult for computing in general. But the bounds $est_{konpap03a}$, $est_{konpap03b}$ are reliable and generally give satisfactory accurate estimates. The bounds est_{has10a}, est_{has10b} are very simple for computing. They were the sharpest in most cases considered. But the lack of a general rule for choosing the matrix *P* is an inconvenience. The bound est_{xu01a} is an elegant bound and does not require the solution to the perturbed or the unperturbed equations. This allows the a priori calculation of the bound est_{xu01a}. But, as it is seen from the results obtained in Section 3, the bound est_{xu01a} either is too conservative, or does not work due to violated conditions of existence. The other considered bounds showed more conservative results than the bounds mentioned above.

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