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Intuitionistic Fuzzy Hypergraphs

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***Abstract*:** *In this paper, the concepts of intuitionistic fuzzy hypergraph (IFHG) and dual intuitionistic fuzzy hypergraph (DIFHG) are introduced. In IFHG, the concepts of* (α, β)*-cut hypergraph, strength of an edge have also been introduced.*

***Keywords*:** *Intuitionistic fuzzy hypergraph,* (α, β)*-cut, strength of an edge, dual intuitionistic fuzzy hypergraph.*

1. Introduction

In [1, 2] the concepts of fuzzy graphs, fuzzy hypergraphs and intuitionistic fuzzy graphs are introduced. In this paper, the concept of IFHG is introduced. Some important concepts have been developed such as the strength of an edge, (α, β)-cut hypergraph, DIFHG. The proposed IFHG is useful to describe visually a fuzzy partition or covering. In section 2, brief definitions and terminologies of hypergraph and fuzzy hypergraph are introduced. In Section 3 and Section 4, the concepts of IFHG, strength of an edge, (α, β)-cut hypergraph, DIFHG are developed. Section 5 concludes the paper.

Definition 1.1. When a fuzzy set A is given in a universal set X the fuzzy set A is represented by A=$\{(x, \mu_A(x)) : \mu_A(x) > 0, x \in X\}$ where $\mu_A(x)$ is the membership function representing the membership degree of element *x* in the fuzzy set A.

Definition 1.2. The support of the fuzzy set A, denoted by supp(A), is defined as supp$(A) = \{x : \mu_A(x) > 0\}$. Note that the support of a fuzzy set is a crisp set.

Definition 1.3. The α- *cut* of a fuzzy set A, denoted by A_α, is defined as

$$A_\alpha = \{x : \mu_A(x) \geq \alpha\}.$$

When two fuzzy sets A and B are given, the operations $\cup$ and $\cap$ are defined on A and B by their membership function $\mu_{A\cup B}(x)$and $\mu_{A\cap B}(x)$ respectively as follows:

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)),$$
$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x)).$$

Definition 1.4. A family of subsets $A_1, A_2, ..., A_m$ (for every i, $A_i \neq \emptyset, A_i \in X$) in a set X is called a partition of X if the following conditions are satisfied:

$$\bigcup A_i = X, \qquad i = 1, 2, ..., m$$
$$A_i \bigcap A_j = \emptyset, \qquad i, j = 1, 2, ..., m(i \neq j)$$

2. Basic concepts

Definition 2.1. A hypergraph H is an ordered pair $H = (V, E)$ where
(i) $V = \{x_1, x_2, ..., x_n\}$, a finite of set of vertices
(ii)$E = \{E_1, E_2, ..., E_m\}$ a family of subsets of V,
(iii) $E_j \neq \emptyset$, *j=1,2,...,m* and
(iv) $\cup_j E_j = V$.
The set V is called the set of vertices and E is the set of edges (or hyperedges).

Definition 2.2. The edge E_j is represented by a solid line surrounding its vertices if $\|E_j\|$=1 by a cycle on the element. If $\|E_j\|$= 2 for all *j*, the hypergraph becomes an ordinary (undirected) graph. The hypergraph (V, E) can also be represented by (V;$E_1, E_2, ..., E_m$).

Definition 2.3. In a hypergraph, two vertices x and y are said to be adjacent if there exists an edge E_j which contains the two vertices $x \in E_j$, $y \in E_j$.

Definition 2.4. In a hypergraph two edges E_i and E_j are said to be adjacent if their intersection is not empty. That is, $E_i \cap E_j \neq \emptyset$, $i \neq j$.

Definition 2.5. In a hypergraph the degree of a vertex x is the number of edges which contain the vertex.

Definition 2.6. In a hypergraph $H = (V, E)$; $V = \{x_1, x_2, ..., x_n\}$ and

$$E = \{E_1, E_2, ..., E_m\},$$

its incidence matrix is a matrix $M_H = (a_{ij})_{n\times m}$ with m columns representing the edges and n rows representing the vertices, where the elements a_{ij} indicates the membership of vertex to hyperedge as follows

$$a_{ij} = \begin{cases} 1, & \text{if } x_i \in E_j, \\ 0, & \text{if } x_i \notin E_j. \end{cases}$$

Example 1. Consider a hypergraph $H = (V, E)$, such that $V = \{x_1, x_2, x_3, x_4\}$, $E = \{E_1, E_2, E_3, E_4\}$ where $E_1 = \{x_1, x_2\}$, $E_2 = \{x_2, x_3, \}$, $E_3 = \{x_3, x_4\}$, $E_4 = \{x_4, x_1\}$.

The graph and hypergraph are shown on Fig. 1.

The corresponding incidence matrix is given below.

H	E_1	E_2	E_3	E_4
x_1	1	0	0	1
x_2	1	1	0	0
x_3	0	1	1	0
x_4	0	0	1	1

In general, a hypergraph represents a covering of the set V. In a hypergraph, if every vertex has its degree 1 (i.e. $E_i \cap E_j = \emptyset$, $i \neq j$), the hypergraph represents a partition of V.

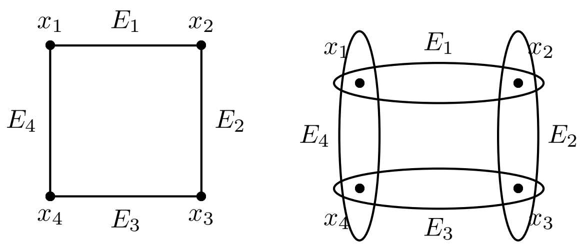

Fig. 1. Graph and Hypergraph

Definition 2.7. A hypergraph $H = (X; E_1, E_2, ..., E_m)$ where $X = \{x_1, ..., x_n\}$ can be mapped to a hypergraph $H^* = (E; x_1, x_2, ..., x_n)$ whose vertices are the points $e_1, e_2, ..., e_m$(corresponding to $E_1, E_2, ..., E_m$ respectively), and whose edges are the sets $X_1, X_2, ..., X_n$ (corresponding to $x_1, x_2, ..., x_n$ respectively) where $X_j = \{e_i | x_j \in E_i, i \leq m\}$, j=1, 2, ..., n. $X_j \neq \emptyset \cup_j X_j = E$. The hypergraph H^* is called the dual hypergraph of H.

Note 1.

(i) The incidence matrix $(a_{ij})_{n\times m}$ of the hypergraph H and that $(b_{ij})_{m\times n}$ of dual hypergraph H^* are transpose to each other, that is, $(a_{ij})^{\mathrm{T}}_{n\times m} = (b_{ij})_{m\times n}$. Thus, $(H^*)^* = H$.

(ii) If two vertices $x_i$and x_j in H are adjacent, the edges X_i and $X_j$in H^* are adjacent. Similarly, if two edges E_i and E_j in H are adjacent, two vertices e_i and e_j in H^* are adjacent.

Example 2. The dual hypergraph H^* of the hypergraph H given in Definition 2.7 is as follows. $H^* = (E; X_1, X_2, X_3, X_4); E = \{e_1, e_2, e_3, e_4\}$,

$$X_1 = \{e_1, e_2\},$$

$$X_2 = \{e_2, e_3\}, X_3 = \{e_3, e_4\},$$

$$X_4 = \{e_4, e_1\}.$$

The dual hypergraph H^* is shown on Fig.2

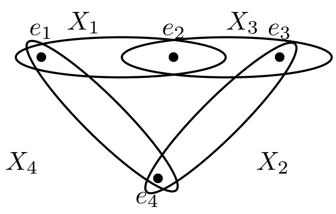

Fig. 2. Dual hypergraph

The corresponding incidence matrix is given below.

H^*	X_1	X_2	X_3	X_4
e_1	1	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	1	0	0	1

3. Intuitionistic fuzzy hypergraph

Definition 3.1. An intuitionistic fuzzy set (IFS) A in X is defined as an object of the form $A = \{(x, \mu_A(x), \nu_A(x))|x \in X\}$ where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in X$, respectively, and for every $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 3.2. The support of an IFS A, denoted by supp(A), is defined as supp(A)= $\{x : \mu_A(x) > 0$ and $\nu_A(x) > 0\}$. The support of the intutionistic fuzzy set is a crisp set.

Definition 3.3. The(α, β)-cut of an IFS A, denoted by $A_{(\alpha,\beta)}$, is defined as

$$A_{(\alpha,\beta)} = \{x : \mu_A(x) \geq \alpha, \text{ and } \nu_A(x) \leq \beta\}.$$

Definition 3.4. The IFHG H is an ordered pair $H = (V, E)$ where
(i)$V = \{x_1, x_2, ..., x_n\}$, is a finite set of vertices
(ii) $E = \{E_1, E_2, ..., E_m t\}$ is a family of intuitionistic fuzzy subsets of V
(iii)$E_j = \{(x_i, \mu_j(x_i), \nu_j(x_i)) : \mu_j(x_i), \nu_j(x_i) \geq 0 \ \& \mu_j(x_i) + \nu_j(x_i) \leq 1\}$, $j = 1, 2, \ldots, m$
(iv)$E_j \neq \emptyset$, $j = 1, 2, ..., m$.
(v)$\cup_j \text{supp}(E_j) = V$, $j = 1, 2, ..., m$
Here, the edges E_j are an IFSs of vertices, $\mu_j(x_i)$ and $\nu_j(x_i)$ denote the degree of membership and non-membership of vertex x_i to edge E_j. Thus, the elements of the incidence matrix of IFHG are of the form $(a_{ij}, \mu_j(x_i), \nu_j(x_i))$. The sets (V, E) are crisp sets.

Example 3. Consider an IFHG $H = (V, E)$ such that $V = \{x_1, x_2, x_3, x_4\}$, $E = \{E_1, E_2, E_3, E_4\}$.Here,

$$E_1 = \{(x_1, 0.2, 0.5), (x_2, 0.3, 0.6)\},\ E_2 = \{(x_2, 0.3, 0.6), (x_3, 0.4, 0.5)\},$$
$$E_3 = \{(x_3, 0.4, 0.5), (x_4, 0.6, 0.2)\} \text{ and } E_4 = \{(x_4, 0.6, 0.2), (x_1, 0.2, 0.5)\}.$$

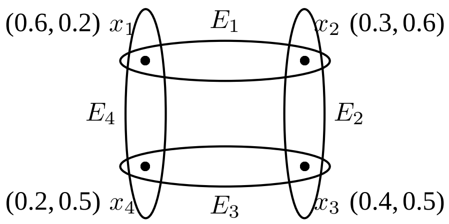

Fig. 3. Intuitionistic fuzzy hypergraph

The corresponding incidence matrix M_H is as follows:

M_H	E_1	E_2	E_3	E_4
x_1	(0.2, 0.5)	(0, 0)	(0, 0)	(0.2, 0.5)
x_2	(0.3, 0.6)	(0.3, 0.6)	(0, 0)	(0, 0)
x_3	(0, 0)	(0.4, 0.5)	(0.4, 0.5)	(0, 0)
x_4	(0, 0)	(0, 0)	(0.6, 0.2)	(0.6, 0.2)

Definition 3.5. In an IFHG, the adjacent level between two vertices x_r and x_s, denoted by $\gamma(x_r, x_s)$, is defined as $\gamma(x_r, x_s) = \max_j(\min(\mu_j(x_r), \mu_j(x_s)))$, $\min_j(\max(\nu_j(x_r), \nu_j(x_s)))$, j=1, 2, ..., m.

Definition 3.6. In an IFHG, the adjacent level between the edges E_j and E_k, denoted by $\sigma(E_j, E_k)$, is defined by

$$\sigma(E_j, E_k) = \max_j(\min(\mu_j(x), \mu_k(x))), \min_j(\max(\nu_j(x), \nu_k(x))).$$

Example 4. In Fig. 3 the adjacent level between vertices $x_1$and x_2 is (0.2, 0.5) and the adjacent level between edges E_1 and E_2 is (0.3, 0.6).

Definition 3.7. The (α, β)- cut of IFHG H, denoted by $H_{(\alpha,\beta)}$, is an ordered pair $H_{(\alpha,\beta)} = (V_{(\alpha,\beta)}, E_{(\alpha,\beta)})$ where:
(i)$V_{(\alpha,\beta)} = \{x_1, ..., x_n\}$,
(ii)$E_{(\alpha,\beta)} = \{x_i : \mu_j(x_i) \geq \alpha$ and $\nu_j(x_i) \geq \beta, j = 1, 2, ..., m\}$,
(iii)$E_{m+1,(\alpha,\beta)} = \{x_i : \mu_j(x_i) < \alpha$ and $\nu_j(x_i) > \beta$ for every j.

Note 2.
(i) The edge $E_{m+1,(\alpha,\beta)}$ is added to group the elements which are not contained in any edge $E_{j,(\alpha,\beta)}$of $H_{(\alpha,\beta)}$.
(ii) The edges in the (α, β)- cut hypergraph are now crisp sets.

Example 5. In Fig. 4, (0.3, 0.6)- cut of IFHG H is given.

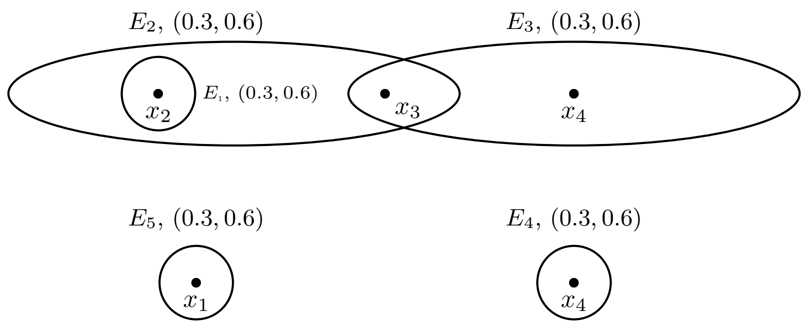

Fig. 4. $H(0.3, 0.6)$

In the (0.3, 0.6)- cut hypergraph a new edge $E_{5,(0.3,\,0.6)}$ is added to contain the element x_1 whose membership value is ≥ 0.3 and whose non-membership value is ≤ 0.6 for all edges. The incidence matrix of $H_{(0.3,\,0.6)}$ is as follows.

$H_{(0.3,\,0.6)}$	$E_{1,(0.3,\,0.6)}$	$E_{2,(0.3,\,0.6)}$	$E_{3,(0.3,\,0.6)}$	$E_{4,(0.3,\,0.6)}$	$E_{5,(0.3,\,0.6)}$
x_1	0	0	0	0	1
x_2	1	1	0	0	0
x_3	0	1	1	0	0
x_4	0	0	1	1	0

Definition 3.8. The strengh δ of an edge E_j is the minimum membership $\mu_j(x)$ and non-membership $\nu_j(x)$ of vertices in the edge E_j. That is,

$$\delta(E_j) = (\min_x(\mu_j(x)), \max_x(\nu_j(x)))$$

for every $\mu_j(x) > 0, \nu_j(x) > 0$.

Example 6. In Fig. 3, the strength of each edge is $\delta(E_1) = (0.2,\ 0.6)$, $\delta(E_2) = (0.3,\ 0.6)$, $\delta(E_3) = (0.4,\ 0.5)$, $\delta(E_4) = (0.2,\ 0.5)$.

Note 3. Among the strength of edges, the edge which possess maximum membership and minimum non-membership is said to be stronger. Thus, the edge E_3 is stronger than E_1, E_2 and E_4 in Example 6.

The incidence matrix corresponding to strength is given below.

δ	E_1	E_2	E_3	E_4
x_1	$(0.2, 0.6)$	$(0, 0)$	$(0, 0)$	$(0.2, 0.5)$
x_2	$(0.2, 0.6)$	$(0.3, 0.6)$	$(0, 0)$	$(0, 0)$
x_3	$(0, 0)$	$(0.3, 0.6)$	$(0.4, 0.5)$	$(0, 0)$
x_4	$(0, 0)$	$(0, 0)$	$(0.4, 0.5)$	$(0.2, 0.5)$

4. Dual intuitionistic fuzzy hypergraph

Definition 4.1. If an IFHG $H = (X; E_1, E_2, ..., E_m)$, X=$(x_1, x_2, ..., x_n)$is given, its DIFHG $H^* = (E; \overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$ is defined as follows. $H^* = (E; \overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$, where

(i) $E = (e_1, e_2, ..., e_m)$, set of vertices corresponding to $E_1, E_2, ..., E_m$ respectively.

(ii) $\overline{x}_j = \{(e_j, \mu_j(e_j), \nu_j(e_j)) : \mu_i(e_j) = \mu_j(x_i), \nu_i(e_j) = \nu_j(x_i)\}$.

Example 7. The DIFHG $H^* = \{(E, \overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4)\}$ of IFHG H in definition 3.5 is given below. Here,

$$E=\{e_1, e_2, e_3, e_4\}\ .$$

$$\overline{x}_1 = \{(e_1, 0.2, 0.5), (e_4, 0.2, 0.5)\},$$

$$\overline{x}_2 = \{(e_1, 0.3, 0.6), (e_2, 0.3, 0.6)\},$$

$$\overline{x}_3 = \{(e_2, 0.4, 0.5), (e_3, 0.4, 0.5)\},$$

$$\overline{x}_4 = \{(e_3, 0.6, 0.2), (e_4, 0.6, 0.2)\}.$$

The DIFHG is shown on Fig. 5

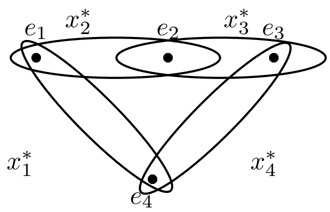

Fig. 5. DIFHG H*

The corresponding incidence matrix is

M_H^*	X_1	X_2	X_3	X_4
e_1	$(0.2, 0.5)$	$(0.3, 0.6)$	$(0, 0)$	$(0, 0)$
e_2	$(0, 0)$	$(0.3, 0.6)$	$(0.4, 0.5)$	$(0, 0)$
e_3	$(0, 0)$	$(0, 0)$	$(0.4, 0.5)$	$(0.6, 0.2)$
e_4	$(0.2, 0.5)$	$(0, 0)$	$(0, 0)$	$(0.6, 0.2)$

The (0.3, 0.6)-cut hypergraph $H^*_{(0.3,\,0.6)}$ is shown on Fig. 6.

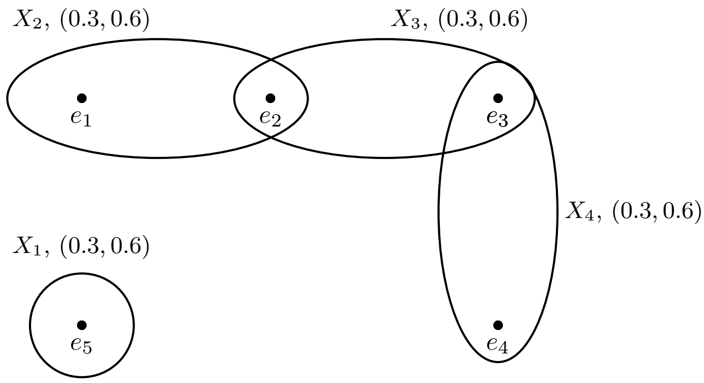

Fig. 6. $M^*_{H_{(0.3,\,0.6)}}$

The corresponding incidence matrix $M^*_{H_{(0.3,0.6)}}$ is

$M^*_{H_{0.3,0.6}}$	X_1	X_2	X_3	X_4
e_1	0	1	0	0
e_2	0	1	1	0
e_3	0	0	1	1
e_4	0	0	0	1
e_5	1	0	0	0

5. Conclusion

The authors further propose to show that the IFHG and (α, β)-cut hypergraph are useful to represent an intuitionistic fuzzy partition. Ultimately, to show that the strength of an edge can be used to decompose the data set in a clustering problem.

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