BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **10**, No 2

Sofia • 2010

On the Sensitivity Estimation of the Matrix Equation* $X^s \pm A^* X^t A = Q$

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Abstract: *The paper deals with the existing methods for estimating the sensitivity of the solution to the nonlinear matrix equation* $X^s \pm A^* X^t A = Q$, *with s and t – real numbers. The perturbation bounds for the complex matrix equation proposed by Yin, Liu, Fang in* [9] *and Yin, Liu in* [8], *when s is a positive and t is a negative integers and by Li, Zhang in [6], when s*=1, $t \in [-1, 0)$, *as well as the estimates proposed by Jia, Wei in* [4] *for the real equation with s and t – non negative integers and the perturbation bounds proposed by Konstantinov et al. in* [5] *for the real and complex equation with s and t real numbers are considered. The effectiveness and the reliability of the perturbation bounds are analysed by several numerical examples with reference models based on Example* 1 *from* [3] *and Example 2 from* [2].

Keywords: *Non-linear matrix equation, perturbation bounds.*

1. Introduction and notation

Consider the matrix equation

(1) $$X^s \pm A^* X^t A = Q,$$

where s and t are real numbers, A, Q and X ($Q>0$, $X > 0$) are $n\times n$ complex or real matrices, A^* stands for the complex conjugate transpose of A in the complex case and for the transpose of A in the real case. Equations of this type are of current practical interest [1, 4, 8, 9, 10]. Several perturbation bounds of the unique positive

* This research work was supported by the Bulgarian Academy of Sciences under Grant No 010093/2010.

definite solution to (1), when s = 1 and $t = -1$ or $t \in [-1, 0)$ are proposed [6] and the references therein. The sensitivity of the more general cases, when s and t are real numbers is discussed as follows:

- the complex equations $X^s \pm A^* X^t A = Q$ with s and t positive integers – in [9] and [8],
- the real equation $X^s + A^{\mathrm{T}} X^t A = Q$ with s and t non negative integers – in [4],
- the complex and the real cases of (1) when s and t are real numbers – in [5].

It would be interesting to estimate the field of application of the different bounds.

In this paper by means of numerical experiments the effectiveness of the bounds proposed in [4, 5, 6, 8, 9] with respect to their sharpness is analyzed. For the experiments we use Example 1 form [3] and Example 2 from [2] which are the most exploited in the literature examples, when refer to equations of type (1). As both examples consider real matrix equations, to compare the behaviour of the bounds related to the complex equations we modify Example 1 [3], so that the coefficient matrices to be complex.

The paper is organized as follow. In Section 2 we mention the purpose of the sensitivity analysis. In Section 3 we give in brief the error bounds, which will be analysed in Section 4. For sake of convenience we keep the original notations of the authors. In Section 5 we carry out several numerical experiments to analyse the effectiveness of the bounds considered.

Throughout the paper the following notations are used: $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ – the sets of natural, real and complex numbers, respectively; $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$; I – the identity matrix; $\mathrm{vec}(A) \in \mathbb{F}^{n^2}$ – the column-wise vector representation of the matrix $A \in \mathbb{F}^{n \times n}$, where $\mathbb{F}^n = \mathbb{F}^{n \times 1}$; $A \otimes B = [A(k,l)B]$ – the Kronecker product of the matrices $A = [A(k,l)]$ and B; $\mathrm{P}_{n^2} \in \mathbb{R}^{n^2 \times n^2}$ – the so called vec-permutation matrix, such that $\mathrm{vec}(Y^T) = \mathrm{P}_{n^2} \mathrm{vec}(Y)$ for each $Y \in \mathbb{F}^{n \times n}$; $\|\cdot\|$ – a vector or a matrix norm; $\|\cdot\|_2$ – the Euclidean vector or the spectral matrix norm; $\|\cdot\|_{\mathrm{F}}$ – the Frobenius norm.

2. Statement of the problem

Consider the non-linear matrix equation (1). The perturbed equation is

(2) $$(X + \delta X)^s \pm (A + \delta A)^* (X + \delta X)^t (A + \delta A) = Q + \delta Q,$$

where δA, δQ are perturbations in the matrix coefficients A and Q, respectively, which represent equivalently the rounding and parameter errors, accompanying the numerical solution of equation (1). The perturbation in the solution $X + \delta X$ of the perturbed equation (2) is δX. The purpose of the sensitivity analysis of (1) is to derive perturbation bounds for $\|\delta X\|_{\mathrm{F}}$ as a linear or nonlinear function of the perturbations in the data.

3. Sensitivity estimates

For sake of convenience, we denote the bounds by their original notations.

3.1. The bounds of Yi n, L i u, F a n g [9]

In [9] the nonlinear matrix equation $X^s + A^* X^t A = Q$, with A – nonsingular n square complex matrix, Q – an Hermitian positive definite matrix and s and t – positive integers is considered. Several bounds for the Frobenius norm of the perturbation in the solution $\|\delta X\|_{\mathrm{F}}$ are proposed. By means of numerical experiments on the base of Example 1 from [3] the effectiveness of the bounds proposed in the paper are compared to the bounds proposed in [3] and [7]. The results show that for this example among the bounds proposed in the three sources the bounds ξ_{err1} (3.5), ξ_{err2} (3.16) and η_{err1} (3.18) from [9] are the sharpest. Therefore we consider the perturbation bounds ξ_{err1} (3.5), ξ_{err2} (3.16) and η_{err1} (3.18) proposed in [9]:

(3) $$\|\delta X\|_{\mathrm{F}} \le \xi_{\mathrm{err1}},$$

$$\xi_{\mathrm{err1}} = \frac{\|T_1^{-1}\|_2}{1 - \|T_1^{-1}\|_2 \|A + \delta A\|_2^2 \|T_2\|_2} \cdot \left(\|\delta Q\|_{\mathrm{F}} + 2\|X^{-t}A\|_2 \|\delta A\|_{\mathrm{F}} + \|X^{-t}\|_2 \|\delta A\|_{\mathrm{F}}^2 \right),$$

(4) $$\|\delta X\|_{\mathrm{F}} \le \xi_{\mathrm{err2}}, \quad \xi_{\mathrm{err2}} = \xi\left(\|\delta Q\|_{\mathrm{F}} + 2\|X^{-t}A\|_2 \|\delta A\|_{\mathrm{F}} + \|X^{-t}\|_2 \|\delta A\|_{\mathrm{F}}^2 \right),$$

$$\xi = \max\left\{ \frac{1}{s\tilde{x}_*^{s-1} - t\|A + \delta A\|_2^2 / \tilde{x}_*^{t+1}}, \frac{1}{s x_*^{s-1} - t\|A\|_2^2 / x_*^{t+1}} \right\}, \quad x_* = \left[\lambda_{\min}(Q) \frac{t}{s+t} \right]^{\frac{1}{s}},$$

$$\tilde{x}_* = \left[\lambda_{\min}(Q + \delta Q) \frac{t}{s+t} \right]^{\frac{1}{s}},$$

where $\lambda_{\min}(H)$ is the minimal eigenvalue of the Hermitian matrix H, $H = Q$, $Q + \delta Q$,

(5) $$\|\delta X\|_{\mathrm{F}} \le \eta_{\mathrm{err1}},$$

$$\eta_{\mathrm{err1}} = \frac{\|L_1^{-1}\|_2}{1 - \|L_1^{-1}\|_2 \|A\|_2 \|A + \delta A\|_2 \|L_2\|_2} \left[\|\delta Q\|_{\mathrm{F}} + \left(\|(A + \delta A)^* (X + \delta X)^{-t}\|_2 + \|X^{-t}A\|_2 \right) \|\delta A\|_{\mathrm{F}} \right],$$

under the conditions

(6) $$\theta = \sqrt{\frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}} - \|A\|_2 \sqrt{\|Q^{-1}\|^{\frac{s+t}{s}}} > 0, \quad \|\delta Q\| \le \frac{1}{\|Q^{-1}\|_2}\left[1 - (1-\theta)^{\frac{2s}{s+t}} \right],$$

$$\|\delta A\| < \frac{\sqrt{\left(\frac{s+t}{s}\right)^{\frac{s+t}{s}}} - \sqrt{\left(\frac{t}{s}\right)^{\frac{t}{s}}}}{\sqrt{\left(\frac{s+t}{s}\right)^{\frac{s+t}{s}}} \cdot \|Q^{-1}\|_2^{\frac{s+t}{s}}} \theta.$$

where

$$T_1 = \sum_{i=1}^{s} (X+\delta X)^{s-i} \otimes X^{i-1}, T_2 = \sum_{i=1}^{t} X^{-t-1+i} \otimes (X+\delta X)^{-i},$$

$$L_1 = \sum_{i=1}^{s} X^{s-i} \otimes (X+\delta X)^{i-1}, L_2 = \sum_{i=1}^{t} X^{-t-1+i} \otimes (X+\delta X)^{-i}.$$

3.2. The bounds of Y i n, L i u [8]

Y i n and L i u [8] have proposed two perturbation bounds $E_{\text{rr}1}$ (3.5) and $E_{\text{rr}2}$ (3.15) for the solution of the nonlinear matrix equation $X^s - A^* X^t A = Q$, with A – nonsingular n square complex matrix, Q – positive definite matrix and s and t – positive integers:

(7) $$\|\delta X\|_{\text{F}} \le E_{\text{rr}1},$$

$$E_{\text{rr}1} = \frac{\|T_1^{-1}\|_2}{1 - \|T_1^{-1}\|_2 \|A+\delta A\|_2^2 \|T_2\|_2} \cdot \left(\|\delta Q\|_{\text{F}} + 2\|X^{-t} A\|_2 \|\delta A\|_F + \|X^{-t}\|_2 \|\delta A\|_{\text{F}}^2\right),$$

where $T_1 = \sum_{i=1}^{s} (X+\delta X)^{s-i} \otimes X^{i-1}, T_2 = \sum_{i=1}^{t} X^{-t-1+i} \otimes (X+\delta X)^{-i}$

and

(8) $$\|\delta X\|_{\text{F}} \le E_{\text{rr}2}, \ E_{rr2} = \xi\left(\|\delta Q\|_{\text{F}} + 2\|X^{-t} A\|_2 \|\delta A\|_{\text{F}} + \|X^{-t}\|_2 \|\delta A\|_{\text{F}}^2\right),$$

where $\xi = \max\left\{ \frac{\|Q^{-1}\|_2^{\frac{s-1}{s}}}{s - t\|A+\delta A\|_2^2 \|Q^{-1}\|_2^{\frac{s+t}{s}}}, \frac{\|(Q+\delta Q)^{-1}\|_2^{\frac{s-1}{s}}}{s - t\|A+\delta A\|_2^2 \|(Q+\delta Q)^{-1}\|_2^{\frac{s+t}{s}}} \right\},$

under the conditions

(9) $$t > s, \ \theta = \sqrt{\frac{s}{t}} - \|A\|_2 \sqrt{\|Q^{-1}\|_2^{\frac{s+t}{s}}} > 0, \ \|\delta Q\|_2 \le \frac{1}{\|Q^{-1}\|_2}\left[1 - (1-\theta)^{\frac{2s}{s+t}}\right],$$

$$\|\delta A\|_2 < \frac{\left(\sqrt{\frac{s}{t}} - 1\right)\theta}{\sqrt{\frac{t}{s}\|Q^{-1}\|_2^{\frac{s+t}{s}}}}.$$

3.3. The bound of L i, Z h a n g [6]

Although the authors consider the particular case of equation (1) when s = 1, for completeness we include their bound in our analysis because it concern the case $t \in [-1, 0)$. For the complex matrix equation $X - A^* X^{-t} A = Q$ with $t \in [-1, 0)$ in [6] is proved the following Theorem 3.1, which gives a perturbation bound for the solution of the equation. Under the assumptions of A, $A + \delta A$, Q, $Q + \delta Q \in \mathbb{C}^{n\times n}$ with Q and δQ positive definite and denoting $\overline{k} = \lambda_{\max}(A^* A)$, $\underline{k} = \lambda_{\min}(A^* A)$, $\overline{q} = \lambda_{\max}(Q)$, $\underline{q} = \lambda_{\min}(Q)$, with (α, β) – the solution of the system $\alpha = \underline{q} + \underline{k}\beta^{-t}$, $\beta = \overline{q} + \overline{k}\alpha^{-t}$, $\varepsilon = (\beta + p\underline{q} - p\beta)\alpha^{1+t} + \beta^2\alpha^{t-1}\|\delta Q\|_2$, $\zeta = \dfrac{\varepsilon^2}{4\beta^3\alpha^t} - \alpha^t\|\delta Q\|_2$. If

(10) $$\|\delta A\|_2 \le \sqrt{\|A^2\|_2 + \zeta} - \|A\|_2,$$

then

(11) $$\frac{\|\delta X\|_2}{\|X\|_2} \le \xi_*,\ \xi_* = \varsigma\|\delta A\|_2 + \omega\|\delta Q\|_2,$$

where

$$\varsigma = \frac{2\left(2\|A\|_2 + \|\delta A\|_2\right)\beta}{\varepsilon + \sqrt{\varepsilon^2 - 4\alpha^t\beta^3\left(\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2 + \alpha^t\|\delta Q\|_2\right)}},$$

$$\omega = \frac{2\alpha^t\beta}{\varepsilon + \sqrt{\varepsilon^2 - 4\alpha^t\beta^3\left(\left(2\|A\|_2 + \|\delta A\|_2\right)\|\delta A\|_2 + \alpha^t\|\delta Q\|_2\right)}}.$$

As it is seen from the expressions above, the perturbation bound ξ_* (11) and the condition (10) of its existence does not use any knowledge of the solution X or $X + \delta X$ of the unperturbed and the perturbed equations. This allows the analyzing of the sensitivity of the equation before solving it, which is an undeniable advantage.

3.4. The bound of J i a, W e i [4]

Considering the real matrix equation $X^s + A^{\mathrm{T}} X^t A = Q$, where s, t are both nonnegative integers, in [4] Jia and Wei make algebraic perturbation analysis of its unique symmetric positive definite solution with respect to perturbations in the data matrices A and Q. The following perturbation bound is obtained for equation $X^s + A^{\mathrm{T}} X^t A = Q$ with unique solution $X \in [\beta_1 I, \alpha_1 I]$,

(12) $$\|A\|_2^2 < s\tilde{\beta}_1^{s-1}\left(t\tilde{\alpha}_1^{t-1}\right)^{-1}$$

and unique solution $X + \delta X \in \left[\hat{\beta}_1 I,\ \hat{\alpha}_1 I\right]$ to the corresponding perturbed equation. For any arbitrary $\varepsilon > 0$, if

(13) $$\|\delta A\|_{\mathrm{F}} < 2\varepsilon\tau\left(\tilde{\beta}_1, \tilde{\alpha}_1\right)\left(3\eta(\varepsilon)\|X + \delta X\|_2^t\right)^{-1}$$

with $\eta(\varepsilon) = \|A\|_2 + \left(\|A\|_2^2 + \frac{2\varepsilon}{3}\tau\left(\widetilde{\beta}_1, \widetilde{\alpha}_1\right) \|X + \delta X\|_2^{-t} \right)^{\frac{1}{2}}$,

$$\tau(\widetilde{\beta}, \widetilde{\alpha}) = s\widetilde{\beta}^{s-1} - t\widetilde{\alpha}^{t-1}\|A\|_2^2$$

and

(14) $$\|\delta Q\|_{\mathrm{F}} < \frac{1}{3}\tau\left(\widetilde{\beta}_1, \widetilde{\alpha}_1\right)\varepsilon ,$$

then

(15) $$\|\delta X\|_{\mathrm{F}} < \varepsilon .$$

Here $\widetilde{\alpha}_1 = \max(\alpha_1, \hat{\alpha}_1)$, $\widetilde{\beta}_1 = \max(\beta_1, \hat{\beta}_1)$, where $\alpha_1, \beta_1, \hat{\alpha}_1, \hat{\beta}_1$ are the unique positive roots of

$$g_1(x) = x^s + \lambda_{\min}(A^{\mathrm{T}}A)x^t - \lambda_{\max}(Q),\ g_2(x) = x^s + \lambda_{\max}(A^{\mathrm{T}}A)x^t - \lambda_{\min}(Q),$$

$$\hat{g}_1(\hat{x}) = \hat{x}^s + \lambda_{\min}((A+\delta A)^{\mathrm{T}}(A+\delta A))\hat{x}^t - \lambda_{\max}(Q+\delta Q),$$

$$\hat{g}_2(\hat{x}) = \hat{x}^s + \lambda_{\max}((A+\delta A)^{\mathrm{T}}(A+\delta A))\hat{x}^t - \lambda_{\min}(Q+\delta Q),$$ respectively.

3.5. The bounds of K o n s t a n t i n o v, P e t k o v, P o p c h e v, A n g e l o v a [5]

K o n s t a n t i n o v et al. [5] consider the sensitivity of the nonlinear matrix equation (1) with *s* and *t* real numbers. Using the technique of Frechet derivatives and applying the method of Lyapunov majorants and the Scauder fixed point principle, local and non-local perturbation bounds for the positive definite solution of equation (1) are obtained. Explicit expressions of local perturbation bounds, when *s, t* = ± *r*, ±1/2, 1/3, ± 1/*p*, ± *r/p* ($r, p \in \mathbb{N}$) and a non-local perturbation bound, when *s, t* = ± *r*, ±1/2, 1/3, 1/*p* ($r, p \in \mathbb{N}$) are proposed.

- Local bound

(16) $$\|\delta X\|_{\mathrm{F}} < \mathrm{est}(\delta),$$

$$\mathrm{est}(\delta) = \min\left\{ \left\| \begin{bmatrix} W_Q^R & M_A \end{bmatrix} \right\|_2 \|\delta\|_2, \sqrt{\delta^{\mathrm{T}} R \delta} \right\} + O\left((\delta_x + \delta)^2\right),$$

where $\delta = \begin{bmatrix} \|\delta A\|_F & \|\delta Q\|_F \end{bmatrix}^{\mathrm{T}}$, $\delta_x = \|\delta X\|_{\mathrm{F}}$. Here $W_Q^R = \begin{bmatrix} W_{Q0} & -W_{Q1} \\ W_{Q1} & W_{Q0} \end{bmatrix}$ is the real representation of the matrix $W_Q = W_{Q0} + iW_{Q1} = L^{-1}$ with $L = L_s \pm (A^{\mathrm{T}} \otimes A^{\mathrm{H}})L_t$.

The matrix $M_A = \begin{bmatrix} W_{A0} + W_{\overline{A}0} & W_{\overline{A}1} - W_{A1} \\ W_{A1} + W_{\overline{A}1} & W_{A0} - W_{\overline{A}0} \end{bmatrix}$ is composed from the matrices $W_A = W_{A0} + iW_{A1} = -L^{-1}(I \otimes A^H X^t)$, $W_{\overline{A}} = W_{\overline{A}0} + W_{\overline{A}1} = -L^{-1}((X^t A^H \otimes I)\mathrm{P}_{n^2}$ and the matrix *R* is given by $R = \begin{bmatrix} \|(W_Q^{\mathrm{R}})^{\mathrm{T}} W_Q^{\mathrm{R}}\|_2 & \|(W_Q^{\mathrm{R}})^{\mathrm{T}} M_A\|_2 \\ \|MW_A)^{\mathrm{T}} W_Q^{\mathrm{R}}\|_2 & \|M_A{}^{\mathrm{T}} M_A\|_2 \end{bmatrix}$.

- Non-local bound

Under the condition on the perturbations in the data

(17) $$\delta \in \Omega = \left\{ \delta \geq 0 : a_1(\delta) + 2\sqrt{a_0(\delta) a_2(\delta)} \leq 0 \right\}$$

for the Frobenius norm of the perturbation in the solution is valid the non-local bound

(18) $$\|\delta X\|_{\mathrm{F}} \leq f(\delta), \quad f(\delta) = \frac{2a_0}{1 - a_1 + \sqrt{(1-a_1)^2 - 4a_0 a_2}},$$

where $a_0(\delta) = est(\delta) + \left\|L^{-1}\right\|_2 \left\|X^t\right\|_2 \delta_A^2$, $\quad a_1(\delta) = \left\|L^{-1}\right\|_2 \|L_t\|_2 \left(2\|A\|_2 \delta_A + \delta_A^2\right)$, $a_2(\delta) = \left\|L^{-1}\right\|_2 \left(\|A\|_2 + \delta_A\right)^2 \varphi(t, X)$. The expressions of L_t and $\varphi(t,X)$ for $t = \pm r$, $t = \pm 1/2$, $t = 1/3$, $t = \pm 1/p$, $t = \pm r/p$, when $r, p \in \mathbb{N}$ are listed below

Table 1

t	L_t	$\varphi(t, X)$
r	$\sum_{k=0}^{r-1} \left(X^k\right)^{\mathrm{T}} \otimes X^{r-1-k}$	$\sum_{j=2}^{r} \left\| \sum_{i=j}^{r} \left(X^{r-i}\right)^{\mathrm{T}} \otimes X^{j-2} \right\|_2 \left\|X^{j-2}\right\|_2$
$-r$	$-\sum_{k=0}^{r-1} \left(X^{k-r}\right)^{\mathrm{T}} \otimes X^{-1-k}$	$\sum_{j=1}^{r} \left\| \sum_{i=1}^{r-j+1} \left(X^{-r+i+j-2}\right)^{\mathrm{T}} \otimes X^{-i} \right\|_2 \left\|X^{-j}\right\|_2 \leq$ $\leq \left(2r - 1 + \binom{r}{2}\right) \left\|X^{-1}\right\|_2^{r+2}$
1/2	$I \otimes X + X^{\mathrm{T}} \otimes I$	$\dfrac{2\left\|L_{1/2}\right\|_2^3}{1 - 2\left\|L_{1/2}\right\|_2^2 \delta_x + \sqrt{1 - 4\left\|L_{1/2}\right\|_2^2 \delta_x}}$
−1/2	$\left(-\left(X^{-1}\right)^{\mathrm{T}} \otimes X^{-1/2} - \left(X^{-1/2}\right)^{\mathrm{T}} \otimes X^{-1}\right)^{-1}$	$\frac{3}{8} \left\| \left(X^{-1}\right)^{\mathrm{T}} \otimes X^{-1/2} \right\|_2 \left\|X^{-1}\right\|_2$
1/3	$I \otimes X^2 + X^{\mathrm{T}} \otimes X + (X^2)^{\mathrm{T}} \otimes I$	$3\|X\|_2^{1/3} \|L_{1/3}\|_2^3$
$1/p$	$\left(\sum_{k=0}^{p-1} (X^{k/p})^{\mathrm{T}} \otimes X^{\frac{p-k-1}{p}} \right)^{-1}$	$\left(\|X\|_2^{\frac{p-2}{p}} \left\|L_{1/p}\right\|_2^3 p(p-1) \right) / 2$
$-1/p$	$\left(-\sum_{k=0}^{p-1} (X^{\frac{k-p}{p}})^{\mathrm{T}} \otimes X^{-\frac{k+1}{p}} \right)^{-1}$	
r/p	$\left(\sum_{k=0}^{r-1} \left(X^{k/p}\right)^{\mathrm{T}} \otimes X^{\frac{r-k-1}{p}} \right) L_{1/p}$	
$-r/p$	$-\left(\left(X^{-r/p}\right)^{\mathrm{T}} \otimes X_{-r/p} \right) L_{r/p}$	

4. Numerical examples

Four models are used for the numerical experiments. Example 1 (Example 1 from [3]) and Example 2 (Example 2 from [2]) are preferred in the literature, when analysing the effectiveness of perturbation bounds for equation (1). Examples 3 and 4 are modifications of Example 1 form [3] for the case of a complex equation (1). The computations are performed on a PC with 2.61 GHz Pentium Dual-Core using MATLAB (MATLAB is a trade mark of MathWorks). To facilitate the analysis of the results in the tables the ratio of the corresponding perturbation bound to the estimated value is given, for example $\xi_{\text{err1}} / \|\delta X\|_{\text{F}}$. Unit value means the full match of the bound and the estimated value. A value above unite indicates how many times the bound exceeds the estimated value. When the necessary assumptions are violated, the results are noted by asterisk.

4.1. Example 1 [3]

Let for the real matrix equation

$$X^s \pm A^{\text{T}} X^t A = Q \quad A = \frac{2\sqrt{3}}{45}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix},$$

$$X = \operatorname{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1), \; Q = X^s \pm A^{\text{T}} X^t A.$$

The perturbed matrix equation is

$$(X + \delta X)^s \pm (A + \delta A)^{\text{T}} (X + \delta X)^t (A + A) = Q + \delta Q,$$

with $\delta A = e_j (I_5 + E)$, $\delta A = e_j (I_5 - E)$, $e_j = 0.1^{2j}$ for j = 2, 3, 4, 5,

$\delta Q = (X + \delta X)^s \pm (A + \delta A)^{\text{T}} (X + \delta X)^t (A + A) - Q,$ and

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Case 4.1.1. Consider equation $X^s + A^{\text{T}} X^t A = Q$, with s = 1 and t = –2 (Case 1 of Example 4.2 [9]). The perturbation in the solution is estimated by the bounds of Yin, Liu, Fang ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5) from Subsection 3.1 and the bounds of Konstantinov et al. $\text{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5 for j = 2, 3, 4, 5. The corresponding conditions for existence (6) and (17) are satisfied. As it is seen from Table 4.1.1 the bounds est(δ) (16), $f(\delta)$ (18) give the tightest results for this example.

Then we modify the example (as in Example 1 from [3]) by choosing the diagonal matrix $X = \mathrm{diag}(0.25 \quad 2 \quad 3 \quad 2 \quad 1)$ for the solution of the unperturbed equation (1). The numerical experiments show that the bounds ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5) cannot be used, because their necessary assumptions (6) are violated. The condition of existence (17) of the bound $f(\delta)$ (18) is violated for j = 2. An asterisk in Table 2 notes this. The bound $\mathrm{est}(\delta)$ (16), as well as the bound $f(\delta)$ (18) for j = 3, 4, 5 are still working, but are quite conservative.

Table 2. Case (4.1.1): $X^s + A^{\mathrm{T}} X^t A = Q$, with s = 1, t = – 2.
Ratio of the estimated value and the perturbation bound

j	j=2	j=3	j=4	j=5
$X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$				
$\xi_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	2.3514	2.3505	2.3505	2.3505
$\xi_{\mathrm{err2}} / \lVert \delta X \rVert_{\mathrm{F}}$	34.7406	35.4032	35.4100	35.4101
$\eta_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	2.3502	2.3505	2.3505	2.3505
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	1.7022	1.7012	1.7012	1.7012
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	1.7078	1.7013	1.7012	1.7012
$X = \mathrm{diag}(0.25 \quad 2 \quad 3 \quad 2 \quad 1)$				
$\xi_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	*	*	*
$\xi_{\mathrm{err2}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	*	*	*
$\eta_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	*	*	*
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	23.874	23.630	23.628	23.628
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	*	24.217	23.634	23.628

Case 4.1.2. Consider equation $X^s + A^{\mathrm{T}} X^t A = Q$, with s = 2 and t = –2 and $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$ (Case 2 of Example 4.2 [9]). The results for j = 2, 3, 4, 5 are given in Table 3. The conditions (6) and (17) of existence of the bounds are satisfied. The bound $\xi_{\mathrm{err2}} / \lVert \delta X \rVert_{\mathrm{F}}$ is considerably more conservative than the others bounds considered in the example. The bounds $\xi_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$, $\eta_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$, $\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$ and $f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$ are relatively sharp.

Table 3. Case $X^s + A^{\mathrm{T}} X^t A = Q$, with s = 2, t = –2. Ratio of perturbation bounds and estimated value

j	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\xi_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	3.7339	3.7340	3.7340	3.7340
$\xi_{\mathrm{err2}} / \lVert \delta X \rVert_{\mathrm{F}}$	18.9405	19.0049	19.0056	19.0056
$\eta_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	3.7334	3.7340	3.7340	3.7340
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.9619	2.9615	2.9615	2.9615
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.9710	2.9615	2.9615	2.9615

Case 4.1.3. Consider the real matrix equation $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 1, $t = -1/2$. Let the solution of the equation be $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$ and then $X = \mathrm{diag}(0.25 \quad 2 \quad 3 \quad 2 \quad 1)$. The sensitivity of this equation is studied by Li, Zhang [6] and Konstantinov et al. [5]. Therefore here the effectiveness of the perturbation bounds ξ_* (11) from Subsection 3.3 and est(δ) (16), $f(\delta)$ (18) from Subsection 3.5 is analysed [5]. The conditions (10) and (17) of existence of the bounds are satisfied. As it is seen, the perturbation bound ξ_* (11) is more conservative than the bounds est(δ) (16), $f(\delta)$ (18), when $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$ and deteriorates significantly, when $X = \mathrm{diag}(0.25 \quad 2 \quad 3 \quad 2 \quad 1)$ while the bounds est(δ) (16), $f(\delta)$ (18) stay sharp (Table 4).

Table 4. Case $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 1, $t = -1/2$. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$				
$\xi_* / \Vert\delta X\Vert_2$	13.0637	12.9448	12.9436	12.9436
$\mathrm{est}(\delta) / \Vert\delta X\Vert_F$	1.5868	1.5866	1.5866	1.5866
$f(\delta) / \Vert\delta X\Vert_{\mathrm{F}}$	1.5893	1.5866	1.5866	1.5866
$X = \mathrm{diag}(0.25 \quad 2 \quad 3 \quad 2 \quad 1)$				
$\xi_* / \Vert\delta X\Vert_2$	85.1575	64.6569	64.5465	64.5454
$\mathrm{est}(\delta) / \Vert\delta X\Vert_{\mathrm{F}}$	1.7000	1.6982	1.6982	1.6982
$f(\delta) / \Vert\delta X\Vert_{\mathrm{F}}$	1.7043	1.6983	1.6982	1.6982

Case 4.1.4. Consider equation $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 2, t = –3 and $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. The perturbation in the solution is estimated by the perturbation bounds E_{rr1} (7) and E_{rr2} (8) from Subsection 3.2 proposed by Yin, Liu [8] and the bounds of Konstantinov et al. [5] est(δ) (16), $f(\delta)$ (18) from Subsection 3.5. The results for j=2, 3, 4, 5 are given in Table 5.

Table 5. Case $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 2, t = –3. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$E_{\mathrm{rr1}} / \Vert\delta X\Vert_{\mathrm{F}}$	5.4561	5.4571	5.4571	5.4571
$E_{\mathrm{rr2}} / \Vert\delta X\Vert_{\mathrm{F}}$	6.1619	6.1619	6.1619	6.1619
$\mathrm{est}(\delta) / \Vert\delta X\Vert_{\mathrm{F}}$	2.9001	2.8995	2.8995	2.8995
$f(\delta) / \Vert\delta X\Vert_{\mathrm{F}}$	2.9150	2.8996	2.8995	2.8995

Case 4.1.5. Consider $X^s + A^{\mathrm{T}} X^t A = Q$, with $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. Perturbation bounds for this case of equation (1) are given by J i a and W e i – the bound (15) from Subsection 3.4 and by K o n s t a n t i n o v e t a l. [5] – the bounds $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5. Expression

(19) $$\|\delta X\|_{\mathrm{F}} \le \varepsilon_*,$$

$$\varepsilon_* := (\tau(\widetilde{\beta}_1, \widetilde{\alpha}_1))^{-1} (2\|A\|_2 \|X + \delta X\|_2^t \|\delta A\|_{\mathrm{F}} + \|X + \delta X\|_2^t \|\delta A\|_{\mathrm{F}}^2 + \|\delta Q\|_{\mathrm{F}})$$

from [4] is used to calculate the bound of Jia and Wei. The results for j=2, 3, 4, 5, s=2, t=3 and for s=3, t=2 are given in Table 6. The submission (12) is not fulfilled, when s=2, t=3. The necessary conditions (13)-(15) are violated and the bound (15) of Jia and Wei cannot be computed. All the conditions of existence are satisfied, when s=3, t=2, but the sharpness of the bound ε_* of Jia and Wei, obtained by expression (19) varies for the different ranges of the perturbations in the data.

Table 6. Case $X^s + A^{\mathrm{T}} X^t A = Q$, with $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$s = 2,\ t = 3$				
$\varepsilon_* / \|\delta X\|_{\mathrm{F}}$	*	*	*	*
$\mathrm{est}(\delta) / \|\delta X\|_{\mathrm{F}}$	4.6438	4.6466	4.6466	4.6466
$f(\delta) / \|\delta X\|_{\mathrm{F}}$	4.6867	4.6470	4.6466	4.6466
$s = 3,\ t = 2$				
$\varepsilon_* / \|\delta X\|_{\mathrm{F}}$	25.2187	5.4619	2.3918	1.3427
$\mathrm{est}(\delta) / \|\delta X\|_{\mathrm{F}}$	7.8299	7.8321	7.8321	7.8321
$f(\delta) / \|\delta X\|_{\mathrm{F}}$	8.2443	7.8358	7.8321	7.8321

Case 4.1.6. Consider equation $X^s - A^{\mathrm{T}} X^t A = Q$, with s=1, t=3/4 and s=1, t=1/3. Let $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. Only K o n s t a n t i n o v et al. [5] propose perturbation bounds for this case of equation (1). The results of the bounds from Subsection 3.5 $\mathrm{est}(\delta)$ (16), when s=1, t=3/4 and $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18), when s=1, t=1/3 for j=2, 3, 4, 5 are given in Table 7. As it is seen, the bounds are quite tight.

Table 7. Case $X^s - A^{\mathrm{T}} X^t A = Q$, with $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
s=1, t=3/4				
$\mathrm{est}(\delta) / \|\delta X\|_{\mathrm{F}}$	2.2888	2.2881	2.2881	2.2881
s=1, t=1/3				
$\mathrm{est}(\delta) / \|\delta X\|_{\mathrm{F}}$	1.8707	1.8703	1.8703	1.8703
$f(\delta) / \|\delta X\|_{\mathrm{F}}$	1.8723	1.8703	1.8703	1.8703

4.2. Example 2

Consider Example 2 from [2] for the real matrix equation $X^s \pm A^{\mathrm{T}} X^t A = Q$.

Let $A = \frac{d_k}{\|A_0\|} A_0$, where $d_k = \frac{19}{20} - 10^{-k}$, k=2, 3, and $A_0 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

Let $X = \begin{bmatrix} 2.5 & 1 & 1 & 1 & 1 \\ 1 & 2.5 & 1 & 1 & 1 \\ 1 & 1 & 2.5 & 1 & 1 \\ 1 & 1 & 1 & 2.5 & 1 \\ 1 & 1 & 1 & 1 & 2.5 \end{bmatrix}$ and $Q = X^s - A^{\mathrm{T}} X^t A$.

Consider the perturbed matrix equation

$$(X + \delta X)^s \pm (A + \delta A)^{\mathrm{T}} (X + \delta X)^t (A + A) = Q + \delta Q,$$

where $\delta A = 10^{-j} \frac{C^* + C}{\|C^* + C\|}$, j = 2, 3, 4, 5, C is a random matrix generated by the MATLAB function **randn,**

$$\delta X = 10^{-j} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and $\delta Q = (X + \delta X)^s \pm (A + \delta A)^{\mathrm{T}} (X + \delta X)^t (A + A) - Q.$

Case 4.2.1. Consider the real matrix equation $X^s + A^{\mathrm{T}} X^t A = Q$ with s = 1, t = –2 and then with s = 2, t = –2. The effectiveness of the bounds ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5) from Subsection 3.1 and the bounds $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5 is analyzed. The results for i = 2, 3, j = 2, 3, 4, 5 are listed in Table 8. The necessary conditions (6) are violated, when s = 1, t = –2 and the bounds of Yin, Liu, Fang cannot be used. All the conditions of existence are satisfied for s = 2, t = –2. As it is seen the perturbation bound ξ_{err2} (4) is quite conservative.

Table 8. Case $X^s + A^{\mathrm{T}} X^t A = Q$, with s=1, t –2 and s=2, t=–2. Ratio of the estimated value and the perturbation bound

j	j=2	j=3	j=4	j=5
$s = 1, t = -2$				
$k = 2$				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	1.7939	1.7939	1.7939	1.7939
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	1.7956	1.7939	1.7939	1.7939
$k = 3$				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	1.8191	1.8191	1.8191	1.8191
$f(\delta)/\lVert\delta X\rVert_{F}$	1.8191	1.8191	1.8191	1.8191
$s = 2, t = -2$				
$k = 2$				
$\xi_{\mathrm{err1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.3240	5.3238	5.3237	5.3237
$\xi_{\mathrm{err2}}/\lVert\delta X\rVert_{\mathrm{F}}$	20.4811	20.4802	20.4802	20.4802
$\eta_{\mathrm{err1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.3240	5.3238	5.3237	5.3237
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.9534	4.9532	4.9532	4.9532
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.9607	4.9533	4.9532	4.9532
$k = 3$				
$\xi_{\mathrm{err1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.3456	5.3456	5.3455	5.2022
$\xi_{\mathrm{err2}}/\lVert\delta X\rVert_{\mathrm{F}}$	21.4253	21.4253	21.4251	20.8507
$\eta_{\mathrm{err1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.6161	5.6161	5.6161	5.4551
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.9654	4.9654	4.9654	4.8323
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.9655	4.9654	4.9654	4.8323

Case 4.2.2. Consider the real matrix equation $X^s - A^{\mathrm{T}} X^t A = Q$ with s = 2, t = –3. The perturbation bounds E_{rr1} (7), E_{rr2} (8) from Subsection 3.2 and $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5 are compared. The conditions (7) and (17) of existence of the bounds are satisfied. The results are too close (Table 9).

Table 9. Case $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 2, t = –3.
Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
k=2				
$E_{\mathrm{rr1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.3012	5.3010	5.3010	5.3010
$E_{\mathrm{rr2}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.7760	5.7758	5.7758	5.7758
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.3707	4.3705	4.3705	4.3705
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.3763	4.3706	4.3705	4.3705
k=2				
$E_{\mathrm{rr1}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.3232	5.3232	5.3232	5.1769
$E_{\mathrm{rr2}}/\lVert\delta X\rVert_{\mathrm{F}}$	5.8174	5.8174	5.8173	5.6575
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.3709	4.3709	4.3709	4.3709
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	4.3710	4.3709	4.3709	4.3708

Case 4.2.3. Consider equation $X^s - A^{\mathrm{T}} X^t A = Q$, with s = 1, t = 3/4 and s = 1, t = 1/3. Only K o n s t a n t i n o v et al. [5] propose perturbation bounds for this case of equation (1). The results of the perturbation bounds $\mathrm{est}(\delta)$ (16), when s = 1,

$t = 3/4$ and $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18), when $s = 1$, $t = 1/3$ for $j = 2, 3, 4, 5$ are given in Table 10.

Table 10. Case $X^s - A^{\mathrm{T}} X^t A = Q$, with $s = 1$, $t = 3/4$ and $s = 1$, $t = 1/3$. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
s=1, t=3/4				
k=2				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	5.3542	5.3540	5.3540	5.3540
k=3				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	5.5241	5.5241	5.5236	5.4609
s=1, t=1/3				
k=2				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.2093	2.2093	2.2093	2.2093
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.2099	2.2093	2.2093	2.2093
k=3				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.2985	2.2985	2.2983	2.3120
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.2985	2.2985	2.2983	2.3120

4.3. Example 3

Let modify Example 1 form Subsection 4.1 by choosing the matrix A as

$$A = \frac{2\sqrt{3}}{45}\begin{bmatrix} 1+i & 0 & 0 & 0 & 1+i \\ -1-i & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1+i \end{bmatrix},$$

where i is the imaginary unit. Let the solution X of the equation be $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$.

Case 4.3.1. Consider the complex matrix equation $X^s + A^{\mathrm{H}} X^t A = Q$, with $s = 1$ and $t = -2$. Perturbation bounds for this type of equation (1) are the proposed by Y i n. L i u and F a n g in [9] – bounds ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5) described in Subsection 3.1 and the local and non-local bounds of K o n s t a n t i n o v et al. [5] – $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5. The necessary conditions (6) for the existence of the bounds ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5) are violated for $j = 2$. The bounds cannot be used and asterisks in Table 11 note this fact. The bound ξ_{err2} (4) give complex values for $j = 3, 4, 5$, although the conditions (6) are satisfied. The other bounds give satisfactorily estimate of the perturbation in the solution.

Table 11. Case $X^s + A^{\mathrm{H}} X^t A = Q$, with s=1, t=–2. Ratio of the estimated value and the perturbation bound

j	j=2	j=3	j=4	j=5
$\xi_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	2.8723	2.8717	2.8717
$\xi_{\mathrm{err2}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	*	*	*
$\eta_{\mathrm{err1}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	2.8716	2.8717	2.8717
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.1273	2.1267	2.1267	2.1267
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.1349	2.1268	2.1267	2.1267

Case 4.3.2. Consider the complex matrix equation $X^s - A^{\mathrm{H}} X^t A = Q$, with s = 1 and t = –1/2. The effectiveness of the bound ξ_* (11) of Li, Zhang described in Subsection 3.3 and the bounds of Konstantinov et al. [5] $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) described in Subsection 3.5 is analysed. All the conditions of existence are satisfied. As it is seen from Table 12 the estimates of Konstantinov et al. are very close.

Table 12. Case $X^s - A^{\mathrm{H}} X^t A = Q$, with s=1, t = –1/2. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$\xi_* / \lVert \delta X \rVert_2$	13.7451	13.6101	13.6088	13.6088
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	1.6376	1.6372	1.6372	1.6372
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	1.6402	1.6373	1.6372	1.6372

Case 4.3.3. Consider the complex matrix equation $X^s - A^{\mathrm{H}} X^t A = Q$, with s = 2 and t = –3. The bounds of Yin, Liu [8] and of Konstantinov et al. [5] refer to this case. The results for E_{rr1} (7), E_{rr2} (8) from Subsection 3.2 and $\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5, when j = 2, 3, 4, 5 are listed in Table 13. The conditions (7) and (17) of existence of the bounds are satisfied except condition (7), when j = 2.

Table 13. Case $X^s - A^{\mathrm{H}} X^t A = Q$, with s=2, t = –3. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$E_{\mathrm{rr1}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	5.7872	5.7873	5.7873
$E_{\mathrm{rr2}} / \lVert \delta X \rVert_{\mathrm{F}}$	*	6.5616	6.5616	6.5616
$\mathrm{est}(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.8693	2.8689	2.8689	2.8689
$f(\delta) / \lVert \delta X \rVert_{\mathrm{F}}$	2.8842	2.8690	2.8689	2.8689

4.4. Example 4

Modify Example 1 form Subsection 4.1 so that the matrix A is chosen to be A=A_0+iA_0, where

$$A_0 = \frac{2\sqrt{3}}{45}\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

and i is the imaginary unit. Let $X = \mathrm{diag}(0.725 \quad 2 \quad 3 \quad 2 \quad 1)$. The bounds described in Subsections 3.1, 3.2, 3.3 and 3.5 are considered. Unfortunately the conditions of existence (6) for the bounds of Yin, Liu, Fang [9] (ξ_{err1} (3), ξ_{err2} (4), η_{err1} (5), Subsection 3.1) and the conditions (9) for the bounds of Yin, Liu [8] (E_{rr1} (7), E_{rr2} (8) from Subsection 3.2) are violated and these bounds cannot be used. The results obtained for the bounds of Li, Zhang [6] (ξ_* (11), Subsection 3.3) and Konstantinov et al. [5] ($\mathrm{est}(\delta)$ (16), $f(\delta)$ (18) from Subsection 3.5) are given in Table 14.

Table 14. Complex case A=A_0+iA_0. Ratio of perturbation bound and estimated value

j	j=2	j=3	j=4	j=5
$X^s + A^{\mathrm{H}} X^t A = Q$, $s = 1$, $t = -2$				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.6709	2.6697	2.6697	2.6697
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	2.6818	2.6698	2.6697	2.6697
$X^s - A^{\mathrm{H}} X^t A = Q$, $s = 1$, $t = -1/2$				
$\xi_*/\lVert\delta X\rVert_2$	16.7957	16.5825	16.5805	16.5805
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	1.9138	1.9133	1.9133	1.9133
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	1.9174	1.9134	1.9133	1.9133
$X^s - A^{\mathrm{H}} X^t A = Q$, $s = 2$, $t = -3$				
$\mathrm{est}(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	3.2615	3.2614	3.2614	3.2614
$f(\delta)/\lVert\delta X\rVert_{\mathrm{F}}$	3.2900	3.2617	3.2614	3.2614

6. Concluding remarks

In this paper the effectiveness of the perturbation bounds proposed in 5 issues for the real and the complex equations $X^s \pm A^* X^t A = Q$ is analysed. The comparison is made on the base of several numerical examples. The results of the experimental analysis show that for the given class of problems the bounds proposed by

Konstantinov et al. [5] for estimating the sensitivity of the solution to equation (1) is superior to the methods of Jia, Wei [4], Li, Zhang [6], Yin, Liu [8] and Yin, Liu, Fang [9] with respect to closeness to the estimated quantity and comprehensive application. The bound ξ_{err2} (4) of Yin, Liu and Fang [9] is quite conservative, when estimating the sensitivity of the real equation and unusable in the complex case of the models considered. The behaviour observed and analysed properties of the bounds considered in the paper hold true for every problem, which belongs to the class of the experimental models used.

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