BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **9**, No 3

Sofia • 2009

Positive Definite Uncertain Homogeneous Matrix Polynomials: Analysis and Application

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Abstract: *The problem of deriving necessary and sufficient conditions for positive definiteness of a Homogeneous Matrix Polynomial (HMP) defined on the unit simplex is considered. The obtained results clearly indicate the superiority of the proposed here solution approach over a classical theorem, which can be viewed as a particular case of them. The main results are applicable for the solution of the stability analysis problem for a class of uncertain linear systems. It is shown that the proved exact conditions generalize and improve (in sense of conservatism reduction) two recent results, aimed at solving the same problem. The presented approach is illustrated by a comparative numerical example.*

Keywords: *Uncertain HMP, M-matrices, Polya's theorem, dynamic polytopic systems.*

1. Introduction

Stability analysis of linear systems subjected to structured real parametric uncertainty, belonging to a compact vector set, has been recognised as a key issue in the analysis of control systems. Usually, a quadratic in the state candidate for a LF is postulated, which is either fixed (quadratic stability), or parameter dependent (robust stability). Quadratic stability approaches lead to conservative results, especially if the uncertainty is known to be constant. On the other hand, robust stability cannot be directly assessed, using convex optimization. In order to reduce the gap between quadratic and robust stability, attempts for reducing the conservatism of LMI methods have been made for more than a decade. Aimed at going beyond parameter-independent LFs, LMI techniques were proposed to derive quadratic in the state candidates for Lyapunov functions, which are affine [6, 7, 14], quadratic [1] and recently polynomial [2, 4, 9, 13, 17], in the uncertain parameter. Robust stability is verified through convex optimization problems formulated in terms of parameterized LMIs, which can be efficiently solved by polynomial-time algorithms. An important result, derived in [3], paved the way to necessary robust stability conditions via homogeneous matrix polynomials (HMPs). More accurate results have been obtained at expense of increased computational effort.

The stability analysis of uncertain linear systems is based mainly on the powerful Lyapunov's second method. It has been proved in [4], that this problem reduces to the determination of non-conservative conditions for positive definiteness of a uncertain HMP, in this case. Therefore, this becomes a problem of outstanding importance.

The objective of this research is to find computable, less conservative, relaxed, necessary and sufficient conditions for positive definiteness of a given HMP, in a case when the uncertain vector α belongs to the unit simplex. It is actually motivated by several recently obtained results [6, 7, 13, 14], aimed at solving the stability analysis problem, which exhibit some common shortcomings (sources of conservatism). The main contributions are: (i) a based on the theory of *M*-matrices, new necessary and sufficient condition for positive definiteness of a HMP of degree two is obtained (Theorem 3), (ii) aimed at taking some additional advantage from the fact that α is a nonnegative vector, some or all pairwise inequalities between its entries are also considered, which results in new alternative necessary and sufficient conditions (Theorems 4, 5 and 7) for positive definiteness of a HMP and stability of a polytopic system, and (iii) three generalizing conditions (Lemmas 1 and 2), proved to be less conservative in comparison with the available ones, are obtained. Contrary to the usual practice, the proposed approach takes into account the contribution of each term of the HMP to the overall positive definiteness, reflecting adequately the various relations between its coefficients.

2. Preliminaries, previous results, open problems

The following notations will be used in the sequel. $\mathbf{N}$ is the set of positive integers and $\mathbf{N}_x$ denotes a set of x positive integers. The *i*-th eigenvalue of a matrix X is $\lambda_i(X)$. The notations $A > (\geq) 0$ and $a > 0$ indicate that A is a positive (semi-) definite matrix and a is a positive vector, $A = [a_{ij}] \in \mathbf{R}_n$ and $a = (a_i) \in \mathbf{R}^N$ denote real $n \times n$ matrix and $N \times 1$ vector with entries a_{ij} and a_i, respectively. The sum of N nonnegative scalars α_i is $|\alpha|$. Define also the vector sets $\mathbf{x}_n \equiv \{x \in \mathbf{R}^n : x^{\mathrm{T}} x = 1\}$ and $\boldsymbol{\omega}_N \equiv \{\alpha = (\alpha_i) \in R^N : |\alpha| = 1\}$.

Consider a HMP in $\alpha \in \boldsymbol{\omega}_N$ of an arbitrary integer degree $k > 1$ with $\chi(k) = \dfrac{(k+N-1)!}{k!(N-1)!}$, $0! = 1$, symmetric matrix coefficients given by

(1) $$\Pi(\alpha,k)=\sum_{|k|=k}\alpha_1^{k_1}\alpha_2^{k_2}...\alpha_N^{k_N}P_{k_1k_2...k_N}\in\mathbf{R}_n.$$

Since $|\alpha|^d=1,\ d=0,1,...$, then the HMP (1) can be equivalently represented as a HMP of degree $d+k$ with $\chi(k+d)$ symmetric matrix coefficients

(2) $$\Pi(\alpha,k)=\Pi(\alpha,k+d)=|\alpha|^d\,\Pi(\alpha,k)= \\ =\sum_{|k|=k+d}\alpha_1^{k_1}\alpha_2^{k_2}...\alpha_N^{k_N}\bar{P}_{k_1k_2...k_N}=\sum_{l=1}^{\chi(k+d)}\tilde{\alpha}_l\Pi_l\in\mathbf{R}_n,$$

where $\tilde{\alpha}_l$ and Π_l, $l=1,...,\chi(k+d)$, denote the lexically ordered monomial $\alpha_1^{k_1}\alpha_2^{k_2}...\alpha_N^{k_N}$ and matrix coefficient $\bar{P}_{k_1k_2...k_N}$, $|k|=k+d$, respectively. Let $k+d$ = 2τ, which makes possible to write

(3) $$\Pi(\alpha,k)=\Pi(\alpha,2\tau)=\sum_{i,j\in\mathbf{N}_{\chi(\tau)},i\le j}\bar{\alpha}_i\bar{\alpha}_j\Pi_{ij},$$

where $\bar{\alpha}_i=\alpha_1^{\tau_1}\alpha_2^{\tau_2}...\alpha_N^{\tau_N},|\tau|=\tau,\ i=1,2,...,\chi(\tau)$, denotes the *i*-th monomial of degree τ, and obviously, $\tilde{\alpha}_l$ = $\bar{\alpha}_i\bar{\alpha}_j$ and Π_l = Π_{ij}, for some subscripts *l*, *i* and *j*.

Define the real uncertain scalar

(4) $$f(\alpha,\tau,x)=x^{\mathrm{T}}\Pi(\alpha,k)x=x^{\mathrm{T}}\Pi(\alpha,2\tau)x=\sum_{i,j\in\mathbf{N}_{\chi(\tau)},i\le j}\bar{\alpha}_i\bar{\alpha}_j\tilde{c}_{ij}(x),\ \ x\in\mathbf{X}_n,$$

where $\tilde{c}_{ij}(x)=x^{\mathrm{T}}\Pi_{ij}x$ and the uncertain vector

$$\bar{\alpha}_v=(\bar{\alpha}_i)^{\mathrm{T}}\in\mathbf{R}^{\chi(\tau)},\ \bar{\alpha}_i=\alpha_1^{\tau_1}\alpha_2^{\tau_2}...\alpha_N^{\tau_N},\ |\tau|=\tau,\ i=1,2,...,\chi(\tau),$$

containing all monomials of degree τ. Then, (4) can be rewritten in a quadratic (with respect to $\bar{\alpha}_v$) compact matrix form as

(5) $$f(\alpha,\tau,x)=\bar{\alpha}_v^{\mathrm{T}}C(x)\bar{\alpha}_v,\ C(x)=[c_{ij}(x)]\in\mathbf{R}_{\chi(\tau)},$$

$$c_{ij}(x)=\pi_{ij}\tilde{c}_{ij}(x),\ \pi_{ij}=\begin{cases}1, & i=j\\ 0.5, & i\ne j\end{cases},$$

where the symmetric matrix $C(x)$ is said to be a Coefficient Matrix (CM) for the uncertain Homogeneous Scalar Polynomial (HSP) $f(\alpha,\tau,x)$.

It is desired to derive conditions under which the HMP in (3) is positive definite on the compact vector set $\boldsymbol{\omega}_N$, i.e., $\Pi(\alpha,k)$ contains only positive definite matrices, or equivalently, the strict scalar inequality

(6) $$f(\alpha,\tau,x)>0\ \ \forall\alpha\in\boldsymbol{\omega}_N,\forall x\in\mathbf{X}_n,$$

holds. Next, consider the following important result concerning the analysis of symmetric HMPs defined on $\boldsymbol{\omega}_N$, obtained in [16].

Theorem 1. Let a given HMP in (1) be positive definite. There exists some sufficiently large integer d^*, such that for all $d \geq d^*$ all $\chi(k+d)$ matrix coefficients of the HMP in (2) are positive definite.

This result generalizes the famous Polya's theorem [8] for the case of HMPs. It is based on the derived in [12] lower bound for *d*, which is proven to be tighter than all previously obtained ones and representing an asymptotically exact condition, it provides a systematic way to decide whether a given HMP is positive definite. Unfortunately, this result is a very conservative one with respect to sufficiency due to the following reasons. It follows that the above stated problem has a solution if and only if for some *d* all coefficients $\tilde{c}_{ij}(x) = x^{\mathrm{T}} P_{ij} x$ of the HSP $f(\alpha, \tau, x)$ are positive for all $x \in \mathbf{x}_n$. It is clear that the scalar inequality in (6) may hold even if some coefficients are not positive definite.

This paper is devoted to the problem of deriving relaxed necessary and sufficient condition for the validity of (6), and is intended to improve and generalize Theorem 1.

3. Relaxed analysis for HMPS

3.1. Case $k = 2$

In this special case, $\chi(2) = 0.5N(N-1)$, $\chi(1) = N$; the HMP in (1) and the HSP in (5) can be represented respectively for $d = 0$, as follows:

$$\Pi(\alpha, 2) = \Pi(\alpha, 2\tau) = \sum_{i,j=1, i \leq j}^{N} \alpha_i \alpha_j \Pi_{ij}, \tag{7}$$

$$f(\alpha, 1, x) = \alpha^{\mathrm{T}} C(x) \alpha, \; C(x) = [c_{ij}(x)] \in \mathbf{R}_N. \tag{8}$$

Let ***L*** denotes the set of real *N* × *N* matrices with nonpositive off-diagonal entries. The set of *M*-matrices consists of all matrices $M \in$ ***L***, which are positive stable and is denoted ***M***. The next theorem presents some of the numerous properties of the matrix set ***M***.

Theorem 2 [10, 11]. A matrix $M \in$ ***M*** if and only if the following equivalent statements hold: (s1) *M* has an eigenvector $\alpha \in \boldsymbol{\omega}_N$, the corresponding to it eigenvalue λ is real and such that $0 < \lambda \leq \mathrm{Re}\,\lambda_i(M)$, *i* = 1,…, *N*; (s2) M^{-1} exists and its entries are nonnegative (nonnegative matrix); (s3) there exists a vector $\beta > 0$, such that $M\beta > 0$; (s4) there exists a positive diagonal matrix *D*, such that $M^{\mathrm{T}} D + DM > 0$.

Positive definiteness of the CM *C*(*x*) in (5) is only a sufficient condition for robust stability. The next result states that under some assumption, it becomes a necessary one, as well.

Theorem 3. Suppose that $MC(x) \in$ ***L*** $\forall x$, $M \in$ ***M***. The following statements are equivalent: (i) the HMP in (7) is positive definite; (ii) for any *x*, there exists a vector $\beta(x) > 0$, such that $C(x)\beta(x) > 0$; (iii) $C(x)$ is a positive definite CM for all x.

P r o o f. Let (i) holds, i.e., $\Pi(\alpha, 2)$ is positive definite on $\boldsymbol{\omega}_N$. In accordance with the assumption that $MC(x) \in \boldsymbol{L}\ \forall x$, and Theorem 2, (s1), there exists some vector $\alpha(x) \in \boldsymbol{\omega}_N$, such that $MC(x)\alpha(x) = \lambda(x)\alpha(x)\ \forall x$. It follows that, $C(x)\alpha(x) = \lambda(x)\ M^{-1}\ \alpha(x)$ and $\alpha^{\mathrm{T}}(x)\,C(x)\,\alpha(x) = \lambda(x)\ \alpha^{\mathrm{T}}(x)\ M^{-1}\ \alpha(x) > 0$, where M^{-1} is a non-negative matrix (Theorem 2, (s2)). The scalar $\alpha^{T}(x)\ M^{-1}\ \alpha(x)$ is positive, then $\lambda(x)$ is also positive, by necessity, and $MC(x) \in \boldsymbol{M}\ \ \forall x$ or equivalently, $MC(x)\beta(x) > 0\ \ \forall x$ for some $\beta(x) > 0$ (Theorem 2, (s3)) and (ii) follows, since $M^{-1}[MC(x)\beta(x)] = C(x)\beta(x) > 0$.

If (ii) holds, i.e.,

$$C(x)M^{\mathrm{T}}M^{-\mathrm{T}}\beta(x) = C(x)M^{\mathrm{T}}\tilde{\beta}(x) > 0,\ \ \tilde{\beta}(x) > 0\ \ \forall x,$$

then, $C(x)M^{\mathrm{T}} \in \boldsymbol{M}\ \ \forall x$, since $C(x)M^{\mathrm{T}} \in \boldsymbol{L}\ \ \forall x$. For any diagonal matrix $D > 0$, matrix $K^{\mathrm{T}}(x) = D^{-1}C(x)M^{\mathrm{T}}D \in \boldsymbol{M}$. In accordance with Theorem 2, (s4), there exist some diagonal matrices $D^{2}(x) > 0$ and $D > 0$, such that $D^{-1}(x)K^{\mathrm{T}}(x)D(x) + D(x)K(x)D^{-1}(x) > 0\ \ \forall x$, i.e.,

$$z^{\mathrm{T}}\{[D^{-1}(x)D^{-1}C(x)D^{-1}(x)][D(x)M^{\mathrm{T}}DD(x)] +$$
$$+[D(x)DMD(x)][D^{-1}(x)C(x)D^{-1}D^{-1}(x)]\}z > 0\ \ \forall z \in \mathbf{x}_{N}$$

and $M^{\mathrm{T}}D + DM > 0$. All eigenvalues of $D^{-1}(x)C(x)D^{-1}D^{-1}(x)$ are real and let $z = z(x)$ be the eigenvector corresponding to the minimal one $\lambda(x)$. The above scalar inequality becomes:

$$\lambda(x)z^{\mathrm{T}}(x)[D(x)(M^{\mathrm{T}}D + DM)D(x)]z(x) > 0$$
$$\forall x \Rightarrow \lambda(x) = \lambda_{\min}[C(x)\tilde{D}(x)] > 0, \forall x \Leftrightarrow C(x) > 0\ \ \forall x,$$

where $\tilde{D}(x) = [DD^{2}(x)]^{-1}$ is a positive diagonal matrix. Finally, (iii) always implies (i). ■

This result shows also that under some assumptions statements (ii) and (iii) are equivalent necessary and sufficient conditions for positive definiteness of a HMP of degree two. From the theory of positive matrices it is known that (i) and (ii) are valid only if $C(x) \in \boldsymbol{L}\ \ \forall x$. Due to Theorem 3, this assumption is not necessarily required any more, in order to have statements (ii) and (iii) applicable for the analysis of positive definiteness. Although a *M*-matrix is used for their proof, it is not actually present in them.

Let $\alpha(s)$ denotes a vector with $s \geq 2$ arbitrarily selected entries from α. If $\mathbf{v}(s)$ is the set of $s \times 1$ vectors with entries representing an arbitrary nondescending sequence, then all possible systems of $s!0.5s(s\text{–}1)$ pairwise inequalities $\alpha_i \leq \alpha_j,\ \alpha_i, \alpha_j \in \alpha(s), i \neq j$, are described by the set of ordered vectors

$\alpha_p(s) \in \mathbf{V}(s)$, p = 1,…, s!. For fixed N, the number of all possible distinct and compatible monomial inequalities $\alpha_i\alpha_j \le \alpha_u\alpha_v$, $i \le j,\ u \le v,\ ij \ne uv$, is given by

$$\mu(s) = 0.5[s(s-1)N + \sum_{i=0}^{s-2}(s-i)(s-i-1) + \sum_{k=2}^{s-1}\sum_{i=k}^{s-1}(s-i+1)(s-i)].$$

For any $\alpha_p(s)$, the $\mu(s)$ scalar inequalities imply $\mu(s)$ matrix inequalities $(\alpha_i\alpha_j - \alpha_u\alpha_v)X_{ijuv,p} \le 0$, where $X_{ijuv,p} = X_{uvij,p} \ge 0$ are arbitrary matrices. Consider the associated with $\alpha_p(s) \in \mathbf{V}(s)$ HMP

(9) $$\tilde{\Pi}_p(\alpha,2) = \sum_{i,j=1,i\le j}^{N}\alpha_i\alpha_j(\sum_{u,v=1,u\le v,uv\ne ij}^{N}\mu_{ijuv,p}X_{ijuv,p}) = \sum_{i,j=1,i\le j}^{N}\alpha_i\alpha_j\tilde{\Pi}_{ij,p},$$

p = 1,…,s!

where $\mu_{ijuv,p} = 1$, if $\alpha_i\alpha_j - \alpha_u\alpha_v \le 0$, $\mu_{ijuv,p} = -1$, otherwise, and $\mu_{ijuv,p} = 0$, if the sign of the monomial difference is indefinite, due to $s < N$. Consider the HMP

(10) $$\Pi_p(\alpha,2) = \Pi(\alpha,2) + \tilde{\Pi}_p(\alpha,2) = \sum_{i,j=1,i\le j}^{N}\alpha_i\alpha_j\Pi_{ij,p}, \quad \Pi_{ij,p} = \Pi_{ij} + \tilde{\Pi}_{ij,p},$$

p = 1,…,s!.

The next theorem provides an alternative necessary and sufficient robust stability condition.

Theorem 4. The HMP in (7) is positive definite on $\boldsymbol{\omega}_N$ if and only if for any $s \ge 2$ there exist $s!\,\mu(s)$ parameter matrices in (9), such that all s! HMPs in (10) are positive definite on $\boldsymbol{\omega}_N$.

P r o o f. Let $\Pi(\alpha,2) > 0\ \ \forall\alpha \in \boldsymbol{\omega}_N$. In accordance with Theorem 1, there exists some positive integer d, such that all coefficients of the HMP $|\alpha|^d\,\Pi(\alpha,2) = \Pi(\alpha,d+2)$ of degree d + 2 are positive definite. Then, the following implication holds:

$$\Pi_p(\alpha,2) > 0\ \ \forall\alpha,\ p \Leftrightarrow |\alpha|^d\,\Pi_p(\alpha,2) = \Pi(\alpha,d+2) + \tilde{\Pi}_p(\alpha,d+2) > 0;$$

$$\Pi(\alpha,d+2) > 0\ \ \forall\alpha,\ p.$$

For any $s \ge 2$ and p, there always exist some appropriate $\mu(s)$ positive semi-definite matrices in (9), such that all coefficients of the HMP $|\alpha|^d\,\Pi_p(\alpha,2)$ are positive definite matrices, which guarantees that $\Pi_p(\alpha,\ 2) > 0\ \ \forall\alpha,\ p.$

Let the converse be true, i.e., $\Pi_p(\alpha,\ 2) > 0\ \ \forall\alpha,\ p,$ and consider an arbitrary vector α. For any $s \ge 2$, there exists some vector $\alpha_p(s)$, such that the s common entries of α and $\alpha_p(s)$ represent one and the same nondescending sequence. Having in mind (3) and (4), one has

$$\tilde{\Pi}_p(\alpha, 2) \le 0 \Rightarrow 0 < \Pi_p(\alpha, 2) \le \Pi(\alpha, 2)$$

for this particular α, but since it has been arbitrarily chosen, it follows that $\Pi(\alpha, 2) > 0 \ \forall \alpha$, and inequality (2) holds. ■

In other words, the function $f(\alpha, 1, x)$ is a positive if and only if there exist $s!\,\mu(s)$ matrices in (8), such that

$$f(\alpha, 1, x) + x^{\mathrm{T}}[\tilde{\Pi}_p(\alpha, 2)]x = x^{\mathrm{T}}\Pi_p(\alpha, 2)x = \alpha^{\mathrm{T}}[C(x) + \tilde{C}_p(x)]\alpha = \alpha^{\mathrm{T}}C_p(x)\alpha > 0 \quad \forall \alpha, x, p, \tag{11}$$

where $\tilde{C}_p(x)$ and $C_p(x)$, p = 1,…, s!, are the CMs for the HMPs in (9) and (10), respectively. If s = 1, then, $\mathbf{v}(1) \equiv \varnothing$, $\mu(1) = 0$ and (11) reduces to the trivial condition $f(\alpha, 1, x) > 0$.

Consider an arbitrary matrix coefficient defined in (10) and denote

$$\Pi_{ij,p} = \Pi_{ij} + \Pi^{+}_{ij,p} + \Pi^{-}_{ij,p}; \ \Pi^{+}_{ij,p} = \sum_{\mu_{ijuv,p}=1} X_{ijuv,p} \ge 0,$$

$$\Pi^{-}_{ij,p} = -\sum_{\mu_{ijuv,p}=-1} X_{ijuv,p} \le 0, i \le j, \ p = 1,\ldots,s!.$$

Remark 1. For any $\alpha_p(s)$, one has $\mu_{ijuv,p} = -\mu_{uvij,p}$. This means that all sums $\Pi^{+}_{ij,p}$, $i \le j$, are composed of distinct matrices (the same refers to all $\Pi^{-}_{ij,p}$) since, if $\mu_{ijuv,p} = 1$, then $X_{ijuv,p}$ participates only in the sum $\Pi^{+}_{ij,p}$ and it appears only once more time, but now as $-X_{uvij,p} = -X_{ijuv,p}$, in the sum $\Pi^{-}_{uv,p}$. Respective conclusions are made when $\mu_{ijuv,p} = -1$.

Theorems 4 and 5 give rise to the following new robust stability conditions.

Lemma 1. For an integer $s \ge 2$ and $s!\,\mu(s)$ parameter matrices in (8) the following statements are distinct sufficient conditions for validity of the inequality in (11):

(i) there exist s! M-matrices M_p and positive vectors $\beta_p = (\beta_{j,p}) \in \mathbf{R}^N$, such that

$$M_p C_p(x) \in \boldsymbol{L}, \forall x; \quad C_p(x)\beta_p > 0, \forall x, \ p = 1,\ldots,s! \Leftrightarrow$$
$$\Leftrightarrow \sum_{j=1, j\ge i}^{N} \beta_{j,p}(\pi_{ij}\Pi_{ij,p}) > 0, i = 1,\ldots,N, \ p = 1,\ldots,s!; \tag{12}$$

(ii) there exist $s!0.5N(N+1)$ scalars $c_{ij,p}$, such that

$$c_{ij,p}(x) \ge c_{ij,p} \quad \forall x, i \le j; \quad C_p = [c_{ij,p}] > 0, \ p = 1,\ldots,s!. \tag{13}$$

P r o o f. The proof of the first statement follows easily from Theorem 3, (ii), and Theorem 4. If (13) holds, then inequality (5) is valid, since $0 < \alpha^{\mathrm{T}}C_p\alpha \le \alpha^{\mathrm{T}}C(x)\alpha \ \ \forall \alpha, x, p.$ ■

As the dimension of the selected vector $\alpha(s)$ increases, one obtains more and more relaxed conditions and the maximal effect is achieved when the whole vector α is selected, i.e., for $s = N$. The next lemma states a condition, which eliminates the awkward dependence of all CMs in (5) on vector x.

Lemma 2. Let α_m denotes the minimal entry of the p-th ordered vector $\alpha_p(N)$, $p = 1,\ldots,N!$. Consider the $N!\mu(N)$ parameter matrices in (9), chosen for any p, as follows:

(14) $$-\Pi^-_{ij,p} = \Pi_{ij} + \Pi^+_{ij,p}, \text{ if } \Pi_{ij} + \Pi^+_{ij,p} \geq 0 \Rightarrow c_{ij,p}(x) = c_{ij,p} = 0, \quad i < j,$$

(15) $$-\Pi^-_{ij,p} = \Pi_{ij} + \Pi^+_{ij,p} - \lambda_{\min}(\Pi_{ij} + \Pi^+_{ij})I \geq 0,$$
$$\text{if } \lambda_{\min}(\Pi_{ij} + \Pi^+_{ij,p}) = c_{ij,p} < 0 \Rightarrow c_{ij,p}(x) = c_{ij,p} < 0, \; i < j,$$

(16) $$-\Pi^-_{ii,p} = \Pi_{ii} + \Pi^+_{ii,p} - \lambda_{\min}(\Pi_{ii} + \Pi^+_{ii})I \geq 0 \Rightarrow$$
$$c_{ii,p}(x) = \lambda_{\min}(\Pi_{ii} + \Pi^+_{ii,p}) = c_{ii,p} \quad \forall\, i \neq m.$$

For this choice, (11) holds if and only if $C_p = [c_{ij,p}] > 0 \;\; \forall p$, where $0.5N(N+1) - 1$ of the entries of C_p are defined in (14), (15) and (16) and $c_{mm,p} = \lambda_{\min}(\Pi_{mm} + \Pi^+_{mm,p}) \;\; \forall p$.

P r o o f. Having in mind Remark 1, all equalities in (14), (15) and (16) are possible, since $s = N$, and $\mu_{ijuv,p} = -\mu_{uvij,p} \neq 0 \;\; \forall p$. For any p, the entries of the CM $C_p(x)$ are equal to some scalars $c_{ij,p}$, except for the entry $c_{mm,p}(x) = x^{\mathrm{T}}(\Pi_{mm} + \Pi^+_{mm,p})x$, since $\mu_{mmuv,p} = 1 \;\; \forall u, v$, i.e., $\Pi^-_{mm,p} = 0 \;\; \forall p$. Taking $c_{mm,p}$ as above is obligatory, since $c_{mm,p}(x) = c_{mm,p} \;\; \forall p$, is an admissible case. Then, inequality (11) holds if and only if $\alpha^{\mathrm{T}} C_p \alpha > 0 \;\; \forall \alpha, p$, which is a necessary and sufficient condition for positive definiteness of the considered HMP, due to Theorem 4. Finally, from Theorem 3, (iii), it follows that this condition is equivalent to $C_p > 0 \;\; \forall p$, since $C_p \in \boldsymbol{L} \;\; \forall p$. ■

3.2. Positive definite HMP of arbitrary degree k

A monomial $\tilde{\alpha}_l = \bar{\alpha}_i \bar{\alpha}_j$ of an arbitrary even degree is said to be even if $i = j$, otherwise it called an odd one. Consider the μ even monomials of variable degree $2|r|$, given by

$$\tilde{\alpha}_f = \alpha_1^{2r_1} \alpha_2^{2r_2} \ldots \alpha_i^{2r_i} \ldots \alpha_j^{2r_j} \ldots \alpha_N^{2r_N}; \; |r| = 0, 1, \ldots, \tau - 1, \;\; \mu = \sum_{|r|=0}^{\tau-1} \chi(|r|).$$

Any scalar inequality $\alpha_i \le \alpha_j, i \ne j, \alpha_i, \alpha_j \in \alpha(s)$, implies μ monomial inequalities of the form

(17) $$\tilde{\alpha}_f \alpha_i^{\gamma+1} = \bar{\alpha}_f^2 \le \tilde{\alpha}_f \alpha_i \alpha_j^{\gamma} = \bar{\alpha}_h \bar{\alpha}_g;\ f, g, h \in \mathbf{N}_{\chi(\tau)}, g \ne h,$$

where $\gamma = 2\tau - 2|r| - 1 = 1, 3, 5, ..., 2\tau - 1$ and $\bar{\alpha}_f, \bar{\alpha}_h, \bar{\alpha}_g$ are some monomials of degree τ.

Any vector $\alpha_p(s) \in \mathbf{V}(s)$ defines $\mu_s = 0.5s(s-1)$ pairwise inequalities (17). Finally, all $s!$ vectors $\alpha_p(s)$ determine $s!\,\mu_s$ systems of such monomial inequalities, which correspond to all possible sets of pairwise inequalities involving the entries of vector $\alpha(s)$. For a given $\alpha_p(s) \in \mathbf{V}(s)$ any odd monomial in (17) serves as an upper bound for at least one even monomial, i.e.,

(18) $$\bar{\alpha}_f^2 \le \bar{\alpha}_h \bar{\alpha}_g;\ f \in \mathbf{N}_{\eta_{gh}} \subset \mathbf{N}_{\chi(\tau)}, \eta_{gh} \ge 1, g \ne h,$$

and any even monomial is a lower bound for at least one odd monomial.

For a given s, consider the associated with some vector $\alpha_p(s) \in \mathbf{V}(s)$ HMP of degree 2τ

(19) $$\tilde{\Pi}_p(\alpha, 2\tau) = \sum_{f,g,h \in \mathbf{N}_{\chi(\tau)}, g \ne h} (\bar{\alpha}_f^2 - \bar{\alpha}_h \bar{\alpha}_g) X_{fgh,p} \in \mathbf{R}_n;\ \bar{\alpha}_f^2 \le \bar{\alpha}_g \bar{\alpha}_h,\ p = 1, 2, ..., s!,$$

with μ_s arbitrary positive semidefinite matrix coefficients $X_{fgh,p}$. Assume that N of the even monomials in (2) are lexically ordered as follows: $\bar{\alpha}_i^2 = \alpha_i^{2\tau},\ i = 1, 2, ..., N$. For $\tau > 1$, any of the rest $\chi(\tau) - N$ even monomials can be represented as a product of two distinct monomials of degree τ, i.e., $\bar{\alpha}_t^2 = \bar{\alpha}_u \bar{\alpha}_v\ \ u, v \in \mathbf{N}_{\chi(\tau)}$, which makes possible the definition of the HMP

(20) $$\tilde{\Pi}_{0p}(\alpha, 2\tau) = \sum_{t=N+1}^{\chi(\tau)} \sum_{u,v \in \mathbf{N}_{\chi(\tau)}, u \ne v} (\bar{\alpha}_t^2 - \bar{\alpha}_u \bar{\alpha}_v) X_{tuv,0p} = 0 \in \mathbf{R}_n\ \forall \alpha,\ p = 1, 2, ..., s!,$$

with $\chi(\tau) - N$ arbitrary symmetric matrix coefficients. Consider the HMP

(21) $$\Pi_p(\alpha, 2\tau) = \Pi(\alpha, 2\tau) + \tilde{\Pi}_p(\alpha, 2\tau) + \tilde{\Pi}_{0p}(\alpha, 2\tau) = \\ = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \bar{\alpha}_i \bar{\alpha}_j \Pi_{ij},\ p = 1, 2, ..., s!,$$

where $\Pi(\alpha, 2\tau)$ is defined in (3), the associated with it Homogeneous Scalar Polynomial (HSP) in α and $x \in \mathbf{X}_n$

(22) $$h_p(\alpha, 2\tau, x) = x^{\mathrm{T}} \Pi_p(\alpha, 2\tau) x = \sum_{i,j \in \mathbf{N}_{\chi(\tau)}, i \le j} \bar{\alpha}_i \bar{\alpha}_j c_{ij,p}(x), \\ c_{ij,p}(x) = x^{\mathrm{T}} \Pi_{ij,p} x,\quad p = 1, 2, ..., s!,$$

and the HSP in α

(23) $$h_p(\alpha, 2\tau) = \sum_{i,j\in \mathbf{N}_{\chi(\tau)}, i\le j} \overline{\alpha}_i \overline{\alpha}_j c_{ij,p}; \; c_{ij,p} I \le \Pi_{ij,p}, \quad p = 1, 2, ..., s!,$$

where $\Pi_{ij,p}$ denotes the respective symmetric matrix coefficient of the HMP in (21), corresponding to the monomial $\overline{\alpha}_i \overline{\alpha}_j$ and vector $\alpha_p(s) \in \mathbf{V}(s)$. Consider the quadratic in $\overline{\alpha}_v$ matrix representation of the p-th HSP in (10)

(24) $$h_p(\alpha, 2\tau) = \overline{\alpha}_v^{\mathrm{T}} C_p \overline{\alpha}_v, \quad C_p = C_p^{\mathrm{T}} \in \mathbf{R}_{\chi(\tau)}, \quad p = 1, 2, ..., s!.$$

The entries of the CM C_p are the $\chi(2\tau)$ scalar valued coefficients of the HSP in (23).

Theorem 5. Consider a given HMP (1) of arbitrary degree. The following statements are equivalent:

(a) there exist scalars d and s, such that $d + k = 2\tau$, $1 \le s \le N$, $s![\mu_s + \chi(\tau) - N]$ parameter matrices in (19) and (20) and $s!\chi(2\tau)$ scalars $c_{ij,p}$ defined in (23), such that all $s!$ CMs in (24) are positive definite;

(b) $\Pi(\alpha, k)$ is a positive definite HMP.

P r o o f. Let assertion (a) holds. Consider an arbitrary given vector $\alpha \in \boldsymbol{\omega}_{\mathrm{N}}$. For any $s \ge 1$, there exists some subscript p, such that the s common entries of the vectors α and $\alpha_p(s) \in \mathbf{V}(s)$ represent one and the same sequence of non-descending scalars. Having in mind (1), (2), (3) and the HMPs in (19), (20) and (21), the following matrix and scalar inequalities are valid:

$$\Pi(\alpha, k) = \Pi(\alpha, 2\tau) \ge \Pi_p(\alpha, 2\tau) \Leftrightarrow h_p(\alpha, 2\tau, x) \ge h_p(\alpha, 2\tau) = \overline{\alpha}_v^{\mathrm{T}} C_p \overline{\alpha}_v \quad \forall x \in \mathbf{X}_n.$$

Since all CMs are positive definite by assumption and vector α has been arbitrarily chosen, it follows that (1) is a positive definite HMP. In the special case when $s = 1$, then $\mathbf{V}(1)$ is an empty set, $\mu_s = 0$ and $\Pi(\alpha, k) = \Pi_1(\alpha, 2\tau) = = \Pi(\alpha, 2\tau) + \Pi_{01}(\alpha, 2\tau)$. Positive definiteness of the single CM C_1 is a sufficient condition for positive definiteness of the HMP. This proves (a) $\Rightarrow$ (b).

Let (b) be valid. According to Theorem 1, there exists some d, such that all $\chi(2\tau)$ matrix coefficients Π_{ij} in (3) are positive definite. Let $X_{tuv,0p} = 0 \;\forall p = 1, 2, ..., s!$, in this case. It will be shown that (a) holds for arbitrary integer s. Let $s = 1$. Then, there always exist some appropriate scalars $c_{ij,1}$ in (23), such that C_1 is a positive definite CM. If s > 1, consider an arbitrary vector $\alpha_p(s) \in \mathbf{V}(s)$ and the associated with it monomial inequality in (18), which implies

$$\sum_{f\in \mathbf{N}_{\eta_{gh}}} \bar{\alpha}_f^2 X_{fgh,p} \le \bar{\alpha}_g \bar{\alpha}_h \tilde{\Pi}_{gh,p};\ \tilde{\Pi}_{gh,p} = \sum_{f\in \mathbf{N}_{\eta_{gh}}} X_{fgh,p},$$

$$X_{fgh,p} \ge 0,\ g \ne h,\ p = 1,\ 2,\ldots,\ s!.$$

Let the coefficient matrices in (19) are chosen, such that $\tilde{\Pi}_{gh,p} = \Pi_{gh}$ for all subscript pairs (g, h) in (18). This choice guarantees that $\theta_{\mathrm{o}}(s)$ (number of distinct odd monomials in (17)) matrix coefficients in (21) become $\Pi_{gh,p} = \tilde{\Pi}_{gh,p} - \Pi_{gh} = 0$, and $\theta_{\mathrm{e}}(s)$ (number of distinct even monomials in (17)) matrix coefficients are $\Pi_{ff,p} > \Pi_{ff}$. The rest of the $\chi(2\tau) - [\theta_{\mathrm{o}}(s) + \theta_{\mathrm{e}}(s)]$ matrix coefficients are not affected and hence they remain positive definite. For a given s, the integers $\theta_{\mathrm{o}}(s)$ and $\theta_{\mathrm{e}}(s)$ are fixed for all p = 1, 2,…, s!. There always exist some appropriate scalars $c_{ij,p}$, such that the respective p-th CM is a positive definite one. The vector $\alpha_p(s)$ has been arbitrarily chosen, which proves that (b) $\Rightarrow$ (a) for any s. ■

Theorem 4 can be viewed as a generalization of Theorem 1, which overcomes its most important shortcoming – the unnecessarily hard requirement for positive definiteness of all matrix coefficients in (2). At the same time, the equivalence of the necessary and sufficient conditions (6) and (11) for positive definiteness of a given HMP of arbitrary degree revealed by Theorem 4 makes possible to achieve flexibility by means of an appropriate choice for the parameter matrices in (19) and (20). Next, it will be shown how these facts become very important for getting relaxed exact solution conditions when the discussed below application problem is faced.

4. Application: Exact stability conditions for uncertain systems

Consider the uncertain linear system

$$\dot{x} = A(\alpha)x, \quad A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i \in \mathbf{R}_n, \quad \alpha \in \boldsymbol{\omega}_N,$$

where all matrices A_i are fixed and Hurwitz (negative stable). The stability analysis problem for this class of uncertain systems is: determine necessary and sufficient conditions, under which the polytope $\mathbf{A} = \{A(\alpha) : \alpha \in \boldsymbol{\omega}_{\mathrm{N}}\}$ contains only Hurwitz matrices.

Theorem 6 [15]. The following statements are equivalent:

(a) $\mathbf{A}$ is a Hurwitz polytope;

(b) there exists a HMP $\Pi(\alpha,\ s) \in \mathbf{R}_n$ in (1) of degree $s \le \bar{s} = b + 1$, $b = 0.5n(n-1)$ such that

(25) $$\Pi(\alpha,k) = -\{A^{\mathrm{T}}(\alpha)\Pi(\alpha,s) + \Pi(\alpha,s)A(\alpha)\} > 0 \quad \forall\, \alpha \in \boldsymbol{\omega}_N .$$

The significance of this result, proved by means of the fundamental result obtained in [5], consists in the determination of a tighter, than some previously derived, upper bound for the degree s of the HMP, via which the robust stability of a given polytopic system can be analyzed. E.g., this bound was determined as $\bar{s} = b + n$ in [4] for HMPs and $\bar{s} = 2nN$ in [9], or $\bar{s} = b + n - 1$ in [17] for general matrix polynomials. Theorem 6 provides a necessary and sufficient condition for stability of a class of uncertain systems, expressed in terms of the positive definiteness of a HMP. Theorem 5 proposes a new relaxed asymptotically exact condition via which it can be tested. The next theorem summarizes them.

Theorem 7. The following statements are equivalent:

(i) assertion (a) of Theorem 5, holds for the HMP $\Pi(\alpha,k)$ in (25);

(ii) $\mathbf{A}$ is a Hurwitz polytope.

P r o o f. It is entirely based on Theorems 5 and 6 and due to self-evidence is omitted.

Theorem 7 represents a generalization of two recent results presented below.

Theorem 8 [6, 7]. Let $d = 0$ and $s = 1$. A HMP $\Pi(\alpha,1)$ of degree one assures robust stability of $\mathbf{A}$, i.e., validity of the matrix inequality in (25), if the single CM C_1 in (24) is positive definite.

Theorem 9 [13]. The polytope $\mathbf{A}$ is Hurwitz stable if and only if there exist a HMP $\Pi(\alpha,s)$ and some sufficiently large integer d, such that all matrix coefficients of the HMP $|\alpha|^d\ \Pi(\alpha,k)$ are positive definite, where $\Pi(\alpha,k)$ denotes the HMP in (25).

Although Theorem 7 and Theorem 9 provide distinct, but asymptotically equivalent conditions, their sufficiency parts differ substantially. For one and the same HMP $\Pi(\alpha,s)$, via which the robust stability of $\mathbf{A}$ is analyzed, the stated conditions are based on Theorems 5 and 1, respectively. Theorem 9 treats the problem as the solution of a rather conservative set of, more all less isolated LMIs, where only the sign of the coefficient matrices is significant. On the contrary, Theorem 7 takes into account their lower bounds and the various relations between them put in a Compact Matrix form (CM). It is clear that if robust stability of $\mathbf{A}$ is concluded via Theorem 9, the same refers to Theorem 7, but not *vice versa*, since all s! CMs may be positive definite, when some or even all matrix coefficients Π_{ij}, $i < j$, are not positive definite. Theorem 1 is based on the assumption that the uncertainty vector is non negative, while Theorem 5 is aimed at taking some additional advantage from this fact. Taking into account the presence of monomial inequalities makes possible to get some relaxations. E.g., if Π_{ij}, $i < j$, is a positive semi-definite matrix, then this fact can be reflected by the choice $X_{ij,p} = \Pi_{ij}$ for the respective parameter matrix in (7), which leads to $\Pi_{ij,p} = 0$ and

$\Pi_{ii,p} = \Pi_{ii} + \Pi_{ij}^{+} \geq \Pi_{ii}$, i.e., this approach reflects the contribution of each positive semi-definite term $\overline{\alpha}_i \overline{\alpha}_j \Pi_{ij}$ to the overall HMPs positive definiteness. For $X_{ij,p} = \Pi_{ij} - \lambda_{\min}(P_{ij}) I \geq 0$, an advantage is obtained, even if Π_{ij} is a sign-indefinite matrix.

5. Numerical example

Example. Consider a polytope $\mathbf{A}$ and a HMP $\Pi(\alpha,\, 1) = \sum_{i=1}^{3} \alpha_i \Pi_i$, described by the vertices:

$$A_1 = \begin{bmatrix} -0.6895 & 4.137 & 4.728 \\ -9.8500 & -3.152 & -0.197 \\ -13.1990 & -13.987 & -15.169 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.3276 & 0.3432 & 0.4836 \\ -2.0280 & -0.5616 & 0.2184 \\ -1.9968 & -1.4976 & -1.2480 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.7056 & 2.856 & 2.352 \\ -4.8496 & -1.344 & -0.224 \\ -5.6000 & -8.232 & -7.168 \end{bmatrix};$$

$$\Pi_1 = \begin{bmatrix} 0.7880 & 0.1970 & 0.2955 \\ 0.1970 & 0.5910 & 0.3940 \\ 0.2955 & 0.3944 & 0.7880 \end{bmatrix}, \qquad \Pi_2 = \begin{bmatrix} 0.1404 & 0.0312 & 0.0156 \\ 0.0312 & 0.0624 & 0.0312 \\ 0.0156 & 0.0312 & 0.0780 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} 0.3864 & 0.0560 & 0.1120 \\ 0.0560 & 0.3920 & 0.2240 \\ 0.1120 & 0.2240 & 0.3360 \end{bmatrix}.$$

For $d = 0$ the HMP in (25) is $\Pi(\alpha,\, 2) = \sum_{i,j=1, i\leq j}^{3} \alpha_i \alpha_j \Pi_{ij}$ and let λ_{ij} denotes the minimal eigenvalue of Π_{ij}. It is desired to find whether the HMP $\Pi(\alpha,\, 2)$ is positive definite on the unit simplex.

The respective eigenvalues of the matrix coefficients are computed as:

$$\lambda_{11} = 0.139626,\ \lambda_{22} = 0.00172,$$

$$\lambda_{33} = 0.05448,\ \lambda_{12} = 0.02284,\ \lambda_{13} = -0.001423,\ \lambda_{23} = -0.001998,$$

i.e., for $d = 0$, the polytope is not stable by Theorem 9. Consider the case when $s = 1$.

Since there does not exist a scalar $c_{12,1} \leq \lambda_{12}$, such that the single CM C_1 is positive definite, the considered HMP does not assure robust stability, in accordance with Theorem 8. Let $s = 2$ and $\alpha(2) = (\alpha_2\ \ \alpha_3)^{\mathrm{T}}$. For the single parameter matrix in (19), chosen as $X_{23,1} = X_{23,2} = X_{23} = \Pi_{23} - \lambda_{23} I$, one gets

$$\Pi_{23,1} = \Pi_{23,2} = \lambda_{23} I,\ \ \Pi_{33,1} = P_{33} + X_{23}, \Pi_{22,2} = P_{22} + X_{23}.$$

Consider the two possible cases:

$p = 1.\ \alpha_2 \ge \alpha_3 \Rightarrow \alpha_3^2 \le \alpha_2\alpha_3 \Rightarrow \lambda_{\min}(\Pi_{33,1}) = 0.0864 = \lambda_{33,1} > \lambda_{33}$,

$p = 2.\ \alpha_3 \ge \alpha_2 \Rightarrow \alpha_2^2 \le \alpha_2\alpha_3 \Rightarrow \lambda_{\min}(\Pi_{22,2}) = 0.08137 = \lambda_{22,2} > \lambda_{22}$.

Let $c_{12,1} = c_{12,2} = 0 < \lambda_{12}$. The respective CMs C_1 and C_2 are:

$$2C_1 = \begin{bmatrix} 2\lambda_{11} & 0 & \lambda_{13} \\ 0 & 2\lambda_{22} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & 2\lambda_{33,1} \end{bmatrix} > 0\ ;\ 2C_2 = \begin{bmatrix} 2\lambda_{11} & 0 & \lambda_{13} \\ 0 & 2\lambda_{22,2} & \lambda_{23} \\ \lambda_{13} & \lambda_{23} & 2\lambda_{33} \end{bmatrix} > 0.$$

According to Theorem 7, **A** is robustly stable via the considered HMP $\Pi(\alpha, 1)$ with $d = 0$. Robust stability by Theorem 9 has been concluded for $d = 2$. This simple example illustrates the advantages of Theorem 7 over Theorems 8 and 9.

Acknowledgements. This research work is supported by the Bulgarian Academy of Sciences under Grant No 010077/2007.

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