

Some Remarks about L. Atanassova's Paper "A New Intuitionistic Fuzzy Implication"

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Abstract: *A new class of intuitionistic fuzzy implications is introduced in the paper. Fulfillment of some of its axioms and properties together with Modus Ponens and Modus Tollens inference rules are investigated.*

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1. Introduction

The concept of Intuitionistic Fuzzy Sets (IFS) was introduced more than 25 years ago [1, 8, 2]. The Intuitionistic Fuzzy Logic (IFL) has been also developed in a series of papers. In this logic the truth-value of a variable x is given by the ordered pair $\langle a, b \rangle$, where $a, b, a+b \in [0, 1]$. The numbers a and b are interpreted as the degrees of validity and non-validity of x . We denote the truth-value of x by $V(x)$.

The variable with a truth-value *true* in the classical logic is denoted by $\underline{1}$ and the variable *false* – by $\underline{0}$. For these variables it holds also that $V(\underline{1}) = \langle 1, 0 \rangle$ and $V(\underline{0}) = \langle 0, 1 \rangle$.

We call the variable x an Intuitionistic Fuzzy Tautology (shortly: IFT), if and only if

$$\text{for } V(x) = \langle a, b \rangle \text{ holds } a \geq b.$$

Similarly we call the variable x an Intuitionistic Fuzzy co-Tautology (IFcT), if and only if

$$\text{for } V(x) = \langle a, b \rangle \text{ holds } a \leq b.$$

For every x we can define the value of negation of x in the typical form $V(\neg x) = \langle b, a \rangle$.

It is clear that a IFcT could be defined by IFT and \neg .

An important operator of IFL is the intuitionistic fuzzy implication.

Definition 1. The fuzzy implication (definition by Czogała and Łeśki [9]) is a mapping $I: [0, 1]^2 \rightarrow [0, 1]$ where for $p_1, p_2, p, q_1, q_2, q \in [0, 1]$ holds:

(i1FL) if $p_1 \leq p_2$ then $I(p_1, q) \geq I(p_2, q)$,

(i2FL) if $q_1 \leq q_2$ then $I(p, q_1) \leq I(p, q_2)$,

(i3FL) $I(\underline{0}, q) = \underline{1}$,

(i4FL) $I(p, \underline{1}) = \underline{1}$,

(i5FL) $I(\underline{1}, \underline{0}) = \underline{0}$.

Applying this definition to IFL firstly, we will introduce some ordering relation for the intuitionistic truth-value.

For $V(x) = \langle a, b \rangle$ and $V(y) = \langle c, d \rangle$ where $a, b, c, d, a+b, c+d \in [0, 1]$, we denote $V(x) \preceq V(y)$ if and only if $a \leq c$ and $b \geq d$.

In the case of IFL conditions (i1FL)-(i5FL) for any implication \Rightarrow are given in the form:

(i1) if $V(x_1) \preceq V(x_2)$ then $V(x_1 \Rightarrow y) \succeq V(x_2 \Rightarrow y)$,

(i2) if $V(y_1) \preceq V(y_2)$ then $V(x \Rightarrow y_1) \preceq V(x \Rightarrow y_2)$,

(i3) $\underline{0} \Rightarrow y$ is an IFT,

(i4) $x \Rightarrow \underline{1}$ is an IFT,

(i5) $\underline{1} \Rightarrow \underline{0}$ is an IFcT.

In [7] Lilija Atanassova has introduced a new intuitionistic fuzzy implication by the formula

$$V(x \rightarrow_{@} y) = \left\langle \frac{b+c}{2}, \frac{a+d}{2} \right\rangle.$$

This implication holds, of course, conditions (i1)-(i5).

Consequently the truth-value of a negation $\neg_{@}$ generated by $\rightarrow_{@}$ is given in the form

$$V(\neg_{@} x) = \left\langle \frac{b}{2}, \frac{a+1}{2} \right\rangle.$$

We notice that this negation is not involutive, i.e., in general

$$V(\neg_{@} (\neg_{@} (x))) \neq V(x).$$

Moreover, for every x , $\neg_{@} x$ is an IFcT, and $\neg_{@} x$ is an IFT only for $a = 0$ and $b = 1$.

Atanassova has also presented some theorems and properties of her implication as follows.

Theorem 1 (Atanassova [7, Theorem 1, p. 22]). Implication $\rightarrow_{@}$

- (a) does not satisfy Modus Ponens in case of tautology;
- (b) satisfies Modus Ponens in IFT-case.

For implication $\rightarrow_{@}$ and negation $\neg_{@}$ none of the following properties are valid:

Property 1. $x \rightarrow_{@} \neg_{@} \neg_{@} x.$

Property 2. $\neg_{@} \neg_{@} x \rightarrow_{@} x.$

Property 3. $\neg_{@} \neg_{@} \neg_{@} x = \neg_{@} x.$

Following [11], Atanassova gives also 17 axioms, that an implication should fulfill. They are:

- (a) $x \Rightarrow x;$
- (b) $x \Rightarrow (y \Rightarrow x);$
- (c) $x \Rightarrow (y \Rightarrow x \wedge y);$
- (d) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow (y \Rightarrow (x \Rightarrow z));$
- (e) $(x \Rightarrow (y \Rightarrow z)) \Rightarrow ((x \Rightarrow y) \Rightarrow (x \Rightarrow z));$
- (f) $x \Rightarrow N(N(x));$
- (g) $N(x \wedge N(x));$
- (h) $(N(x) \vee y) \Rightarrow (x \Rightarrow y);$
- (i) $N(x \vee y) \Rightarrow (N(x) \wedge N(y));$
- (j) $(N(x) \wedge N(y)) \Rightarrow N(x \vee y);$
- (k) $(N(x) \vee N(y)) \Rightarrow N(x \wedge y);$
- (l) $(x \Rightarrow y) \Rightarrow (N(y) \Rightarrow N(x));$
- (m) $(x \Rightarrow N(y)) \Rightarrow (y \Rightarrow N(x));$
- (n) $N(N(N(x))) \Rightarrow N(x);$
- (o) $N(x) \Rightarrow N(N(N(x)));$
- (p) $N(N(x \Rightarrow y)) \Rightarrow (x \Rightarrow N(N(y)));$
- (q) $(z \Rightarrow x) \Rightarrow ((z \Rightarrow (x \Rightarrow y)) \Rightarrow (z \Rightarrow y))$

where \wedge and \vee denote conjunction and disjunction respectively, given by the formulas:

$$V(x \wedge y) = \langle \min(a, c), \max(b, d) \rangle,$$

$$V(x \vee y) = \langle \max(a, c), \min(b, d) \rangle,$$

while N is a negation.

Implication $\rightarrow_{@}$ satisfies the Axioms (a), (i), (j), (k) in IFL-case with $N = \neg_{@}$,

while it does not satisfy none from this axioms in the classical logic case (see [7]).

Besides conditions (i1)-(i5) Atanassova, following Klir and Yuan [10, p. 308, 310], gives also the axioms:

(i6) $V(\underline{1} \Rightarrow y) = V(y),$

(i7) $V(x \Rightarrow x) = \underline{1},$

(i8) $V(x \Rightarrow (y \Rightarrow z)) = V(y \Rightarrow (x \Rightarrow z)),$

$$(i9) \quad V(x \Rightarrow y) = \underline{1} \Leftrightarrow V(x) \preceq V(y),$$

$$(i10) \quad V(x \Rightarrow y) = V(N(y) \Rightarrow N(x)),$$

$$(i11) \quad \Rightarrow \text{ is a continuous function,}$$

together with the theorem: implication $\rightarrow_{@}$ satisfies Axioms (i8) and (i9) and Axiom

$$(i7') \quad x \Rightarrow x \text{ is an IFT.}$$

2. Main results

Now we introduce a parametric class of fuzzy intuitionistic implications Atanassova-type.

Theorem 2. An intuitionistic logical connective with a truth-value

$$V(x \rightarrow_{@}^{\lambda} y) = \left\langle \frac{b+c+\lambda-1}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda} \right\rangle$$

where $\lambda \geq 1$, $\lambda \in \mathfrak{R}$ is an Intuitionistic Fuzzy Implication, fulfilling Definition 1 with (i1)-(i5) where implication $\rightarrow_{@}^{\lambda}$ is not presented in the previous bibliography (known to the author).

Proof. Preliminary note:

$$\frac{b+c+\lambda-1}{2\lambda}, \frac{a+d+\lambda-1}{2\lambda} \in \left[\frac{\lambda-1}{2\lambda}, \frac{\lambda+1}{2\lambda} \right] \subseteq [0, 1] \text{ and}$$

$$\frac{b+c+\lambda-1}{2\lambda} + \frac{a+d+\lambda-1}{2\lambda} \in \left[\frac{2\lambda-2}{2\lambda}, \frac{2\lambda}{2\lambda} \right] \subseteq [0, 1],$$

(i1) if $V(x_1) \preceq V(x_2)$ then $a_1 \leq a_2$ and $b_1 \geq b_2$ therefore

$$\left\langle \frac{b_1+c+\lambda-1}{2\lambda}, \frac{a_1+d+\lambda-1}{2\lambda} \right\rangle \succeq \left\langle \frac{b_2+c+\lambda-1}{2\lambda}, \frac{a_2+d+\lambda-1}{2\lambda} \right\rangle \text{ so}$$

$$V(x_1 \rightarrow_{@}^{\lambda} y) \succeq V(x_2 \rightarrow_{@}^{\lambda} y),$$

(i2) if $V(y_1) \preceq V(y_2)$ then $c_1 \leq c_2$ and $d_1 \geq d_2$, therefore

$$\left\langle \frac{b+c_1+\lambda-1}{2\lambda}, \frac{a+d_1+\lambda-1}{2\lambda} \right\rangle \preceq \left\langle \frac{b+c_2+\lambda-1}{2\lambda}, \frac{a+d_2+\lambda-1}{2\lambda} \right\rangle \text{ so}$$

$$V(x \rightarrow_{@}^{\lambda} y_1) \preceq V(x \rightarrow_{@}^{\lambda} y_2),$$

$$(i3) \quad V(\underline{0} \rightarrow_{@}^{\lambda} y) = \left\langle \frac{c+\lambda}{2\lambda}, \frac{d+\lambda-1}{2\lambda} \right\rangle, \text{ and that } \frac{c+\lambda}{2\lambda} \geq \frac{d+\lambda-1}{2\lambda}$$

because $c-d \geq -1$ so

$$\underline{0} \rightarrow_{@}^{\lambda} y \text{ is an IFT,}$$

$$(i4) \quad V(x \rightarrow_{@}^{\lambda} \underline{1}) = \left\langle \frac{b+\lambda}{2\lambda}, \frac{a+\lambda-1}{2\lambda} \right\rangle, \text{ and that } \frac{b+\lambda}{2\lambda} \geq \frac{a+\lambda-1}{2\lambda}$$

because $b-a \geq -1$ so

$$(i5) \quad x \rightarrow_{@}^{\lambda} \underline{1} \text{ is an IFT,}$$

$$V(\underline{1} \rightarrow_{@}^{\lambda} \underline{0}) = \left\langle \frac{\lambda-1}{2\lambda}, \frac{\lambda+1}{2\lambda} \right\rangle, \text{ so } \underline{1} \rightarrow_{@}^{\lambda} \underline{0} \text{ is an IFcT.} \quad \blacksquare$$

Remark. Atanassova's implication $\rightarrow_{@}$ is a special case of $\rightarrow_{@}^{\lambda}$ implication.

Negation $\neg_{@}^{\lambda}$ generated by $\rightarrow_{@}^{\lambda}$ is expressed by the formula

$$V(\neg_{@}^{\lambda} x) = \left\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda} \right\rangle.$$

Negation $\neg_{@}^{\lambda}$ is not involutive.

Theorem 3. For the implication $\rightarrow_{@}^{\lambda}$ and negation $N = \neg_{@}^{\lambda}$ holds:

- 1) (a), (i), (j), (k) is an IFT;
- 2) (b), (c), (d), (f), (g), (h), (l), (m), (n), (o), (p), (q): none of this properties is valid even in the IFT-case;

$$3) \quad V(\neg_{@}^{\lambda}(x \vee y)) = V(\neg_{@}^{\lambda}(x) \wedge \neg_{@}^{\lambda}(y)),$$

$$V(\neg_{@}^{\lambda}(x \wedge y)) = V(\neg_{@}^{\lambda}(x) \vee \neg_{@}^{\lambda}(y)).$$

Theorem 4. Implication $\rightarrow_{@}^{\lambda}$

1) does not satisfies property (i6), (i7), (i8), (i8'), (i8''), (i9), (i9'), (i10, with $N = \neg_{@}^{\lambda}$), where

- (i8') $x \rightarrow_{@}^{\lambda}(y \rightarrow_{@}^{\lambda} z)$ is an IFT iff $y \rightarrow_{@}^{\lambda}(x \rightarrow_{@}^{\lambda} z)$ is an IFT,
- (i8'') $x \rightarrow_{@}^{\lambda}(y \rightarrow_{@}^{\lambda} z)$ is an IFcT iff $y \rightarrow_{@}^{\lambda}(x \rightarrow_{@}^{\lambda} z)$ is an IFcT,
- (i9') if $x \rightarrow_{@}^{\lambda} y$ is an IFT then $V(x) \leq V(y)$,
- 2) satisfies (i10, with $N = \neg$), (i11) and
 - (i6') if $\underline{1} \rightarrow_{@}^{\lambda} y$ is an IFT then y is an IFT,
 - (i6'') if y is an IFcT then $\underline{1} \rightarrow_{@}^{\lambda} y$ is an IFcT,
 - (i7') $x \rightarrow_{@}^{\lambda} x$ is an IFT,
 - (i7'') $x \rightarrow_{@}^{\lambda} x$ is an IFcT,
 - (i9'') if $V(x) \leq V(y)$ then $x \rightarrow_{@}^{\lambda} y$ is an IFT,
 - (i9''') if $V(x \rightarrow_{@}^{\lambda} y) = \underline{1}$ then $V(x) \leq V(y)$,
 - (i10') $x \rightarrow_{@}^{\lambda} y$ is an IFT iff $\neg_{@}^{\lambda}(y) \rightarrow_{@}^{\lambda} \neg_{@}^{\lambda}(x)$ is an IFT,
 - (i10'') $x \rightarrow_{@}^{\lambda} y$ is an IFcT iff $\neg_{@}^{\lambda}(y) \rightarrow_{@}^{\lambda} \neg_{@}^{\lambda}(x)$ is an IFcT.

Theorem 5. Implication $\rightarrow_{@}^{\lambda}$ satisfies Modus Ponens in the IFL-case (proof is elementary, analogical to the proof in [7, Theorem 1, p. 22]).

Theorem 6. Implication $\rightarrow_{@}^{\lambda}$ satisfies Modus Tollens in the IFL-case with the negation \neg .

Proof. Let $x \rightarrow_{@}^{\lambda} y$ and $\neg y$ be IFTs.

Then $a-b \leq c-d$ and $c \leq d$. Hence $a-b \leq 0$ therefore $\neg x$ is an IFT. ■

Theorem 7. Implication $\rightarrow_{@}^{\lambda}$ satisfies Modus Tollens in the IFL-case with the negation $\neg_{@}^{\lambda}$, i.e. if $x \rightarrow_{@}^{\lambda} y$ is an IFT and $\neg_{@}^{\lambda} y$ is an IFT then $\neg_{@}^{\lambda} x$ is an IFT.

Proof. Let $x \rightarrow_{@}^{\lambda} y$ and $\neg_{@}^{\lambda} y$ be IFTs.

Then $a-b \leq c-d$ and $d-c \geq 1$. Hence $b-a \geq d-c \geq 1$ what holds only if $b=1$ and $a=0$.

Then $V(\neg_{@}^{\lambda} x) = \langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda} \rangle = \langle \frac{1}{2}, \frac{1}{2} \rangle$ therefore $\neg_{@}^{\lambda} x$ is an IFT. ■

Negation $\neg_{@}^{\lambda}$ is not involutive. Multiple use of this negation gives generally a lot of truth-values.

We denote $\neg_{@}^{\lambda,1} x = \neg_{@}^{\lambda} x$ and $\neg_{@}^{\lambda,n+1} x = \neg_{@}^{\lambda}(\neg_{@}^{\lambda,n} x)$.

Theorem 8. Negation $\neg_{@}^{\lambda}$ holds for a natural number $n \geq 1$ the relationships:

- 1)
$$V(\neg_{@}^{\lambda,2n-1} x) = \langle \frac{b}{(2\lambda)^{2n-1}} + \frac{(\lambda+1)((2\lambda)^{2n}-1)}{(2\lambda+1)(2\lambda)^{2n-1}} - \lambda, \frac{a}{(2\lambda)^{2n-1}} + \frac{\lambda((2\lambda)^{2n}-1)}{(2\lambda+1)(2\lambda)^{2n-1}} + 1 - \lambda \rangle;$$
- 2)
$$V(\neg_{@}^{\lambda,2n} x) = \langle \frac{a}{(2\lambda)^{2n}} + \frac{\lambda((2\lambda)^{2n}-1)}{(2\lambda+1)(2\lambda)^{2n}}, \frac{b}{(2\lambda)^{2n}} + \frac{(\lambda+1)((2\lambda)^{2n}-1)}{(2\lambda+1)(2\lambda)^{2n}} \rangle.$$

Proof. Based on the principle of mathematical induction (the result for $\lambda=1$, given by A t a n a s o v a in [7, Theorem 2, p. 23]).

Corollary 1. $\lim_{n \rightarrow \infty} V(\neg_{@}^{\lambda,n} x) = \langle \frac{\lambda}{2\lambda+1}, \frac{\lambda+1}{2\lambda+1} \rangle.$

Corollary 2. $\lim_{\lambda \rightarrow \infty} (\lim_{n \rightarrow \infty} V(\neg_{@}^{\lambda,n} x)) = \langle \frac{1}{2}, \frac{1}{2} \rangle.$

3. Conclusion

The paper presents a new class of fuzzy intuitionistic implications with their basic properties. These implications may be the subject of further research, both in terms of their properties or comparisons with other intuitionistic fuzzy implications, and possible applications also. For example, in the broad field of economics

applications they may solve problems related to fuzzy control, reasoning with incomplete or uncertain information, or multiple criteria decision making, especially with varying degrees of criteria importance. One can have doubts whether the introduction of new methods of information processing makes sense in a situation when the existing ones give satisfactory results and require simpler mathematical tools. Surely this is a topic for discussion. Similar doubts had, I think, the creators, introducing about 40 years ago, the now classical, Zadeh's fuzzy sets to the field of economy, technology and social sciences. Today we can say that this tool, used wisely, can provide tangible benefits.

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