

Subnorms, Non-Archimedean Field Norms and Their Corresponding d -FS and d -IFS

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Abstract: *In this paper the notion subnorm on arbitrary additive Abelian group is introduced. And with its help the corresponding d -FS and d -IFS are obtained for the case when the Abelian group coincides with each one of the infinitely many fields generated by \mathbb{Q}^2 . In the paper is shown that this subnorm is an appropriate non-Archimedean field norm over these fields.*

Keywords: *Fuzzy Sets (FS), Intuitionistic Fuzzy Set (IFS), d -FS, d -IFS, subnorm, norm, field norm, non-Archimedean norm, p -adic norm.*

Used denotations: \mathbb{Q} – the field of all rational numbers; \mathbb{Q}^2 – the Cartesian product $\mathbb{Q} \times \mathbb{Q}$, $a \equiv b \pmod{c}$ – means that c divides $a - b$; $\text{ord}_p x$, x integer – denotes the greatest integer γ such that p^γ divides x ; $\text{ord}_p z$, $z \in \mathbb{Q}$ – denotes $\text{ord}_p a - \text{ord}_p b$ where $z = \frac{a}{b}$, a, b integers, \mathbb{R}^+ – the set of all positive real numbers; iff = if and only if.

1. Introduction

Let us remind some definitions and notions.

Let E be the universe set and I be the unit interval. Following Zadeh [1], we call the ordered couple x with the mapping $\mu: E \rightarrow I$ Fuzzy Set (FS). For $x \in E$ the number $\mu(x)$ is called degree of membership of x to the fuzzy set.

Further, we understand FS as the set:

$$(1) \quad \{ \langle x, \mu(x) \rangle \mid x \in E \}.$$

If $\nu: E \rightarrow I$ is the mapping given by

$$\nu(x) = 1 - \mu(x), x \in E,$$

then we call the number $\nu(x)$ degree of non-membership of x to the fuzzy set (1). Therefore, FS admits the following representation

$$(2) \quad \{ \langle x, \mu(x), \nu(x) \rangle \mid x \in E \},$$

where

$$(3) \quad \forall x \in E (\mu(x) + \nu(x) = 1).$$

Atanassov [2] considered the sets (2) with the condition

$$\forall x \in E (\mu(x) + \nu(x) \leq 1)$$

instead of condition (3). He called these sets Intuitionistic Fuzzy Sets (IFS).

A wider point of view on the Fuzzy Sets and Intuitionistic Fuzzy Sets depending on an arbitrary metric on R^2 was proposed in [3] and [4], where FS and IFS are generated by a particular metric on R^2 .

Namely, let $d: R^2 \times R^2 \rightarrow [0, +\infty)$ be an arbitrary metric on R^2 and $\mu: E \rightarrow I$, $\nu: E \rightarrow I$ be arbitrary mappings.

The set

$$(4) \quad \{ \langle x, \mu(x), \nu(x) \rangle \mid x \in E \}$$

is said to be d -fuzzy set or d -FS, if it is fulfilled

$$\forall x \in E (d((\mu(x), \nu(x)), (0, 0)) = 1).$$

The set (4) is said to be d -Intuitionistic Fuzzy Set or d -IFS, if it is fulfilled:

$$\forall x \in E (d((\mu(x), \nu(x)), (0, 0)) \leq 1)$$

Remark 1. If we choose $d = d_1$ to be the Hamming's metric on R^2 , i.e.

$$d_1((\mu, \nu), 0) = |\mu| + |\nu|$$

then d_1 -FS is the usual FS, and d_1 -IFS is the usual IFS.

A modification of the concepts for d -FS and d -IFS was proposed in [5] and [6]. There for the universe set E is chosen \mathbb{Q} and for μ and ν are used arbitrary mappings of the form:

$$(5) \quad \mu: \mathbb{Q} \rightarrow \mathbb{Q} \cap I; \nu: \mathbb{Q} \rightarrow \mathbb{Q} \cap I.$$

Also, there d is a metric on \mathbb{Q}^2 (but not necessarily on \mathbb{R}^2).

Further we agree that d is a metric on \mathbb{Q}^2 only and we shall consider only the mappings μ and ν of the form (5).

If F is a number field with operations: \oplus – addition; \odot – multiplication and with zero element 0_F , then φ is said to be field norm on F if it is fulfilled:

$$1) \forall x \in F (\varphi(x) \geq 0)$$

and

$$\varphi(x) = 0 \text{ iff } x = 0_F;$$

$$2) \forall x, y \in F (\varphi(x \odot y) = \varphi(x) \cdot \varphi(y))$$

Also φ is said to be non-Archimedean norm on F if it is fulfilled

$$3) \varphi(x \oplus y) \leq \max(\varphi(x), \varphi(y)).$$

Let us remind that in \mathbb{Q} , considered as a field, there are infinitely many different non-equivalent non-Archimedean field norms (so-called p -adic norms). Each one of them with precision up to equivalence, according to Ostrowski's theorem (see [7]), is generated by a suitable prime p by the formula:

$$(6) \quad \varphi_p(x) = \begin{cases} \left(\frac{1}{p}\right)^{\text{ord}_p x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

We shall use the fact that \mathbb{Q}^2 may be transformed into a field in infinitely many ways (as has been shown in [6]) to prove that any of these fields admits infinitely many non-Archimedean field norms, depending on an infinite class of distinct prime numbers.

Also the notion subnorm on additive Abelian group will be introduced in order to extend the class of metrics generated by field norms.

2. Transforming \mathbb{Q}^2 into a field with non-Archimedean field norms

In \mathbb{Q}^2 we introduce two operations. The first of them is additive (which we will call addition). For (a_1, b_1) and $(a_2, b_2) \in \mathbb{Q}^2$ it is given by

$$(7) \quad (a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

The second of them is multiplicative (which we will call multiplication). It is given by

$$(8) \quad (a_1, b_1) \odot (a_2, b_2) = (a_1 a_2 + D b_1 b_2, a_1 b_2 + a_2 b_1)$$

where $D \neq 1$ is a non-zero rational number and $\sqrt{|D|}$ is an irrational number if $D \neq -1$.

It is easy to see that if $D \neq -1$ then \mathbb{Q}^2 with operations (7) and (8) is a field that is isomorphic to the well-known field $\mathbb{Q}(\sqrt{D})$. When $D = -1$, \mathbb{Q}^2 with operations (7) and (8) is isomorphic to the field whose elements are $x + i \cdot y$, where $x, y \in \mathbb{Q}$ and $i = \sqrt{-1}$.

Further the above mentioned fields are denoted by $\mathbb{Q}^2(D)$.

In [6] was shown that if φ is a field norm on \mathbb{Q} , then

$$\Phi(a, b) = \sqrt{\varphi(a^2 - Db^2)}$$

is a field norm on $\mathbb{Q}^2(D)$.

Let q be arbitrary fixed prime number and φ_p is taken from (6). In [6] are established the following important results:

Theorem 1. Let p be a prime number such that q is not a quadratic residue mod p . If Φ_p is given by the formula

$$\Phi_p(a, b) = \sqrt{\varphi_p(a^2 - qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, then Φ_p is a non-Archimedean field norm on $\mathbb{Q}^2(q)$.

Theorem 2. Let p be a prime number such that $-q$ is a quadratic nonresidue mod p . If Φ_p^* is given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, then Φ_p^* is a non-Archimedean field norm on $\mathbb{Q}^2(-q)$.

Theorem 3. Let p be a prime number such that -1 is a quadratic nonresidue mod p . If Φ_p^* is given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + b^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, then Φ_p^* is a non-Archimedean field norm on $\mathbb{Q}^2(-1)$.

3. An infinite class of primes generating non-Archimedean norms on $\mathbb{Q}^2(D)$

Let $A \neq 0$, $B \neq 0$ be relative prime integers. Then a well-known Dirichlet's theorem (see [8]) states that the infinite arithmetic progression $A + k.B$, $k = 0, 1, 2, \dots$, contains infinitely many primes.

Further for brevity we will cite this theorem as Dirichlet's theorem.

Let p and q be odd primes and $\left(\frac{q}{p}\right)$ is the Legendre symbol. Then we remind the Gauss quadratic reciprocity law:

$$\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \frac{q-1}{2}} \left(\frac{p}{q}\right).$$

Also the facts:

$$\begin{aligned} \left(\frac{-1}{p}\right) &= (-1)^{\frac{p-1}{2}}, \\ \left(\frac{2}{p}\right) &= (-1)^{\frac{p^2-1}{8}} \end{aligned}$$

are well known [9].

Using the above and the basic properties of Legendre symbol as well as the fact that a is a quadratic residue $(\text{mod } p)$ iff $\left(\frac{a}{p}\right) = 1$ and a is quadratic nonresidue $(\text{mod } p)$ iff $\left(\frac{a}{p}\right) = -1$, we may easily verify that the following assertions are true:

Lemma 1. Let $q \equiv 3(\text{mod } 8)$ be a prime number. If p is a prime number that belongs to either of the infinite arithmetic progressions:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots,$$

$$q + 2 + k.(4q), k = 0, 1, 2, \dots,$$

then q is a quadratic nonresidue $(\text{mod } p)$.

Lemma 2. Let $q \equiv 5(\text{mod } 8)$ be a prime number. If p is a prime number that belongs to the infinite arithmetic progression:

$$q + 2 + k.(2q), k = 0, 1, 2, \dots,$$

then q is a quadratic nonresidue $(\text{mod } p)$.

Lemma 3. Let $q \equiv 7(\text{mod } 8)$ be a prime number. If p is a prime number that belongs to the infinite arithmetic progression:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots,$$

then q is a quadratic nonresidue $(\text{mod } p)$.

Open Problem 1. Let $q \equiv 1(\text{mod } 8)$ be a prime number. Find an infinite arithmetic progression containing infinitely many primes p such that q is a quadratic nonresidue $(\text{mod } p)$.

Lemma 4. Let $q \equiv 1(\text{mod } 8)$ be a prime number. If p is a prime number belonging to the infinite arithmetic progression $2q + 1 + k.(4q), k = 0, 1, 2, \dots$, then $-q$ is a quadratic nonresidue $(\text{mod } p)$.

Lemma 5. Let $q \equiv 3(\text{mod } 8)$ be a prime number. If p is a prime number belonging to the infinite arithmetic progression $q + 2 + k.(4q), k = 0, 1, 2, \dots$, then $-q$ is a quadratic nonresidue $(\text{mod } p)$.

Lemma 6. Let $q \equiv 5(\text{mod } 8)$ be a prime number. If p is a prime number belonging to the infinite arithmetic progression $2q + 1 + k.(4q), k = 0, 1, 2, \dots$, then $-q$ is a quadratic nonresidue $(\text{mod } p)$.

Lemma 7. Let $q \equiv 7(\text{mod } 8)$ be a prime number. If p is a prime number belonging to the infinite arithmetic progression $2q - 1 + k.(4q), k = 0, 1, 2, \dots$, then $-q$ is a quadratic nonresidue $(\text{mod } p)$.

Lemma 8. Let p be a prime number. Then -1 is a quadratic nonresidue $(\text{mod } p)$ iff p belongs to the infinite arithmetic progression $3 + k.4, k = 0, 1, 2, \dots$.

As corollaries from the Dirichlet's theorem and the above lemmas we obtain the following:

Theorem 4. Let $p \equiv 3(\text{mod } 4)$ be a prime number. Then Φ_p^* given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + b^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(-1)$.

Remark 2. According to Dirichlet theorem there are infinitely many such primes p .

Theorem 5. Let $q \equiv 3(\text{mod } 8)$ be a prime number and p is a prime number that belongs to either of the infinite arithmetic progressions:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots,$$

$$q + 2 + k.(4q), k = 0, 1, 2, \dots$$

Then Φ_p given by

$$\Phi_p(a, b) = \sqrt{\varphi_p(a^2 - qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(q)$.

Remark 3. According to Dirichlet theorem there are infinitely many such primes p .

Theorem 6. Let $q \equiv 5 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$q + 2 + k.(2q), k = 0, 1, 2, \dots$$

Then Φ_p given by

$$\Phi_p(a, b) = \sqrt{\varphi_p(a^2 - qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(q)$.

Remark 4. According to Dirichlet theorem there are infinitely many such primes p .

Theorem 7. Let $q \equiv 7 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots$$

Then Φ_p given by

$$\Phi_p(a, b) = \sqrt{\varphi_p(a^2 - qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(q)$.

Remark 5. According to Dirichlet theorem there are infinitely many such primes p .

Open Problem 2. Let $q \equiv 1 \pmod{8}$ be a prime number. Find infinitely many primes p such that Φ_p given by

$$\Phi_p(a, b) = \sqrt{\varphi_p(a^2 - qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(q)$.

Theorem 8. Let $q \equiv 1 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots$$

Then Φ_p^* given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(-q)$.

Remark 6. According to Dirichlet's theorem there are infinitely many such primes p .

Theorem 9. Let $q \equiv 3 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$q + 2 + k.(4q), k = 0, 1, 2, \dots$$

Then Φ_p^* given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(-q)$.

Remark 7. According to Dirichlet theorem there are infinitely many such primes p .

Theorem 10. Let $q \equiv 5 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$2q + 1 + k.(4q), k = 0, 1, 2, \dots$$

Then Φ_p^* given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(-q)$.

Remark 8. According to Dirichlet theorem there are infinitely many such primes p .

Theorem 11. Let $q \equiv 7 \pmod{8}$ be a prime number and p is a prime number that belongs to the infinite arithmetic progression:

$$2q - 1 + k.(2q), k = 0, 1, 2, \dots$$

Then Φ_p^* given by the formula

$$\Phi_p^*(a, b) = \sqrt{\varphi_p(a^2 + qb^2)}$$

for all $(a, b) \in \mathbb{Q}^2$, is a non-Archimedean field norm on $\mathbb{Q}^2(-q)$.

Remark 9. According to Dirichlet theorem there are infinitely many such primes p .

4. Subnorms on Abelian additive groups

It is possible to provide a general approach ensuring a wide selection of d -FS and d -IFS. This approach is the following.

Let G be an additive Abelian group with group operation \oplus and $\varphi: G \rightarrow R^+ \cup \{0\}$ be a map.

Definition. We call φ a subnorm on G if the following properties are satisfied:

$$1) \varphi(g) \geq 0 \text{ if } g \neq \hat{0} \text{ and } g(\hat{0}) = 0,$$

where $\hat{0}$ is the zero of G .

$$2) \varphi(g) = \varphi(-g),$$

where $-g$ is the inverse element of g in G .

$$3) \varphi(g_1 \oplus g_2) \leq \varphi(g_1) + \varphi(g_2).$$

Every subnorm φ on G introduces a metric d on G which is given by:

$$d(g_1, g_2) = \varphi(g_1 \oplus (-g_2)).$$

Indeed the following properties of d :

- I) $d(g_1, g_2) = 0$ iff $g_1 = g_2$ and $d(g_1, g_2) > 0$ for $g_1 \neq g_2$,
- II) $d(g_1, g_2) = d(g_2, g_1)$,
- III) $d(g_1, g_2) + d(g_2, g_3) \geq d(g_1, g_3)$

are obvious and guarantee that d is a metric on G .

If G is an arbitrary field, then G is an additive Abelian group with respect to the operation addition in the field. Therefore, subnorm on G is every field norm on G satisfying property 3) (also every norm on a vector space over the field of real (complex) numbers is a subnorm on the same vector space).

In particular, if G is a field and φ is a non-Archimedean field norm on G , then φ satisfies 3) and hence φ is a subnorm on G . Therefore, every non-Archimedean field norm φ on G generates a metric d mentioned above (when we consider φ as a subnorm on the additive Abelian group G) and the corresponding d -FS and d -IFS.

In conclusion, we note that in Section 3 from the present paper were described infinitely many examples of non-Archimedean field norms on the following fields: $\mathbb{Q}^2(-1)$; $\mathbb{Q}^2(q)$; $\mathbb{Q}^2(-q)$, with q a fixed prime number. Considering all these fields as additive Abelian groups and their non-Archimedean field norms as subnorms, we can introduce the corresponding metrics and their d -FS and d -IFS and study them. This general scheme is very potent but the question about the corresponding modal operators depending on the described above non-Archimedean field norms remains still open.

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