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Multicriteria Decision Making by Fuzzy Relations and Weighting Functions for the Criteria [1]

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Abstract: *A multicriteria decision making problem with criteria giving fuzzy relations between the couples of alternatives is considered. The importance of each of the criteria is given as weighting function depending on the membership degrees of the corresponding fuzzy relation. Transformed membership degrees with the help of these weighting functions are used in the aggregation procedure for fusing the relations. The properties of the weighted relations, required to decide the problems of choice, ranking or clustering of the alternatives' set are proved. An illustrative numerical example solving the problem of alternatives' ranking is presented as well.*

Keywords: *Multicriteria decision making, fuzzy relations, aggregation operators, fuzzy relations' properties, weighting functions.*

1. Introduction

The following multicriteria decision making problem is considered here. A finite set of alternatives is evaluated by a set of fuzzy criteria, i.e., fuzzy relations, which may be either fuzzy preference or similarity, or likeness ones. The criteria importance is computed with the help of weighting functions in the unit interval. In order to compute the multicriteria score of the fuzzy relations an aggregation operator will be used to fuse the membership degrees of two alternatives according to all relations (fuzzy criteria) taking into account the respective weighting functions. The

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purpose is to aggregate the individual fuzzy relations in order to get a fusion relation as a fuzzy one, giving a possibility to decide a ranking, choice, or cluster problem.

Procedures for determining the weights have been the aim of many investigations and discussions [2, 8, 20, 24, 26]. The weights of the criteria may be given as constants in most cases; besides this, they can be fuzzy numbers [9, 15, 19], fuzzy relation between the criteria importance [18] or weighting functions [11, 21, 1, 10].

The idea of considering weighting functions that depend continuously on the criterion satisfaction values (i.e., good or bad criteria performances) is supported by common sense reasoning and experience in the context of decision theory [11, 21]. The introduction of weighting functions depending continuously on criterion satisfaction values produces weighted aggregation operators with complex dependence on these values. The more complicated case when the weighting functions depend continuously on the membership degrees to the fuzzy relations, is considered herein. Then the dependence of the chosen aggregation operator on the membership degrees is more complex.

Let $A=\{a, b, c, ..., n\}$ be a set of alternatives evaluated by fuzzy criteria $K=\{k_1, ..., k_m\}$. These criteria compare the couples of alternatives and assign membership degrees to the fuzzy relations corresponding to the different criteria. Let $R_1, ..., R_i, ..., R_m$ are the matrices of these fuzzy relations, i.e.,

$$R_i=\begin{pmatrix} \mu_i(a,a) & \mu_i(a,b) & ... & \mu_i(a,n) \\ \mu_i(b,a) & \mu_i(b,b) & ... & \mu_i(b,n) \\ ... & ... & ... & ... \\ \mu_i(n,a) & \mu_i(n,b) & ... & \mu_i(n,n) \end{pmatrix}, \quad i=1,...,m, \quad m\geq 2,$$

where $\mu_i(a,b)\in[0,1]$ is the membership degree of the couple of alternative $a, b\in A$ to the fuzzy relation R_i (R_i means a fuzzy relation and a matrix corresponding to this relation as well, for simplicity). The weights of the criteria are computed using weighting functions $f_1(x), ..., f_m(x), x\in[0,1]$ of the membership degrees to the corresponding relation. An approach for fusing fuzzy relations consists in the usage of aggregation operators. A very good overview of the aggregation operators, by presenting the characteristics, the advantages and disadvantages of each operator and the relations between them, is available in [3, 7].

According to Y a g e r [25] and taking into account the problem so determined, each of the membership degrees may be transformed via weighting functions of the criteria as follows:

(1) $$\mu_i^w(a,b)=g(w_i(\mu_i(a,b)), \mu_i(a,b)) \ \ \forall a, b\in A, i=1,...,m,$$

where $w_i(\mu_i(a,b))\in[0,1]$ is the weighting coefficient computed with the help of the function $f_i(x)$ for the membership degree of the given couple of the alternatives to the fuzzy relation R_i and $g(w, x)$ satisfies the following properties [25]:

$$x > y \to g(w, x) \geq g(w, y); \ g(w, x) \text{ is monotone in } w;$$
$$g(0, x) = \text{id}, \ g(1, x) = x,$$

with the identity element, id, such that it does not change the aggregated value by adding it to the aggregation. The weighted aggregation is obtained as

(2) $$\mu^w(a, b) = \text{Agg}(\mu_1^w(a, b), ..., \mu_i^w(a, b), ..., \mu_m^w(a, b)) \ \ \forall a, b \in A,$$

where Agg denotes some aggregation operator. The form of g depends on the type of aggregation performed, e.g., it may be *t*-norm or *t*-conorm [25], taking into account their properties.

The properties of the new relations characterized by membership functions $\mu_i^w(a, b), \ a, b \in A, \ i = 1, ..., m$, with a product *t*-norm for g and arithmetic mean operator for Agg, weighted by using the functions $f_i(x), \ i = 1, ..., m$ (also called a generalized mixture operator [11]), will be studied.

2. Weighting functions and weighted relations

Let the membership degrees from the comparison of the alternatives $a, b \in A$ to the fuzzy relations $R_1, ..., R_i, ..., R_m$ be $\mu_1(a, b) = x_1, ..., \mu_i(a, b) = x_i, ..., \ \mu_m(a, b) = x_m$. The generalized mixture operator is defined as

(3) $$\text{Agg}(x_1, ..., x_i, ..., x_m) = \begin{cases} 1 & a = b, \\ \dfrac{\sum_{i=1}^m f_i(x_i) x_i}{\sum_{i=1}^m f_i(x_i)} & a \neq b \end{cases}$$

$$\forall a, b \in A, \ i = 1, ..., m,$$

where $f_i : [0, 1] \to [0, \infty), \ i = 1, ..., m,$ are weighting functions, which are supposed to be continuous.

Introduce the following notations for all $a, b \in A$:

(4) $$S(a, b) = \sum_{i=1}^m f_i(x_i),$$

(5) $$w_i(x_i) = \frac{f_i(x_i)}{\sum_{i=1}^m f_i(x_i)} = \frac{f_i(x_i)}{S(a, b)},$$

i.e. $w_i(x_i)$ is a weighting coefficient with the normalization condition $\sum_{i=1}^m w_i(x_i) = 1$.

The aggregated relation obtained as a result of such fusion has the following membership degrees for all $a, \ b \in A$:

(6) $$\mu^w(a,b) = \frac{\sum_{i=1}^{m} f_i(x_i)x_i}{\sum_{i=1}^{m} f_i(x_i)} = \frac{\sum_{i=1}^{m} f_i(x_i)x_i}{S(a,b)} = \sum_{i=1}^{m} w_i(x_i)x_i = \sum_{i=1}^{m} \mu_i^w(a,b).$$

The functions $f_i(x),\ i=1,\dots,m,$ have to be monotonic and sensitive. These continuous functions are defined in the unit interval for $x\in[0,1]$ and they have continuous derivatives $f_i'(x),\ i=1,\dots,m,$ in this interval. It is proved [11] that the sufficient condition for the strict monotonicity of the operator (3) (i. e., to be an aggregation operator) is $0\le f_i'(x)\le f_i(x),\ i=1,\dots,m,\ x\in[0,1]$. New sufficient conditions for weighting functions are introduced in [13], in order to ensure the monotonicity of the generalized mixture operator (3). Let $f_i(x),\ i=1,\dots,m,$ be monotone smooth weighting functions. Then the generalized mixture operator is monotone whenever all non-decreasing $f_i(x)$ fulfill

(7) $$0\le f_i'(x)\le f_i(x)\quad \forall x\in[0,1]$$

or

(8) $$0\le f_i'(x)(1-x)\le f_i(x)\quad \forall x\in[0,1]$$

and all non-increasing $f_i(x)$ fulfill

(9) $$f_i(x)+f_i'(x)\ge 0\quad \forall x\in[0,1]$$

or

(10) $$f_i(x)+f_i'(x)x\ge 0\quad \forall x\in[0,1].$$

The following fitting [13] weighting functions will be considered here [11]:

- Linear weighting functions

(11) $f_i(x)=1+\beta_i x$ with parameters $0\le\beta_i\le 1,\ i=1,\dots,m,\ m\ge 2$.

- Parametric linear weighting functions

(12) $$f_i(x)=\alpha_i\frac{1+\beta_i x}{1+\beta_i}=\gamma_i(1+\beta_i x),\ 0<\alpha_i\le 1,\ 0\le\beta_i\le 1,\ \gamma_i=\frac{\alpha_i}{1+\beta_i},\ i=1,\dots,m.$$

It is obvious that $0<f_i(0)=\alpha_i/(1+\beta_i)\le f_i(1)=\alpha_i\le 1$ and therefore the parameters α_i control the value $f_i(1)$, when the criteria satisfaction values are one. The parameters β_i controls the ratio between the largest and smallest values of the function (12), when the criteria satisfaction values are zero and one, i.e., $1\le f_i(1)/f_i(0)=1+\beta_i\le 2,\ i=1,\dots,m$.

- Quadratic weighting functions

(13) $$f_i(x)=1+(\beta_i-\gamma_i)x+\gamma_i x^2,\quad \beta_i\ge 0,\ \gamma_i\ge 0.$$

The linear weighting functions (11) correspond to $\gamma_i=0$ in (13).

The above conditions for monotonicity of (3) are performed for weighting functions (13) if [11]:

(14) $$0\le\gamma_i\le 1,\ \gamma_i\le\beta_i\le\beta_c(\gamma_i),\ i=1,\dots,m,$$

with

(15) $$\beta_c(\gamma_i) = \begin{cases} \gamma_i + 1 & \text{for } 0 \le \gamma_i \le 0.5, \\ \gamma_i + 2\sqrt{\gamma_i(1-\gamma_i)} & \text{for } 0.5 \le \gamma_i \le 1. \end{cases}$$

Taking into account the operator (3) and the product *t*-norm for *g* in (1), the new membership degrees of the weighted relations are:

(16) $$\mu_i^w(a,b) = \begin{cases} 1 & \text{if } a = b, \\ w_i(x_i)x_i = \dfrac{f_i(x_i)x_i}{S(a,b)} & \text{if } a \ne b \end{cases}$$

$$\forall a, b \in A, \ i = 1, ..., m,$$

where $f_i(x_i)$ is one of the above weighting functions.

The properties of (16) and (3) will be studied.

3. Some definitions of fuzzy relations properties

Let $\mu : X \times X \to [0, 1]$ be a membership function of a binary fuzzy relation R, $x, y, z \in X$ and T be a *t*-norm. The relation R is called:

(i) reflexive if and only if $\forall\, x \in X : \mu(x, x) = 1;$

(ii) symmetric if and only if $\forall\, x, y \in X : \mu(x, y) = \mu(y, x);$

(iii) perfect antisymmetric if $\mu(x, y) > 0$ then $\mu(y, x) = 0 \quad \forall x, y \in X, \quad x \ne y$ [14];

(iv) *T*-transitive if and only if $\forall\, x, y, z \in X : \ T(\mu(x, y), \mu(y, z)) \le \mu(x, z)$.

The following definitions of transitivity are used here:

$$x, y, z \in X : \min(\mu(x, y), \mu(y, z)) \le \mu(x, z).$$

Z a d e h [27] suggested several useful definitions for transitivity, which are compared in [23]. The weakest of them is the max-Δ transitivity, i.e. $\mu(a, c) \ge \max(0, \mu(a, b) + \mu(b, c) - 1)$. It is shown [23] that this is the most suitable notion of transitivity for fuzzy ordering.

(vi) The relation R is max-Δ transitive if and only if

$$\forall\, x, y, z \in X : \max(0, \mu(x, y) + \mu(y, z) - 1) \le \mu(x, z).$$

The above properties are composed to generate new classes of fuzzy relations.

(vii) A similarity relation is reflexive, symmetrical and max-min transitive fuzzy relation [6].

(viii) A fuzzy preorder is reflexive and max-Δ transitive fuzzy relation [6].

(ix) A likeness relation is reflexive, symmetrical and max-Δ transitive fuzzy relation [6].

(x) A fuzzy partial ordering is reflexive, perfect antisymmetrical and max-Δ transitive fuzzy relation [23].

(xi) A fuzzy linear ordering is a fuzzy partial ordering such that $\forall\, x, y \in X$ if $x \ne y$ either $\mu(x, y) > 0$ or $\mu(y, x) > 0$ [6].

4. Properties of the weighted relations and the generalized mixture operator

The dependence between the properties of an aggregated relation and the corresponding individual relations is useful information, because it is required for solving a ranking, choice, or cluster problem. The preservation of fuzzy relations' properties by weighted transformations of these relations and by their aggregation process is considered in [4, 5, 12, 16, 17].

The dependence between the properties of $R_1, ..., R_i, ..., R_m$ and transformations (16) and aggregation relations (3) will be studied.

If the relations with membership functions $\mu_i(a, b),\ a, b \in A,\ i = 1, ..., m,$ are reflexive and symmetrical it is obvious that the weighted relations (16) and the generalized mixture operator (3) preserve these properties. The transformations (16) and (3) do, however, not preserve the property of T-transitivity of the initial relations as it is seen from the following.

Let $R_i,\ i = 1, ..., m,$ be T-transitive relations, i.e.,

$$\mu_i(a,\ c) \geq T(\mu_i(a,\ b),\ \mu_i(b,\ c))\quad \forall a,\ b,\ c \in A,\quad i = 1, ..., m.$$

Introducing the notations $\mu_i(a, c) = z_i,\ \mu_i(a, b) = x_i,\ \mu_i(b, c) = y_i,$ the above inequality becomes $z_i \geq T(x_i, y_i),\ i = 1, ..., m.$ Two ways to show that the generalized mixture operator preserves or does not preserve the T-transitivity, are proposed. The first one consists in proving the preservation of T-transitivity of the weighted relations (16) and then use the results (3), obtained in [16]. The other way is to use the sufficient condition for preserving the T-transitivity suggested in [22], i.e.,

$$\frac{\sum_{i=1}^{m} f_i(T(x_i, y_i))T(x_i, y_i)}{\sum_{i=1}^{m} f_i(T(x_i, y_i))} \geq T\left(\frac{\sum_{i=1}^{m} f_i(x_i)x_i}{\sum_{i=1}^{m} f_i(x_i)}, \frac{\sum_{i=1}^{m} f_i(y_i)y_i}{\sum_{i=1}^{m} f_i(y_i)} \right). \tag{17}$$

Some examples show that (3) does not preserve the T-transitivity of fuzzy relations. The test is done only for max-min and max-Δ transitivity.

Example 1. Let R_1 and R_2 be reflexive and max-min transitive relations, i.e.,

$$x_{ii} = 1,\ x_{ij} \geq \min(x_{ik}, x_{kj}),\ y_{ii} = 1,\ y_{ij} \geq \min(y_{ik}, y_{kj})\ \ \forall\ k,\ i,\ j,\ k = 1,\ 2,\ 3:$$

$$R_1 = \left(x_{ij}\right) = \begin{pmatrix} 1 & 0.6 & 0.8 \\ 0.3 & 1 & 0.7 \\ 0.2 & 0.2 & 1 \end{pmatrix},\ R_2 = \left(y_{ij}\right) = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.7 & 0.5 & 1 \end{pmatrix}.$$

Compute the weighted relations (16) with one of the above weighting functions.

- Let $f_1(x_{ij}) = 1 + 0.3x_{ij}$, $\qquad f_2(y_{ij}) = 1 + 0.7y_{ij}$ from (11).

The transformed weighted relations R_1^w, R_2^w according to (4), (5), (16) and the aggregated relation R by (3) are:

$$R_1^w = \begin{pmatrix} 1 & 0.294 & 0.408 \\ 0.126 & 1 & 0.329 \\ 0.084 & 0.088 & 1 \end{pmatrix}, \quad R_2^w = \begin{pmatrix} 1 & 0.153 & 0.147 \\ 0.406 & 1 & 0.265 \\ 0.406 & 0.280 & 1 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 0.447 & 0.555 \\ 0.632 & 1 & 0.594 \\ 0.490 & 0.368 & 1 \end{pmatrix}.$$

The examination shows that the two relations are not max-min transitive, because for R_1^w one has 0.084 < min(0.088, 0.126), for R_2^w $\to$ 0.147 < min(0.153, 0.265). Therefore, R is not a max-min transitive relation, due to the results in [16], which is confirmed by the test done here, e.g. $0.368 < \min(0.490, 0.447)$.

- Let $f_1(x_{ij}) = 0.6\dfrac{1+0.2x_{ij}}{1+0.2}$, $\quad f_2(y_{ij}) = 0.5\dfrac{1+0.8y_{ij}}{1+0.8}$ from (12).

In this case, R_1^w, R_2^w, according to (4), (5), (16) and (3), are

$$R_1^w = \begin{pmatrix} 1 & 0.37 & 0.50 \\ 0.16 & 1 & 0.42 \\ 0.11 & 0.11 & 1 \end{pmatrix}, \quad R_2^w = \begin{pmatrix} 1 & 0.12 & 0.11 \\ 0.32 & 1 & 0.20 \\ 0.32 & 0.21 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0.49 & 0.61 \\ 0.48 & 1 & 0.62 \\ 0.43 & 0.32 & 1 \end{pmatrix}.$$

R_2^w is not max-min transitive, because $y_{13} < \min(y_{12}, y_{23})$ and therefore, R is not max-min transitive as well, e.g. $0.32 < \min(0.43, 0.49)$.

- Let $f_1(x_{ij}) = 1 + (0.6-0.2)x_{ij} + 0.2x_{ij}^2$, $\quad f_2(y_{ij}) = 1 + (0.5-0.4)y_{ij} + 0.4y_{ij}^2$

from (13). Their parameters satisfy the conditions (14) and (15). In this case, according to (4), (5), (16) and (3):

$$R_1^w = \begin{pmatrix} 1 & 0.33 & 0.46 \\ 0.14 & 1 & 0.40 \\ 0.09 & 0.10 & 1 \end{pmatrix}, \quad R_2^w = \begin{pmatrix} 1 & 0.14 & 0.13 \\ 0.37 & 1 & 0.24 \\ 0.38 & 0.26 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0.47 & 0.59 \\ 0.51 & 1 & 0.64 \\ 0.47 & 0.36 & 1 \end{pmatrix}.$$

The relations R_1^w, R_2^w and R are not max-min transitive, because $x_{31} < \min(x_{32}, x_{21})$, $y_{13} < \min(y_{12}, y_{23})$ and for $R - 0.36 < \min(0.47, 0.47)$.

Example 2. Let R_1 and R_2 be reflexive and max-Δ transitive relations, i.e., $x_{ii} = 1,\ x_{ij} \ge \max(0, x_{ik} + x_{kj} - 1),\ y_{ii} = 1,\ y_{ij} \ge \max(0, y_{ik} + y_{kj} - 1)\ \forall k, i, j, k = 1,\ 2,\ 3$:

$$R_1 = (x_{ij}) = \begin{pmatrix} 1 & 0.6 & 0.9 \\ 0.7 & 1 & 0.7 \\ 0.1 & 0.4 & 1 \end{pmatrix}, \quad R_2 = (y_{ij}) = \begin{pmatrix} 1 & 0.7 & 0.3 \\ 0.7 & 1 & 0.5 \\ 0.2 & 0.5 & 1 \end{pmatrix},$$

and $f_1(x_{ij}) = 1 + 0.8x_{ij}$, $\quad f_2(y_{ij}) = 1 + 0.9y_{ij}$ from (11).

The sufficient condition for preservation of this transitivity is the validity of (17) for all $x_{ij}, y_{ij}, i, j = 1, 2, 3,$ e.g. for the element (31) of the aggregated relation one has:

$$x_{31} = 0.1 \geq \max(0, x_{32} + x_{21} - 1) = \max(0, 0.4 + 0.7 - 1) = 0.1 = T(x_{32}, x_{21}),$$
$$y_{31} = 0.2 \geq \max(0, y_{32} + y_{21} - 1) = \max(0, 0.5 + 0.7 - 1) = 0.2 = T(y_{32}, y_{21}),$$
$$f_1(T(x_{32}, x_{21})) = 1.08, \quad f_2(T(y_{32}, y_{21})) = 1.18,$$

and then it has to be proved that

$$\frac{f_1(T(x_{32}, x_{21}))T(x_{32}, x_{21}) + f_2(T(y_{32}, y_{21}))T(y_{32}, y_{21})}{f_1(T(x_{32}, x_{21})) + f_2(T(y_{32}, y_{21}))} \geq$$
$$\geq \max\left(0, \ \frac{f_1(x_{32})x_{32} + f_2(y_{32})y_{32}}{f_1(x_{32}) + f_2(y_{32})} + \frac{f_1(x_{21})x_{21} + f_2(y_{21})y_{21}}{f_1(x_{21}) + f_2(y_{21})} - 1\right).$$

The computations show that the left side of this inequality is equal to 0.15 and the right side is equal to 0.1523, i.e. the operator (3) does not always preserve the max-Δ transitivity of fuzzy relations. From the practical point of view, comparing the two ways for examination of the preservation of the T-transitivity it is obvious that the first way is easy for the computations.

Therefore, the operator (3) does not preserve the *T*-transitivity of fuzzy relations and the following proposition is very useful, in this case.

Proposition 1. If the relations $R_i, \ i = 1, ..., m,$ are max-min transitive, then (16) transforms them to max-Δ transitive fuzzy relations.

P r o o f. Let $R_i, \ i = 1, ..., m$, be max-min transitive relations, i.e.

$$\mu_i(a, c) \geq \min(\mu_i(a, b), \mu_i(b, c)) \quad \forall a, b, c \in A, \quad i = 1, ..., m.$$

It has to be proved that

$$\mu_i^w(a, c) \geq \max(0, \ \mu_i^w(a, b) + \mu_i^w(b, c) - 1) \ \ \forall a, b, c \in A, \ i = 1, ..., m, \tag{18}$$

where according to (16) and the notations $\mu_i(a, b) = x_i,$ $\mu_i(b, c) = y_i,$ $\mu_i(a, c) = z_i, \ i = 1, ..., m,$

$$\mu_i^w(a, b) = \frac{f_i(x_i)}{S(a, b)} x_i, \ \ \mu_i^w(b, c) = \frac{f_i(y_i)}{S(b, c)} y_i, \ \ \mu_i^w(a, c) = \frac{f_i(z_i)}{S(a, c)} z_i.$$

a) Case with linear weighting functions (11)

This case is a special case of the quadratic weighting functions for $\gamma_i = 0.$

b) Case with parametric linear weighting functions (12)

The inequality (18) becomes

$$\frac{\gamma_i(1 + \beta_i z_i)z_i}{S(a, c)} \geq \max\left(0, \ \frac{\gamma_i(1 + \beta_i x_i)x_i}{S(a, b)} + \frac{\gamma_i(1 + \beta_i y_i)y_i}{S(b, c)} - 1\right), \tag{19}$$

where according to (4)

$$S(a,\ b)=\sum_{i=1}^{m}\gamma_i(1+\beta_i x_i),\ \ S(b,\ c)=\sum_{i=1}^{m}\gamma_i(1+\beta_i y_i),\ \ S(a,\ c)=\sum_{i=1}^{m}\gamma_i(1+\beta_i z_i).$$

- If $x_i + y_i \le 1,$ then taking into account (5), $w_i(x_i)x_i + w_i(y_i)y_i \le 1,$ because $w_i(x_i)\le 1,\ w_i(y_i)\le 1.$ Therefore (18) holds.

- If $x_i + y_i > 1,$ then it has to be proved, that

(20) $$\frac{\gamma_i(1+\beta_i z_i)z_i}{S(a,c)} \ge \frac{\gamma_i(1+\beta_i x_i)x_i}{S(a,b)} + \frac{\gamma_i(1+\beta_i y_i)y_i}{S(b,c)} - 1.$$

According to $z_i \ge \min(x_i, y_i),$ for all $x_i, y_i, z_i,\ i=1,...,m,...,z_i,$ can be preordered in such a way that

$$x_i \le z_i \le y_i,\quad i=1,...,k;\quad y_i \le z_i \le x_i,\quad i=k+1,...,m.$$

Let $j\in[1,k],$ i.e. $x_j \le z_j \le y_j.$ Introducing the notations

$$(1+\beta_j z_j)z_j = Z_j,\ \ (1+\beta_j x_j)x_j = X_j,\ \ (1+\beta_j y_j)y_j = Y_j$$

(20) becomes to

(21) $$\frac{\gamma_j Z_j}{S(a,c)} \ge \frac{\gamma_j X_j}{S(a,b)} + \frac{\gamma_j Y_j}{S(b,c)} - 1.$$

If $\dfrac{Z_j}{S(a,\ c)} \ge \dfrac{X_j}{S(a,b)}$ or $\dfrac{Z_j}{S(a,\ c)} \ge \dfrac{Y_j}{S(b,\ c)},$ then

$$\frac{\gamma_j Z_j}{S(a,\ c)} \ge \frac{\gamma_j X_j}{S(a,\ b)}\ \frac{\gamma_j Y_j}{S(b,\ c)} \ge \max\left(0,\ \frac{\gamma_j X_j}{S(a,\ b)} + \frac{\gamma_j Y_j}{S(b,\ c)} - 1\right)$$

and (20) is proved.

The more complicated case is when $\dfrac{Z_j}{S(a,c)} < \dfrac{X_j}{S(a,b)}$ and $\dfrac{Z_j}{S(a,c)} < \dfrac{Y_j}{S(b,c)},$ i.e.,

(22) $$S(a,b)Z_j < S(a,c)X_j \ \text{ and } \ S(b,c)Z_j < S(a,c)Y_j.$$

From $j\in[1,k],$ i.e. $x_j \le z_j \le y_j,$ it follows that $X_j \le Z_j$ and therefore from (22) $S(a,c) > S(b,c) > S(a,b),$ for example. According to (21) one has to prove that

(23) $$\frac{\gamma_j Z_j}{S(a,c)} + 1 \ge \frac{\gamma_j X_j}{S(a,b)} + \frac{\gamma_j Y_j}{S(b,c)},$$

i.e., $S(a,b)S(b,c)[\gamma_j Z_j + S(a,c)] \ge S(b,c)S(a,c)\gamma_j X_j + S(a,b)S(a,c)\gamma_j Y_j,$ but

(24) $S(a,b)S(b,c)S(a,c) - S(a,b)S(a,c)\gamma_j Y_j = S(a,b)S(a,c)[S(b,c) - \gamma_j Y_j] \ge 0,$

(25) $S(a,b)S(b,c)\gamma_j Z_j - S(b,c)S(a,c)\gamma_j X_j = \gamma_j S(b,c)[S(a,b)Z_j - S(a,c)X_j] \le 0.$

Compare (24) and (25). If (24) is greater or equal to (25), then (23) is proved. The results from the comparisons of the separated multipliers are:

$S(a,c) > S(b,c), \quad S(a,b) > \gamma_j$. According to the third multiplier one has:

(26) $$S(b,c) - \gamma_j Y_j = \sum_{i=1}^{j-1} \gamma_i (1+\beta_i y_i) + \gamma_j (1+\beta_j y_j)(1-y_j) + \sum_{i=j+1}^{m} \gamma_i (1+\beta_i y_i),$$

(27) $$S(a,c)X_j - S(a,b)Z_j = \sum_{i=1}^{j-1} [\gamma_i (1+\beta_i z_i)(1+\beta_j x_j)x_j] + \gamma_j (1+\beta_j z_j)(1+\beta_j x_j)x_j +$$
$$+ \sum_{i=j+1}^{m} [\gamma_i (1+\beta_i z_i)(1+\beta_j x_j)x_j] - \sum_{i=1}^{j-1} [\gamma_i (1+\beta_i x_i)(1+\beta_j z_j)z_j] -$$
$$-\gamma_j (1+\beta_j x_j)(1+\beta_j z_j)z_j - \sum_{i=j+1}^{m} [\gamma_i (1+\beta_i x_i)(1+\beta_j z_j)z_j].$$

Compare the addends from (26) and (27), separately. One has:

$$\sum_{i=1}^{j-1} \gamma_i (1+\beta_i y_i) \geq \sum_{i=1}^{j-1} \gamma_i (1+\beta_i z_i)(1+\beta_j x_j)x_j - \sum_{i=1}^{j-1} \gamma_i (1+\beta_i x_i)(1+\beta_j z_j)z_j,$$

because

$$\gamma_i (1+\beta_i y_i) \geq \gamma_i (1+\beta_i z_i)(1+\beta_j x_j)x_j - \gamma_i (1+\beta_i x_i)(1+\beta_j z_j)z_j$$

from

(28) $$1+\beta_i y_i + z_j + \beta_i x_i z_j + \beta_j z_j^2 + \beta_i \beta_j x_i z_j^2 \geq x_j + \beta_i z_i x_j + \beta_j x_j^2 + \beta_i \beta_j z_i x_j^2,$$

and

$$x_i \leq z_i \leq y_i \leq 1 \quad \text{for} \quad i = 1, ..., k, \quad j \in [1,\ k],$$
$$\beta_i y_i \geq \beta_i z_i x_j, \quad \beta_j z_j^2 \geq \beta_j x_j^2, \quad 1+\beta_i x_i z_j + \beta_i \beta_j x_i z_j^2 \geq \beta_i \beta_j z_i x_j^2.$$

Besides,

$$\gamma_j (1+\beta_j y_j)(1-y_j) \geq \gamma_j (1+\beta_j z_j)(1+\beta_j x_j)x_j - \gamma_j (1+\beta_j x_j)(1+\beta_j z_j)z_j,$$

because

$$(1+\beta_j y_j)(1-y_j) \geq (1+\beta_j z_j)(1+\beta_j x_j)(x_j - z_j),$$

taking into account that the left side of the above inequality is positive and the right-hand side is negative ($x_j \leq z_j$).

Finally,

$$\sum_{i=j+1}^{m} \gamma_i (1+\beta_i y_i) \geq \sum_{i=j+1}^{m} \gamma_i (1+\beta_i z_i)(1+\beta_j x_j)x_j - \sum_{i=j+1}^{m} \gamma_i (1+\beta_i x_i)(1+\beta_j z_j)z_j,$$

because:

- if $i \leq k$, this is analogical to the case, when $i \in [1,\ j-1]$,
- if $i > k$, then $y_i \leq z_i \leq x_i$, but $x_j \leq z_i \leq y_j$ and from (28) $x_j \leq z_j$,

$\beta_i x_i z_j \geq \beta_i z_i x_j$, $\beta_j z_j^2 \geq \beta_j x_j^2$, $1+\beta_i y_i + \beta_i \beta_j x_i z_j^2 \geq \beta_i \beta_j z_i x_j^2$, i.e., the inequality (23) holds and hence (19) is proved.

c) Case with quadratic weighting functions (13)

In this case

(29)
$$S(a,b)=\sum_{i=1}^{m} f_i(x_i)=\sum_{i=1}^{m}(1+(\beta_i-\gamma_i)x_i+\gamma_i x_i^2)=m+\sum_{i=1}^{m} X_i,$$
$$S(b,c)=\sum_{i=1}^{m} f_i(y_i)=\sum_{i=1}^{m}(1+(\beta_i-\gamma_i)y_i+\gamma_i y_i^2)=m+\sum_{i=1}^{m} Y_i,$$
$$S(a,c)=\sum_{i=1}^{m} f_i(z_i)=\sum_{i=1}^{m}(1+(\beta_i-\gamma_i)z_i+\gamma_i z_i^2)=m+\sum_{i=1}^{m} Z_i.$$

It has to be proved that (18) holds.

Let $j\in[1,m]$ and $x_j\le z_j\le y_j$, e.g. $z_i\ge \min(x_i,y_i)$, for all x_i, y_i, z_i, $i=1,...,m.$ If it can be proved that

(30)
$$\frac{f_j(z_j)z_j}{S(a,c)}\ge \frac{f_j(x_j)x_j}{S(a,b)}\frac{f_j(y_j)y_j}{S(b,c)},$$

then (18) is valid in view of the inequality $z\ge xy\ge \max(0,x+y-1)$, $x,y,z\in[0,1]$.

From $x_j\le z_j\le y_j$, it follows that $f_j(z_j)z_j\ge f_j(x_j)x_j$.

If $S(a,c)\le S(a,b)$, $S(a,c)\le S(b,c)$ or $S(a,c)\le S(a,b)S(b,c)$, then (18) is valid.

Let $S(a,c)>S(a,b)$, $S(a,c)>S(b,c)$ and $S(a,c)>S(a,b)S(b,c)$, then it has to be proved in (30), that

(31)
$$\frac{1}{S(a,c)}\ge \frac{f_j(y_j)y_j}{S(a,b)S(b,c)} \Rightarrow S(a,b)S(b,c)\ge S(a,c)f_j(y_j)y_j.$$

This inequality may be rewritten taking into account (29), as follows:
$$(m+\sum_{i=1}^{m}X_i)(m+\sum_{i=1}^{m}Y_i)\ge f_j(y_j)y_j(m+\sum_{i=1}^{m}Z_i).$$

But $m^2\ge mf_j(y_j)y_j$, because $m\ge 2$, $(1+(\beta_j-\gamma_j)y_j+\gamma_j y_j^2)y_j\le 1+\beta_j\le 2$ and $m\left(\sum_{i=1}^{m}X_i+\sum_{i=1}^{m}Y_i\right)\ge f_j(y_j)y_j\sum_{i=1}^{m}Z_i$, which follows from $x_i\le z_i\le y_i$ or $y_i\le z_i\le x_i$ by assumption. Therefore, (31) is valid.

The properties reflexivity, symmetry and max-Δ transitivity of the aggregation operator (3) gives a possibility to decide the problems of choice, ranking or clustering after the computation of the aggregated relation. The following propositions are useful for this purpose.

Corollary 1. If R_i, $i=1,...,m,$ are reflexive and max-min transitive relations, then the relations R_i^w, $i=1,...,m,$ with membership functions (16) are fuzzy preorders.

The proof follows from Proposition 1 and the definition for fuzzy preorder (see Section 3).

Corollary 2. If $R_i,\ i=1,...,m,$ are similarity relations, then the relations $R_i^w,\ i=1,...,m,$ with membership function (16) are likeness relations.

The proof follows from Proposition 1 and the definition of likeness relation given in Section 3.

Corollary 3 [16]. If the relations $R_i^w,\ i=1,...,m,$ are fuzzy preorders, then the aggregation relation with the membership function (3) is a fuzzy preorder relation as well.

Corollary 4 [16]. If $R_i^w,\ i=1,...,m,$ are likeness relations, then the aggregation relation with membership function (3) is a likeness relation as well.

5. Numerical example

This example illustrates the application of the results obtained in deciding the problem of alternatives' ranking. The computed aggregated relations R from the example 1 and weighting functions (11), (12) and (13) are:

$$R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0.447 & 0.555 \\ 0.632 & 1 & 0.594 \\ 0.490 & 0.368 & 1 \end{pmatrix} \end{array}, \quad R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0.49 & 0.61 \\ 0.48 & 1 & 0.62 \\ 0.43 & 0.32 & 1 \end{pmatrix} \end{array},$$

$$R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0.47 & 0.59 \\ 0.51 & 1 & 0.64 \\ 0.47 & 0.36 & 1 \end{pmatrix} \end{array}.$$

These fuzzy preference relations are fuzzy preorders (Corollary 3). Then, the perfect antisymmetry relations R' of R are computed (see the definition in Section 3), i.e., if $R(a,b) \geq R(b,a) \rightarrow R'(a,b) = R(a,b) \vee R'(b,a) = 0,$

$$R' = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0.555 \\ 0.632 & 1 & 0.594 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad R' = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0.49 & 0.61 \\ 0 & 1 & 0.62 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad R' = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0.59 \\ 0.51 & 1 & 0.64 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The relations R' are fuzzy partial orderings, because they are reflexive, perfect antisymmetrical and max-Δ transitive. It is obvious that these relations are fuzzy linear orderings, i.e.,

- for the linear weighting function (11) this ordering is:

$$b \xrightarrow{0.632} a \xrightarrow{0.555} c;$$

- for the weighting function (12) this ordering is:

$$a \xrightarrow{0.49} b \xrightarrow{0.62} c;$$

- for the weighting function (13) this ordering is:

$$b \xrightarrow{0.51} a \xrightarrow{0.59} c.$$

It can be seen that, the orderings in the cases with linear and quadratic weighting functions are equal and they differ from the ones with parametric linear function.

6. Concluding remarks

Weighting functions and fuzzy relations are considered herein instead of constant weights of the criteria and crisp evaluations of the criteria. The introduction of weighting functions, depending continuously on the criterion satisfaction values produces weighted aggregation operators with complex dependency on these values. The advantage of these functions consists in their ability to fine the small values and to reward the great values of the membership degrees. The proved properties of the weighted relations give a possibility to use transformed relations in aggregation procedures. The aggregated preorder relation may be used in the decision making problems for choice or ordering among the set of alternatives. The property of likeness of the aggregated relation is useful for solving a clustering problem of the alternatives' set.

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