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Observables on Intuitionistic Fuzzy Sets: An Elementary Approach

Magdaléna Renčová , Beloslav Riečan

Faculty of Natural Sciences, Matej Bel University
Department of Mathematics
Tajovského 40, SK-974 01 Banská Bystrica, Slovakia
and
Mathematical Institute of Slovak Academy of Sciences
Štefánikova 49, SK-81473 Bratislava
E-mails: rencova@fpv.umb.sk riecan@fpv.umb.sk

Abstract**:** *In preceding paper [6] we have presented an elementary approach to the notion of the probability on intuitionistic fuzzy events (IF-events). In the present paper we present a similar approach to the notion of an observable what is an analogue of the notion of random variable in the classical Kolmogorov theory.*

Keywords**:** *Probability, Intuitionistic fuzzy events, random variable.*

1. Introduction

An IF-set is a mapping $A = (\mu_A, \nu_A) : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1$. The function μ_A is called the membership function, ν_A the non - membership function. If $\nu_A = 1 - \mu_A$, we obtain simply a fuzzy set $\mu_A : \Omega \to [0,1]$.

Let us recall basic definitions of the Kolmogorov probability theory. The basic notion of the probability theory is the notion of a σ-algebra $\mathcal{S}$, i.e. a family of subsets of Ω satisfying the following properties:

(i) $\Omega \in \mathcal{S}$,
(ii) $A \in \mathcal{S} \Rightarrow \Omega - A \in \mathcal{S}$,

(iii) $A_n \in \mathcal{S}\,(n = 1, 2, 3...) \Rightarrow \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{S}$.

The set belonging to $\mathcal{S}$ are called events.

Example 1.1. Consider $\mathcal{S} = \{\emptyset, \Omega\}$. Evidently $\mathcal{S}$ is a σ-algebra. It presents an extreme situation, the smallest possible σ-algebra of subsets of Ω: the only measurable sets are the empty set $\emptyset$ and the whole space Ω.

Example 1.2. Let $\mathcal{S}$ consists of all subsets of Ω ($\mathcal{S} = \mathcal{P}(\Omega)$). It presents the second extreme situation: every subsets of Ω is measurable.

Example 1.3. Let $\Omega = R$, $\mathcal{J}$ be the family of all closed bounded intervals $[a, b]$. They do not form a σ-algebra, but the convenient σ-algebra is the σ-algebra $\mathcal{B}(R)$, the smallest σ-algebra containing $\mathcal{J}$. The sets belonging to $\mathcal{B}(R)$ are called Borel sets.

Recall a possible (equivalent) definition of a probability P defined on a σ-algebra $\mathcal{S}$:

(i) $P(\Omega) = 1$,

(ii) $P(A \cup B) + P(A \cap B) = P(A) + P(B)$ for any $A, B \in \mathcal{S}$,

(iii) $A_n \searrow \emptyset$ (i.e. $A_n \supset A_{n+1} (n = 1, 2, ...)$, $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset) \Rightarrow P(A_n) \searrow 0$.

A similar notion is the notion of a random variable $\eta : \Omega \to R$ what is a measurable function, i.e.

$$\xi^{-1}(I) \in \mathcal{S}$$

for any interval $I \subset R$. This mapping is theoretically described by the distribution function $F : R \to [0, 1]$ defined by the equality

$$F(t) = P(\xi^{-1}((-\infty, t))).$$

In IF-probability theory we shall work with IF-events. An IF-set $A = (\mu_A, \nu_A)$ is called an IF-event, if $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}-$measurable, i.e.

$$\mu_A^{-1}(I) \in \mathcal{S}, \nu_A^{-1}(I) \in \mathcal{S}$$

for any interval $I \subset R$.

If $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ are events, then we define

$$A \oplus B = (\mu_A \oplus \mu_B, \mu_A \odot \mu_B),$$
$$A \odot B = (\mu_A \odot \mu_B, \mu_A \oplus \mu_B),$$
$$\neg A = (1 - \mu_A, 1 - \nu_A),$$

where $a \oplus b = min(a + b, 1), a \odot b = max(a + b - 1, 0)$.

Further

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

It is easy to see, that

$$A_n \nearrow A \Leftrightarrow \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A.$$

Recall that the operations $\oplus, \odot, \neg$ play an important role in the multivalued logic. Namely, $f \oplus g$ corresponds to the disjunction, $f \odot g$ to the conjunction and $\neg f$ to the negation. Also in the set theory they have a similar meaning: if $f = \chi_A, g = \chi_B$, then

$$\chi_A \oplus \chi_B = \chi_{A \cup B}, \chi_A \odot \chi_B = \chi_{A \cap B}, \neg \chi_A = \chi_{A'}.$$

2. IF-observables

Instead of probability or random variable respectively we shall define in our IF theory a notion of a state corresponding to probability and a notion of an observable corresponding to notion of a random variable. Denote by $\mathcal{F}$ the family of all IF-events. The paper contains an elementary approach to the theory of IF-observables. If somebody wants to use the theory, he need not known profound mathematical backgrounds, since they are translated to a simpler language.

Definition 2.1. An IF state is a mapping $m : \mathcal{F} \to [0, 1]$ satisfying the following conditions:
(i) $m((1, 0)) = 1, m((0, 1)) = 0,$
(ii) $m(A \oplus B) + m(A \odot B) = m(A) + m(B)$, whenever $A, B \in \mathcal{F}$,
(iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$, where $A_n \in \mathcal{F}$,
(iv) $A_n \in \mathcal{F}(n = 1, 2, ...), A_n \searrow (0, 1) \Rightarrow m(A_n) \searrow 0.$

Definition 2.2. Let $\mathcal{C}$ be the family of all intervals of the form $(-\infty, t), t \in R$. An IF-observable is a mapping $x : \mathcal{C} \to \mathcal{F}$ satisfying the following conditions:
(i) $A \subset B \Rightarrow x(A) \leq x(B)$,
(ii) $A_n \nearrow R \Rightarrow x(A_n) \nearrow (1_\Omega, 0_\Omega)$,
(iii) $A_n \nearrow A \Rightarrow x(A_n) \nearrow x(A)$,
(iv) $A_n \searrow \emptyset \Rightarrow x(A_n) \searrow (0_\Omega, 1_\Omega)$

Theorem 2.3. Let $x : \mathcal{C} \to \mathcal{F}$ be an IF-observable, $m : \mathcal{F} \to [0, 1]$ be an IF-state. Then the function $F : R \to [0, 1]$ defined by

$$F(t) = m(x((-\infty, t)))$$

is a distribution function.

P r o o f: By (iii) of Definition 2.1 and (i) of Definition 2.2 we obtain that F is non decreasing. The properties (ii),(iii) of Definition 2.1 and (ii) of Definition 2.2 imply $\lim_{n\to\infty} F(n) = 1$, the properties (iv) imply $\lim_{n\to-\infty} F(n) = 0$. Finally (iii) of Definitions 2.1 and 2.2 imply also that F is left continuous in every point $t \in R$ $\blacksquare$.

3. Joint IF-observable

Definition 3.1. Let $\mathcal{D} = \{(-\infty, u) \times (-\infty, v); u, v \in R\}, x, y : \mathcal{C} \to \mathcal{F}$ be IF-observables. The joint IF-observable of x, y is a mapping $h : \mathcal{D} \to \mathcal{F}$ satisfying
(i) $A \subset B \Rightarrow h(A) \leq h(B)$,
(ii) $A_n \nearrow R^2 \Rightarrow h(A_n) \nearrow (1_\Omega, 0_\Omega)$,

(iii) $A_n \nearrow A \Rightarrow h(A_n) \nearrow h(A)$,
(iv) $A_n \searrow \emptyset \Rightarrow h(A_n) \searrow (0_\Omega, 1_\Omega)$
(v) $u_n \nearrow \infty, v_n \nearrow \infty \Rightarrow$
$h((-\infty, u) \times (-\infty, v_n)) \nearrow x((-\infty, u)), h((-\infty, u_n) \times (-\infty, v)) \nearrow y((-\infty, v))$.

Theorem 3.2. Define $F : R^2 \to [0,1] F(u,v) = m(h((-\infty, u) \times (-\infty, v)))$. Then $F : R^2 \to [0,1]$ is a distribution function $F(u,\infty) = F_1(u), F(-\infty, v) = F_2(v)$, where F_1 is the distribution function of x, F_2 is distribution function of y.

P r o o f: It was proved in [4] and [8] that to any observables $x, y : \mathcal{C} \to \mathcal{F}$ there exists mappings

$$\widehat{x}, \widehat{y} : \mathcal{B}(R) \to \mathcal{F}$$

such that

$$\widehat{x}|\mathcal{C} = x, \widehat{y}|\mathcal{C} = y$$

and $\widehat{x}$, $\widehat{y}$ have all properties of observables. Also it was proved that there is $\widehat{h} : \mathcal{B}(R^2) \to \mathcal{F}$ having all the properties of observables and such that

$$\widehat{h}(C \times D) = \widehat{x}(C).\widehat{y}(D)$$

(here the product $A.B$ is defined by the equality
$A.B = (\mu_A.\mu_B, 1-(1-\nu_A)(1-\nu_B))$). Put

$$h = \widehat{h}|\mathcal{D}.$$

Then

$$h((-\infty, u) \times (-\infty, v)) = x((-\infty, u).y(-\infty, v))$$

hence

$$F(u,v) = m(x((-\infty, u)).y((-\infty, v))), u, v \in R.$$

Therefore

$$\begin{aligned}
\lim_{v\to\infty} F(u,v) &= \lim_{n\to\infty} m(x((-\infty,u)).y((-\infty,n))) = \\
&= m(\bigvee_{n=1}^{\infty} x((-\infty,u)).y((-\infty,n))) = \\
&= m(x((-\infty,u)).\bigvee_{n=1}^{\infty} y((-\infty,n))) = \\
&= m(x((-\infty,u)).(1_\Omega, 0_\Omega)) = \\
&= m(x((-\infty,u))) = F_1(u).
\end{aligned}$$

■

4. Conclusion

There is very well organized probability theory on IF-events (with respect to the Łukasiewicz operations) or more generally in MV-algebras. Of course, in the theory some notions and proofs are quite complicated. Therefore, we have presented in the paper some simple formulations describing some important results without mentioned difficulties.

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