A Heuristic Method for Obtaining Upper and Lower Bounds for the Minimum of a Linear Function over the Efficient Set

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Abstract: *The paper proposes the use of the Reference Point (RP) method in a heuristic approach for estimating the minimal value of a linear function over the efficient set of a MultiObjective Linear Programming (MOLP) problem.*

Keywords: *Multiobjective optimization problem, optimization over the efficient set.*

1. Introduction

The Multiobjective Optimization Problems (MOOP) are used as fruitful models in a wide range of areas. As a consequence, the problem of optimizing a real function over the efficient set of a MOOP has various applications. The number of papers concerning the optimization over the efficient set increases constantly. The paper of Y a m a m o t o (2002) gives a clear look over this class of problems. As some examples we will propose here two papers. The paper of H o r s t, T h o a i et al. (2007) considers the problem of optimizing over the efficient set of a multiobjective linear programming problem as a type of global optimization techniques, called reverse convex programming. The authors propose a method for constructing a concave function that can help in the procedure for optimization over the efficient set. The paper of J o r g e (2005) presents an algorithm that gives a series of values that monotonically increase and finally gives an exact solution of the considered problem. The algorithm uses a disjoint bilinear program and the solution is found by the use of specifically designed method for nonconvex optimization. The purpose of the presented herein paper is to draw attention to the possibility to use some more simple techniques for treatment of the linear case.

2. Some notions, some definitions, some basic facts

Let $x \in R^n$. The feasible set $S \subset R^n$ in the MultiObjective Linear Programming (MOLP) problem is defined by a system of linear constraints:

$$S = \{x \in R^n | c_j(x) \leq q_j; \ j = 1, 2, \ldots, m\}.$$

S is a convex set. In this paper S will be a bounded and closed set. Some linear functions $f_i(x)$, $x \in R^n$, $i = 1, 2, \ldots, k$, are given, too. The MOLP problem can be written in the following manner

(1) $\qquad\qquad\qquad$ "max"$f_i(x)$, $i = 1, 2, \ldots, k$,

s.t.

$$x \in S.$$

The sense of this formulation is that we are looking for x such that the corresponding $f_i(x)$ are as high as possible.

For a fixed $x \in S$ we obtain the vector $z = f(x)$; here z is a image of x and x is the corresponding original. So

$$Z = \{z \in R^k \mid z = f(x), x \in S\}.$$

The set Z is the set of all images of x, when $x \in S$; S is a polyhedron, Z is a polyhedron, too, with the same properties. We write $Z = f(S)$.

Having in mind the MOLP problem (1) let us consider $z^1 \in Z$ and $z^2 \in Z$. Then z^1 dominates z^2 if $z^1 \geq z^2$ and $z^1 \neq z^2$, i.e. $z_i^1 \geq z_i^2$ for all i and $z_i^1 > z_i^2$ for one i at least. Let $z^{nd} \in Z$, then z^{nd} is nondominated if there does not exist another vector $z \in Z$, such that $z \geq z^{nd}$ and $z \neq z^{nd}$.

The point $x^e \in S$ is efficient if there does not exist another vector $x \in S$, such that $f(x) \geq f(x^e)$ and $f(x) \neq f(x^e)$. If x^e is efficient this means that the vector $z = f(x^e)$ is nondominated.

The set of all efficient points $x \in S$ is denoted by E, $E = \{x \in S | x \text{ is an efficient point}\}$. In some cases it is possible that $E = S$, but our case is $E \subset S$. In MOLP problems the set E is connected, but E is not convex in the general case. E is bounded, because S is bounded.

Here we shall use the term a wall W_i of the set S. So

$$W_i = \{x \in S | c_i(x) = q_i\} \text{ for a chosen } i.$$

W_i is a part of the boundary of S. The set E consists of parts that belong to different walls of S. Set E contains points that belong simultaneously to different walls of S.

As an addition to the MOLP problem we have a linear function $\varphi(x)$, $x \in R^n$. In this paper we consider the problem $\min\{\varphi(x) | x \in E\}$. We denote this minimal value by d,

$$\min_{x \in E.} \varphi(x) = d,$$

This problem has an exact solution because E is bounded. The difficulty with this problem is that E is not convex.

Following W i e r z b i c k i (1980) – with respect to the MOLP problem (1) – the reference point method recommends to solve the following linear programming problem

(2)   min D

s.t.

$$D > b_i \, (r_i - f_i(x)) - l\Sigma_i \, f_i \, (x), \; i = 1, 2, \ldots, k,$$

$$x \in S.$$

Here the set S and the functions $f_i \, (x)$ are defined as in definition of the MOLP problem, l is a small positive number and all b_i are positive real numbers. The variable D is unrestricted in sign. This LP problem has a well known and remarkable property**:** for an arbitrary reference point $r \in R^k$ the obtained solution determines an efficient point from S.

This paper proposes a heuristic approach for obtaining upper and lower bounds for the number d. For this purpose the following polyhedron S_1 will be used:

$$S_1 = \{x \in R^n \,|c_j(x) \le q_j + \delta, \; j = 1, 2, \ldots, m\}.$$

Here δ is a (relatively) small positive number. $S \subset S_1$, obviously. We define a wall of S_1 as follows:

$$W'_j = \{x \in S_1|c_j(x) = q_j + \delta \} \; \text{ for a chosen } j.$$

The wall W_i (of S) and the wall W'_i (of S_1) are "corresponding".

The wall V_i of $Z = f(S)$ is

$$V_i = \{z \in Z| \; z = f \, (x), \; x \in W_i \subset S\}.$$

By analogy we have

$$V'_i = \{z \in Z_1 = f(S_1)| \; z = f(x), \; x \in W'_i \subset S_1\}.$$

3. Some additional data and a description of the method

Having in mind that d denotes the searched minimal value of φ, we choose a number $d_1 < d$. In what follows we will consider the set Q:

$$Q = \{x \in S \mid \varphi(x) \le d_1\}.$$

For convenience we will use the symbol E_S to denote the set E of all efficient points of S. By analogy the symbol E_Q denotes the set of all efficient points of Q. By W_Q we will denote the following wall:

$$W_Q = \{x \in S \mid \varphi \, (x) = d_1\} = \{x \in Q| \, \varphi(x) = d_1\}$$

(for all other points $x \in Q$ we have $\varphi(x) < d_1$, obviously).

It is clear now that $Q \cap E_S = \varnothing$. Therefore for each point $x' \in Q$ there exists a point $y' \in E_S$ such that

$$f(y') \ge f(x').$$

Remember that all functions $f_i(x)$ are linear.

Theorem. All efficient points of Q belong to the wall W_Q, i.e.

$$E_Q \subseteq W_Q.$$

P r o o f. Suppose the opposite, i.e. suppose that there exists a point t such that

$$t \in E_Q \text{ and } \varphi(t) = d_2 < d_1.$$

But $t \notin E_S$ and therefore there exists a point $y^t \in E_S$ such that $f(y^t) \geq f(t)$. In addition $\varphi(y^t) \geq d$ and $\varphi(t) = d_2 < d_1 < d$.

The segment $y^t - t$ belong to S. Therefore $y^t - t \cap W_Q \neq \varnothing$. Let $u \in y^t - t$ and $u \in W_Q$. Because the functions $f_i(x)$ do not decrease we have $f(u) \geq f(t)$. This means that $t \notin E_Q$. This is a contradiction. Therefore

$$E_Q \subseteq W_Q. \quad \blacksquare$$

This theorem shows that all efficient points of Q belong to the wall W_Q. Note that (in general) different walls of S can contain (different) efficient points from $E \subset S$.

Let we have an upper bound d^{up} for d, i.e.

$$\min\{\varphi(x) \mid x \in E\} \leq d^{up}.$$

The theorem allows to make the following **assumption**. We can choose another number $d_1 < d^{up}$ and we can consider the set $Q = \{x \in S \mid \varphi(x) \leq d_1\}$. If we find that $E_Q \subseteq W_Q$, then we have the inequality

$$d_1 \leq d = \min\{\varphi(x) \mid x \in E\}.$$

Remember that we will use the following sets:

$$S \subset R^n, W_i \text{ are the walls of } S;$$
$$S_1 \supset S, S_1 \subset R^n, W'_i \text{ are the walls of } S_1;$$
$$Z = f(S), Z \subset R^k, V_i \text{ are the walls of } Z;$$
$$Z_1 = f(S_1), \quad Z_1 \subset R^k \quad V'_i \text{ are the walls of } Z_1.$$

Assertion (without a proof).

Suppose that the wall W_i of S contains an efficient point. (The corresponding wall V_i of Z contains a (corresponding) nondominated point.) In this case if the reference point is an arbitrary point of the wall V'_i of Z_1, then the solution of the reference point problem (2) in R^k belongs to the wall V_i of Z (this solution is a nondominated point) and the corresponding solution in S belongs to the wall W_i of S and it is an efficient point of S.

A description of the method

1. We consider all walls W'_i of S_1 in a sequence. For each wall W'_i we find a point x^i such that $\varphi(x^i) = \min\{\varphi(x) \mid x \in W'_i\}$.

We use all points $f(x^i)$ as reference points in the RP problem. (All $f(x^i)$ belong to Z_1.) If the wall V_i of Z contains nondominated points then the solution of the reference point problem (with a reference point belonging to V'_i) is such a nondominated point from Z, the corresponding point from S will be an efficient point and the corresponding value of $\varphi(x)$ will be "small". The smaller value of the so found values (through the sequential considerations of the walls V'_i) is an upper bound for d. We denote this value by φ^{up}, and we have

$$\min\{\varphi(x) \mid x \in E\} \leq \varphi^{up}.$$

2. We choose a number $d_1 < \varphi^{up}$ and we consider the set

$$Q = \{x \in S \mid \varphi(x) \leq d_1\}.$$

For the set Q we solve the RP problem under additional constraint that RP belongs to the wall V'_i of Z_1 (for all i in a sequence). If we have that the equality $\varphi(x^i) = d_1$ is satisfied for all found solutions x^i, then the set Q does not contain efficient points of S i.e. $Q \cap E_S = \varnothing$. Therefore $d_1 < d$ and we have the interval

$$d_1 = \varphi^{\text{low}} \leq d \leq \varphi^{\text{up}} \quad \blacksquare$$

4. An example and a possible application

Firstly we will consider the following MOLP problem (D a u e r, (1991))

$$\max f_1(x) = 9x_1 + x_3,$$
$$\max f_2(x) = 9x_2 + x_3$$

s.t.

$$9x_1 + 9x_2 + 2x_3 \leq 81;$$
$$8x_1 + x_2 + 8x_3 \leq 72;$$
$$x_1 + 8x_2 + 8x_3 \leq 72;$$
$$7x_1 + x_2 + x_3 \geq 9;$$
$$x_1 + 7x_2 + x_3 \geq 9;$$
$$x_1 + x_2 + 7x_3 \geq 9;$$
$$x_1 \leq 8; \ \ x_2 \leq 8;$$
$$x_i \geq 0, \ \ i = 1, 2, 3.$$

The constraints of this example form the set S. In addition we have

$$\varphi(x) = 4x_1 + 5x_2 + 2x_3.$$

D a u e r (1991) has given the list of all efficient extreme points of S. Here they are in the following Table 1.

Table 1. The list of all efficient extreme points of S

x_1	x_2	x_3	$\varphi(x)$
0.8	8	0.9	45
1	8	0.0	44
8	1	0.0	37
8	0.8	0.9	37.8
4	4	4.5	45
0.0	8	1	42
8	0.0	1	34

The last column of this table contains the corresponding values of $\varphi(x)$. The minimal value is 34 and it is in the last line.

The set S_1 is formed by the following constraints

$$9x_1 + 9x_2 + 2x_3 \leq 82;$$
$$8x_1 + x_2 + 8x_3 \leq 73;$$
$$x_1 + 8x_2 + 8x_3 \leq 73;$$
$$7x_1 + x_2 + x_3 \geq 8;$$
$$x_1 + 7x_2 + x_3 \geq 8;$$
$$x_1 + x_2 + 7x_3 \geq 8;$$
$$x_1 \leq 9; x_2 \leq 9;$$

$$x_i \geq -1, \ i = 1, 2, 3.$$

The next step is to minimize $\varphi(x)$ on each wall of S_1, i. e., to solve the following problems:

$$\min \varphi(x)$$

s. t.

$$x \in S_1$$

$x \in W'_i$ (for all i in a sequence;

for example W'_1 is $9x_1 + 9x_2 + 2x_3 = 82$, W'_2 is $8x_1 + x_2 + 8x_3 = 73$, etc.)

The solutions of these problems give different vectors in Z_1 (Table 2).

Table 2. Vectors from Z_1 used as reference points

Names of vectors	f_1	f_2
Na)	80.83607	1.163934
Nb)	7.836364	7.836364
Nc)	8.888889	8.888889
Nd)	80.875	−1.125
Ne)	−1.125	80.875

These five vectors have been used as reference points in the reference point problem (2) with respect to the set S. Table 3 contains the obtained efficient vectors in S and the corresponding values of $\varphi(x)$.

Table 3. The efficient points from S, obtained by the reference point method and the corresponding values of $\varphi(x)$

Reference points (Names of vectors)	Components of obtained effic vectors			$\varphi(x)$
	x_1	x_2	x_3	
Na)	8	0	1	34
Nb)	4.522388	4.477612	0	40.47761
Nc)	4.522388	4.477612	0	40.47761
Nd)	8	0	1	34
Ne)	0	8	1	42

Table 3 shows that the exact values of $\min\{\varphi(x)| \ x \in E\}$ are very quickly found.

But the method does not give a proof that 34 is the needed minimum. Now the following problem is used with the purpose to estimate the value of 33 as a lower bound.

A LINGO program for obtaining a lower bound for $\min\{\varphi(x)| \ x \in E\}$:

```
!29 NOV 09, estimation of the lower bound;
!;
min = D;
!min = φ;
f1 = 9*x1 + x3;
f2 = 9*x2 + x3;
9*x1 + 9*x2 + 2*x3 < 81;!The first constraint describing the set S;
8*x1 + x2 + 8*x3 < 72;
x1 + 8*x2 + 8*x3 < 72;
7*x1 + x2 + x3 > 9;
x1 + 7*x2 + x3 > 9;
x1 + x2 + 7*x3 > 9;
x1 < 8;
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```
x2 < 8;
x1 > 0;
x2 > 0;
x3 > 0;
φ = 4*x1 + 5*x2 + 2*x3;
φ < 33;
!;
D > r1 – f1 – 0.01*f1 – 0.01*f2;
D > r2 – f2 – 0.01*f1 – 0.02*f2;
!;
!@free( x1);
!@free( x2);
!@free( x3);
@free( f1);
@free( f2);
@free( φ);
!;
@free( D);
!;
!The reference point belong to the walls of Z₁ – the vector  g is used;
!to describe  S₁;
r1 = 9*g1 + g3;
r2 = 9*g2 + g3;
9*g1 + 9*g2 + 2*g3 < 82;!This is the first constraint for S₁;
8*g1 + g2 + 8*g3 < 73;
g1 + 8* g2 + 8* g3 < 73;
7* g1 + g2 + g3 > 8;
g1 + 7* g2 + g3 > 8;
g1 + g2 + 7* g3 > 8;
g1 < 9;
g2 < 9;
g1 > –1;
g2 > –1;
g3 > –1;!The last constraint for S₁;
!;
@free( g1);
@free( g2);
@free( g3);
@free( r1);
@free( r2);
end
```

Solving this problem several times under the condition   that one of the constraints describing S_1 is taken as equality and all other constraints (for S_1) rest as inequalities, we obtain vectors that are efficient in Q (Table 4).

Table 4. The obtained efficient vectors in Q

x_1	x_2	x_3
8	0.1515	0.1212
3.6304	3.5942	0.2536
8	0.1111	0.2222
0.3529	6.1764	0.3529

For each one of these vectors the corresponding value of $\varphi(x)$ is 33. Our conclusion is that the wall $\varphi(x) = 33$ contains all efficient points of Q. Therefore, we have a confirmation that

$$33 < \min_{x \in E} \varphi \leq 34.$$

Now we can demonstrate a **possible application**. We will consider again the above MOLP problem. Can we use the described approach with the purpose to estimate the nadir point? We begin with computation of the nondominated extreme points. We know the efficient extreme points, so it is very easy to obtain Table 5.

Table 5. The nondominated extreme points for the considered MOLP problem

f_1	8.1	9.0	72.0	72.9	40.5	1	73
f_2	72.9	72.0	9.0	8.1	40.5	73	1

In Table 5 the first nondominated extreme point is [8.1, 72.9]. The second one is [9, 72] and so on. This table shows that

$$\min_{x \in E} f_1(x) = \min_{x \in E} f_2(x) = 1.$$

Minimizing $f_1(x)$ over the walls of S_1 we obtain the following different points of Z_1.

Table 6. The reference points for searching the minimal value of $f_1(x)$ over E

f_1	f_2
0.885714	81.11429
7.836364	7.836364
−1.327273	81.14545
−1.25	80.875
80.83607	1.163934

The above points from Z_1 will be used as reference points with respect to the set S in problem (2). It is very easy to obtain that the first one of these points determines the point [1; 73] from the set Z and this is the point that gives the minimal value of $f_1(x)$ over the set E.

Minimizing $f_2(x)$ over the walls of S_1 we obtain new points of Z_1 (Table 7).

Table 7. The reference points for searching the minimal value of $f_2(x)$ over E

f_1	f_2
81.11429	0.8857143
81.14545	−1.327273
7.836364	7.836364
80.875	−1.125
1.163934	80.83607

Using the first one of the above points as a reference point (problem (2)), we obtain the point [73; 1] of Z and this point gives the minimal value of $f_2(x)$ over the set E.

So in this example the first part of the here described method (for obtaining upper bounds only) gives the exact values we need. Using the second part (obtaining lower bounds), we get acceptable values. So we have demonstrated that the method can be applied for estimation of the nadir point in MOLP problems.

A DEMO version of LINGO software for solving optimization problems has been used for all computations.

5. Conclusion

The next steps would be to obtain proofs for all used assertions in the described procedure.

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