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Chebyshev Type Inequalities for Sugeno Integrals with Respect to Intuitionistic Fuzzy Measures

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***Abstract*:** *We prove a general version of the Chebyshev inequality for the Sugeno integral type of intuitionistic fuzzy-valued functions with respect to intuitionistic fuzzy-valued fuzzy measures.*

***Keywords*:** *Intuitionistic fuzzy value, Fuzzy measure, Fuzzy integral, Integral inequality.*

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by S u g e n o [16] and intensively studied. Monographs [15] and [18] are dedicated to this topic. Recently, several classical inequalities were generalized to fuzzy integral. F l o r e s-F r a n u l i č and R o m á n-F l o r e s [11] provided a Chebyshev type inequality for fuzzy integral of continuous and strictly monotone functions based on Lebesgue measure, then O u y a n g, F a n g and W a n g [13] generalized the result and obtained a Chebyshev type inequality for fuzzy integral of monotone functions based on an arbitrary fuzzy measure. The series of papers on this topic was recently closed by M e s i a r and O u y a n g ([14]) with a general version of Chebyshev inequality for the Sugeno integral on an abstract fuzzy measure space, based on a product-like operation.

The theory of intuitionistic fuzzy measures was developed in the last years (see [3-8]). In this contribution we use the theorem of decomposition of the Sugeno integral with respect to intuitionistic fuzzy-valued fuzzy measures ([8]) and the general result in [14] to obtain a general Chebyshev type inequality for intuitionistic fuzzy-valued fuzzy integrals of measurable intuitionistic fuzzy-valued functions with respect to intuitionistic fuzzy-valued fuzzy measures.

2. Preliminaries

We recall some basic definitions and previous results which will be used in the sequel.

Throughout this paper, X is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of X, $\overline{\mathbb{R}_+} = [0, +\infty]$ and all considered subsets belong to $\mathcal{A}$.

Definition 1 ([17]). A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}_+}$ is called a fuzzy measure if the following properties are satisfied:

(i) $\mu(\emptyset) = 0$;

(ii) $A \subseteq B$ implies $\mu(A) \leq \mu(B)$;

(iii) $A_1 \subseteq A_2 \subseteq ...$ implies $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$;

(iv) $A_1 \supseteq A_2 \supseteq ...$ and $\mu(A_1) < +\infty$ imply $\mu\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty} \mu(A_n)$.

If $\mu(X) = 1$ then μ is called a normalized fuzzy measure.

When μ is a fuzzy measure, the triple $(X, \mathcal{A}, \mu)$ is called a fuzzy measure space and we denote by $\mathcal{A}_+^{\mu}(X)$ the set of all non-negative μ-measurable functions with respect to $\mathcal{A}$.

Definition 2 ([15-18]). Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space and $A \in \mathcal{A}$. The Sugeno integral of $f \in \mathcal{A}_+^{\mu}(X)$ on A, with respect to the fuzzy measure μ, is defined by

$$(S)\int_A f\, d\mu = \bigvee_{\alpha\geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $F_\alpha = \{x \in X : f(x) \geq \alpha\}$, $\alpha \geq 0$.

We recall that two functions $f_1, f_2 : X \to \mathbb{R}$ are said to be comonotone (we denote $f_1 \sim f_2$) if for all $x, y \in X$

$$(f_1(x) - f_1(y))(f_2(x) - f_2(y)) \geq 0.$$

The set of intuitionistic fuzzy values

$$\mathcal{L} = \{(x, y) : x, y \in [0, 1], x + y \leq 1\}$$

is important in intuitionistic fuzzy set theory ([1,2]). The set $\mathcal{L}$ is a complete lattice ([9]),

$$(x_1, y_1) \leq_{\mathcal{L}} (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \geq y_2,$$

$$
\begin{aligned}
\sup_{\mathcal{L}} W &= (\sup \{x \in [0,1] \,|\exists y \in [0,1] : (x,y) \in W\}, \\
&\qquad \inf \{y \in [0,1] \,|\exists x \in [0,1] : (x,y) \in W\}), \\
\inf_{\mathcal{L}} W &= (\inf \{x \in [0,1] \,|\exists y \in [0,1] : (x,y) \in W\}, \\
&\qquad \sup \{y \in [0,1] \,|\exists x \in [0,1] : (x,y) \in W\}),
\end{aligned}
$$

for each $W \subseteq L$. If $(a_n)_{n\in\mathbb{N}} \subset \mathcal{L}, a_n = (x_n, y_n)$ is increasing, i.e. $a_n \leq_{\mathcal{L}} a_{n+1}$, for every $n \in \mathbb{N}$, or decreasing, that is $a_{n+1} \leq_{\mathcal{L}} a_n$, for every $n \in \mathbb{N}$, then it is convergent and (see [6, p. 168])

$$
\lim_{\substack{\mathcal{L} \\ n\to\infty}} a_n = \left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right).
$$

Definition 3 (see [6]). A mapping $v : \mathcal{A} \to \mathcal{L}$ is called an *intuitionistic fuzzy-valued fuzzy measure* or *intuitionistic fuzzy measure* if the following properties are satisfied:

(i) $v(\emptyset) = (0,1)$ and $v(X) = (1,0)$;

(ii) $A \subseteq B$ implies $v(A) \leq_{\mathcal{L}} v(B)$;

(iii) $A_1 \subseteq A_2 \subseteq ...$ implies $v\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty}{}_{\mathcal{L}} v(A_n)$;

(iv) $A_1 \supseteq A_2 \supseteq ...$ implies $v\left(\bigcap\limits_{n=1}^{\infty} A_n\right) = \lim\limits_{n\to\infty}{}_{\mathcal{L}} v(A_n)$.

The triple $(X, \mathcal{A}, v)$ is called an intuitionistic fuzzy measure space.

Theorem 1. If $v : \mathcal{A} \to \mathcal{L}, v(A) = (v_1(A), v_2(A))$ is an intuitionistic fuzzy measure then $v_1, v_2^c : \mathcal{A} \to [0,1], v_2^c(A) = 1 - v_2(A)$, for every $A \in \mathcal{A}$, are normalized fuzzy measures.

P r o o f: See [6, Definition 3.3, p.174 and Theorem 3.6, p.177.] ■

Definition 4 ([6, p. 206], see also [8]). Let $(X, \mathcal{A}, v)$ be an intuitionistic fuzzy measure space. A function $f : X \to \mathcal{L}$ is called v*-measurable with respect to* $\mathcal{A}$ if

$$
\{x \in X : f(x) \leq_{\mathcal{L}} \alpha\} \in \mathcal{A}
$$

and

$$
\{x \in X : f(x) \geq_{\mathcal{L}} \alpha\} \in \mathcal{A},
$$

for every $\alpha \in \mathcal{L}$.

We denote by $\mathcal{A}^v(X)$ the set of all v-measurable functions with respect to $\mathcal{A}$.

As a matter of fact, the measurability of an intuitionistic fuzzy-valued function reduces to the measurability of its components.

Theorem 2 ([6, p. 206], see also [8]). Let $f : X \to \mathcal{L}, f(x) = (g(x), h(x))$, $x \in X$. Then $f \in \mathcal{A}^v(X)$ if and only if $g \in \mathcal{A}_+^{v_1}(X)$ and $h \in \mathcal{A}_+^{v_2^c}(X)$.

The Sugeno integral of an intuitionistic fuzzy-valued mapping, on a crisp set, with respect to an intuitionistic fuzzy measure is defined as follows.

Defintion 5 ([6, p. 206] or [3]). Let $(X, \mathcal{A}, v)$ be an intuitionistic fuzzy measure space and $f \in \mathcal{A}^v(X)$. The Sugeno integral type of f on $A \in \mathcal{A}$ with respect to v, denoted by $(S)\int_X^{\mathcal{L}} f dv$, is defined by

$$(S)\int_X^{\mathcal{L}} f\, dv = \sup_{a \in \mathcal{L}}{}_{\mathcal{L}} \inf{}_{\mathcal{L}} \left(a, v\left(A \cap F_a\right)\right),$$

where $F_a = \{x \in X : f(x) \geq_{\mathcal{L}} a\}$, for every $a \in \mathcal{L}$.

3. Main result

The following results help us to reach the Chebyshev type inequalities for Sugeno integrals with respect to intuitionistic fuzzy measures.

Theorem 3 ([6, p. 215], see also [8]). Let $v : \mathcal{A} \to \mathcal{L}, v = (v_1, v_2)$ be an intuitionistic fuzzy measure and $f \in \mathcal{A}^v(X), f(x) = (g(x), h(x))$. Then

$$(S)\int_X^{\mathcal{L}} f\, dv = \left((S)\int_X g\, dv_1, 1 - (S)\int_X h^c\, dv_2^c\right)$$

where $h^c(x) = 1 - h(x), x \in X, v_2^c(A) = 1 - v_2(A), A \in \mathcal{A}$.

Theorem 4 ([14]). Let $f, g \in \mathcal{A}_+^{\mu}(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_X f d\mu$ and $(S)\int_X g d\mu$ are finite and let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comontone, then the inequality

$$(S)\int_X f \star g, d\mu \geq \left((S)\int_X f d\mu\right) \star \left((S)\int_X g\, d\mu\right)$$

holds.

The notion of comonotonicity can be extended in the following way.

Definition 6. Two intuitionistic fuzzy-valued functions $f_1, f_2 : X \to \mathcal{L}$ are called comonotone if, for all $x, y \in X$,

$$f_1(x) \leq_{\mathcal{L}} f_1(y) \text{ and } f_2(x) \leq_{\mathcal{L}} f_2(y)$$

or

$$f_1(x) \geq_{\mathcal{L}} f_1(y) \text{ and } f_2(x) \geq_{\mathcal{L}} f_2(y).$$

Proposition 1. If $f_1, f_2 : X \to \mathcal{L}$ are comonotone and $f_1 = (g_1, h_1), f_2 = (g_2, h_2)$ then $g_1 \sim g_2$ and $h_1 \sim h_2$.

P r o o f. It is immediate. ■

Theorem 5. Let v be an intuitionistic fuzzy measure and $f_1, f_2 \in \mathcal{A}^v(X)$. Let

$$* : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$$

be non-decreasing in both arguments with respect to $\leq_{\mathcal{L}}$, such that

$$(a, b) * (c, d) = (a \circ c, b \diamond d),$$

where $x \circ y \leq \min(x, y), x \diamond y \geq \max(x, y) \, \forall x, y \in [0, 1]$ and $\circ, \diamond$ are continuous. If f_1 and f_2 are comonotone then the inequality

$$(S) \int_X^{\mathcal{L}} f_1 * f_2 dv \geq_{\mathcal{L}} (S) \int_X^{\mathcal{L}} f_1 dv * (S) \int_X^{\mathcal{L}} f_2 dv$$

holds.

P r o o f. If $v = (v_1, v_2), f_1 = (g_1, h_1), f_2 = (g_2, h_2)$ then, taking into account Theorem 3, the above inequality is equivalent to

$$\left((S) \int_X g_1 \circ g_2 \, dv_1, 1 - (S) \int_X (1 - h_1 \diamond h_2) \, dv_2^c \right) \geq_{\mathcal{L}}$$
$$\geq_{\mathcal{L}} \left((S) \int_X g_1 dv_1 \circ (S) \int_X g_2 dv_1, \right.$$
$$\left. \left(1 - (S) \int_X (1 - h_1) \, dv_2^c \right) \diamond \left(1 - (S) \int_X (1 - h_2) \, dv_2^c \right) \right).$$

Because g_1 and g_2 are comonotone (Proposition 1), v_1 is a finite fuzzy measure (Theorem 1), $\circ$ is nondecreasing in both arguments and bounded from above by minimum we obtain (Theorem 4)

$$(S) \int_X g_1 \circ g_2 \, dv_1 \geq (S) \int_X g_1 \, dv_1 \circ (S) \int_X g_2 \, dv_1.$$

We introduce $\Box : [0, 1] \times [0, 1] \to [0, 1]$ by

$$x \Box y = 1 - (1 - x) \diamond (1 - y) \, \forall x, y \in [0, 1].$$

Then $\Box$ is nondecreasing in both arguments and bounded from above by minimum. If h_1 and h_2 are comonotone (Proposition 1) then $1 - h_1$ and $1 - h_2$ are comonotone too. Because v_2^c is a finite fuzzy measure we obtain (Theorem 4)

$$(S) \int_X (1 - h_1) \Box (1 - h_2) \, dv_2^c \geq (S) \int_X (1 - h_1) \, dv_2^c \Box (S) \int_X (1 - h_2) \, dv_2^c$$

which is equivalent to

$$(S)\int_X (1-h_1 \diamond h_2)\,dv_2^c \geq$$

$$\geq 1-\left(1-(S)\int_X (1-h_1)\,dv_2^c\right)\diamond\left(1-(S)\int_X (1-h_2)\,dv_2^c\right)$$

and the proof is complete.■

Remark 1. If

$$\begin{gathered} v_2(x)=1-v_1(x)\ \forall x\in X,\\ h_1(x)=1-g_1(x)\ \forall x\in X,\\ h_2(x)=1-g_2(x)\ \forall x\in X,\\ (1-a)\diamond(1-c)=1-a\circ c\,\forall a,c\in[0,1] \end{gathered}$$

then Theorem 5 gives a general Chebyshev inequality for Sugeno integrals in the fuzzy case (see [14, Theorem 3.1]).

4. Examples

If T is a triangular norm and S is its dual triangular conorm, that is (see e. g. [12])

$$S(x,y)=1-T(1-x,1-y)\,\forall x,y\in[0,1]$$

then $\mathcal{T}:\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ defined by

$$\mathcal{T}((a,b),(c,d))=(T(a,c),S(b,d))$$

is an intuitionistic fuzzy norm (see [10] or [6, p. 7]), that is $\mathcal{T}$ is nondecreasing in both arguments with respect to $\leq_{\mathcal{L}}$. If, in addition, T is continuous and

$$T(x,y)\leq\min(x,y)\ \forall x,y\in[0,1],$$

then S is continuous and

$$S(x,y)\geq\max(x,y)\ \forall x,y\in[0,1]$$

such that the hypothesis in Theorem 5 are satisfied for $\circ=T$ and $\diamond=S$. In this way we have the possibility to obtain many examples.

Example 1. If v is an intuitionistic fuzzy measure and $f_1,f_2\in\mathcal{A}^v(X)$ are comonotone then

$$(S)\int_X^{\mathcal{L}} f_1\cdot f_2dv\geq_{\mathcal{L}}(S)\int_X^{\mathcal{L}} f_1dv\cdot(S)\int_X^{\mathcal{L}} f_2dv,$$

where $\cdot$ is generated as above by triangular norm $T(x,y)=xy,\ x,y\in[0,1]$. If with the above notations $X=[0,1]$, v_1 is the Lebesgue measure on $\mathbb{R}$, $v_2=1-v_1$, g_1 and

g_2 are both continuous strictly increasing or continuous strictly decreasing functions, $h_1 = 1 - g_1, h_2 = 1 - g_2$ then the previous inequality becomes the Chebyshev inequality in the fuzzy case (see [11]).

Example 2. If v is an intuitionistic fuzzy measure and $f_1, f_2 \in \mathcal{A}^v(X)$ are comonotone then

$$(S)\int_X^{\mathcal{L}} f_1 \wedge f_2 dv \geq_{\mathcal{L}} (S)\int_X^{\mathcal{L}} f_1 dv \wedge (S)\int_X^{\mathcal{L}} f_2 dv,$$

where $\wedge$ is generated as above by triangular norm $T(x,y) = \min(x,y), x,y \in [0,1]$. Together with the monotonicity of Sugeno integral with respect to intuitionistic fuzzy measures (see [6, Theorem 3.45, p. 217]) we obtain

$$(S)\int_X^{\mathcal{L}} f_1 \wedge f_2\, dv = (S)\int_X^{\mathcal{L}} f_1\, dv \wedge (S)\int_X^{\mathcal{L}} f_2\, dv$$

that is the property of comonotone minimitive property is valid for Sugeno integral with respect to intuitionistic fuzzy measures.

The following example proves that the inequality in Theorem 5 can be strict.

Example 3. Let $X = [0,1], f_1 = (g_1, h_1), f_2 = (g_2, h_2), g_1(x) = x^2, h_1(x) = 1 - x, g_2(x) = \frac{x}{3}, h_2(x) = \frac{1}{2}$ and $\cdot$ as in Example 1. Let the intuitionistic fuzzy measure $v = (v_1, v_2)$ be defined as $v_1(A) = m^2(A), v_2(A) = 1 - m(A)$, where m is the Lebesgue measure. Then we have

$$(S)\int_{[0,1]} g_1\, dv_1 = \bigvee_{\alpha\in[0,1]} \left(\alpha \wedge \left(1 - \sqrt{\alpha}\right)^2\right) = \frac{1}{4},$$

$$(S)\int_{[0,1]} g_2\, dv_1 = \bigvee_{\alpha\in[0,1/3]} \left(\alpha \wedge (1 - 3\alpha)^2\right) = \frac{7 - \sqrt{13}}{18},$$

$$(S)\int_{[0,1]} h_1^c dv_2^c = \bigvee_{\alpha\in[0,1]} (\alpha \wedge (1 - \alpha)) = \frac{1}{2}$$

and

$$(S)\int_{[0,1]} h_2^c dv_2^c = \bigvee_{\alpha\in[0,1/2]} (\alpha \wedge 1) = \frac{1}{2}$$

that is

$$(S)\int_X^{\mathcal{L}} f_1\, dv \cdot (S)\int_X^{\mathcal{L}} f_2\, dv = \left(\frac{1}{4}, \frac{1}{2}\right) \cdot \left(\frac{7 - \sqrt{13}}{18}, \frac{1}{2}\right) = \left(\frac{7 - \sqrt{13}}{72}, \frac{3}{4}\right).$$

On the other hand,

$$f_1 \cdot f_2 = \left(\frac{x^3}{3}, 1 - \frac{x}{2} \right)$$

therefore

$$\begin{aligned}
(S)\int_X^{\mathcal{L}} f_1 \cdot f_2 dv &= \left((S)\int_{[0,1]} \frac{x^3}{3} dv_1, 1 - \int_{[0,1]} \frac{x}{2} dv_2^c \right) \\
&= \left(\bigvee_{\alpha\in[0,1/3]} \left(\alpha \wedge \left(1 - \sqrt[3]{3\alpha} \right)^2 \right), 1 - \bigvee_{\alpha\in[0,1/2]} (\alpha \wedge (1 - 2\alpha)) \right) \\
&= \left(\alpha_0, \frac{2}{3} \right),
\end{aligned}$$

where $\alpha_0 \in \left(\frac{1}{10}, \frac{1}{5} \right)$. We obtain

$$(S)\int_{[0,1]}^{\mathcal{L}} f_1 \cdot f_2 \, dv >_{\mathcal{L}} (S)\int_{[0,1]}^{\mathcal{L}} f_1 \, dv \cdot (S)\int_{[0,1]}^{\mathcal{L}} f_2 \, dv.$$

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