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Poincaré Reccurence Theorem on IF Events

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***Abstract*:** *The classical Poincaré weak reccurence theorem states that for any probability space* $(\Omega, \mathcal{S}, P)$, *any* P*-measure preserving transformation* T, *and any* $A \in \mathcal{S}$, *almost all points of* A *return to* A. *In the present paper the Poincaré theorem is proved when the* σ*-algebra* $\mathcal{S}$ *is substituted by a family* $\mathcal{F}$ *of all measurable IF subsets of* Ω.

***Keywords*:** *Poincaré reccurence theorem, Probability space, Intuitionistic fuzzy set.*

1. Introduction

Let $(\Omega, \mathcal{S}, P)$ be a probability space, i.e. $\Omega \neq \emptyset$, $\mathcal{S}$ is a σ-algebra of subsets of Ω (i.e. $\Omega \in \mathcal{S}; A \in \mathcal{S} \Longrightarrow \Omega \setminus A \in \mathcal{S}$, and $A_n \in \mathcal{S}\,(n = 1, 2, ...) \Longrightarrow \bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{S}$), and $P : \mathcal{S} \to [0, 1]$ is such that $P(\Omega) = 1$, and $P(\bigcup\limits_{n=1}^{\infty} A_n) = \sum\limits_{n=1}^{\infty} P(A_n)$, whenever $A_n \in \mathcal{S}\,(n = 1, 2, ...)$ and $A_n \cap A_m = \emptyset (n \neq m)$. Let $T : \Omega \to \Omega$ be such that

$A \in \mathcal{S}$ implies $T^{-1}(A) \in \mathcal{S}$, and

$$P(T^{-1}(A)) = P(A),$$

whenever $A \in \mathcal{S}$; such transformations T are called measure - preserving. The Poincaré reccurence theorem states that almost every point $x \in A$ will return to A, i.e.

$$P(A \setminus \bigcup_{n=1}^{\infty} T^{-1}(A_n)) = 0,$$

whenever $A \in \mathcal{S}$.

2. IF-events

Consider again the probability space $(\Omega, \mathcal{S}, P)$, and a measure preserving mapping T, of course, instead of the σ-algebra $\mathcal{S}$ the family of all IF-events will be studied. An IF event is a pair

$$A = (\mu_A, \nu_A)$$

of $\mathcal{S}$-measurable mappings $\mu_A, \nu_A : \Omega \to [0,1]$ such that

$$\mu_A(\omega) + \nu_A(\omega) \le 1$$

for all $\omega \in \Omega$. The function μ_A is called the membership function of A, the function ν_A is called the nonmembership function of A. Denote by $\mathcal{F}$ the family of all IF-events, i.e.

$$\mathcal{F} = \{A = (\mu_A, \nu_A); \mu_A \nu_A : \Omega \to [0,1], \mu_A + \nu_A \le 1, \mu_A, \nu_A \text{ are } \mathcal{S}\text{– measurable}\}$$

On the set $\mathcal{F}$ a partial ordering is usually defined by the equivalence

$$A = (\mu_A, \nu_A) \le B = (\mu_B, \nu_B) \Longleftrightarrow \mu_A \le \mu_B,\ \nu_B \le \nu_A.$$

Evidently, the element $(0_\Omega, 1_\Omega)$ is the least element of $\mathcal{F}$, the element $(1_\Omega, 0_\Omega)$ is the greatest element of $\mathcal{F}$. We shall consider two binary operations on the family $\mathcal{F}$, so-called Lukasiewicz operations:

$$A \oplus B = ((\mu_A + \mu_B) \wedge 1, (\nu_A + \nu_B - 1) \vee 0),$$

and

$$A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (\nu_A + \nu_B) \wedge 1).$$

Recall that corresponding to the structure, the following operation $+$ can be defined on the family of all pairs $A = (\mu_A, \nu_A)$ of functions from Ω to R:

$$A + B = (\mu_A + \mu_B, 1 - (1 - \nu_A + 1 - \nu_B)) = (\mu_A + \mu_B, \nu_A + \nu_B - 1).$$

With respect to the operation $+$ the following equality holds:

$$A_1 + A_2 + ... + A_n = (\mu_{A_1} + \mu_{A_2} + ... + \mu_{A_n}, \nu_{A_1} + \nu_{A_2} + ... + \nu_{A_n} - (n-1)).$$

Analogously as in quantum structures (see [2]), instead of a probability, a state $m : \mathcal{F} \to [0,1]$ can be defined.

2.1. Definition. A state on $\mathcal{F}$ is a mapping $m : \mathcal{F} \to [0,1]$ satisfying the following conditions:

(i) $m((0,1)) = 1$;

(ii) m is additive, i.e.

$A \odot B = (0_\Omega, 1_\Omega) \Longrightarrow m(A \oplus B) = m(A) + m(B)$;

(iii) m is continuous, i.e.

$A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$.

2.2. Proposition. Any state $m : M \to [0,1]$ is strongly additive, i.e. it satisfies the following stronger condition:

(ii′) for any $n \in N$ there holds the implication

$A_1 + A_2 + ... + A_n \leq (1_\Omega, 0_\Omega) \Longrightarrow m(A_1 + A_2 + ... + A_n) = \sum_{i=1}^n m(A_i)$.

P r o o f: It can be proved by induction.

The following proposition is evident.

1.3. Proposition. Any mapping $m : M \to [0,1]$ is a state if and only if it satisfies the following conditions

(i) $m((1_\Omega, 0_\Omega)) = 1$;

(ii) if (A_n) is a sequence of elements of M such that $A_1 + A_2 + ... + A_n \leq (1_\Omega, 0_\Omega)$ for any $n \in N$, then

$$m(\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} A_i) = \sum_{i=1}^{\infty} m(A_i).$$

3. Poincaré reccurence theorem

3.1. Definition. Let $\Omega \to \Omega$ be measure preserving mapping, i.e.

$$A \in \mathcal{S} \Longrightarrow T^{-1}(A) \in \mathcal{S}, P(T^{-1}(A)) = P(A).$$

Then we define a mapping $\tau : \mathcal{F} \to \mathcal{F}$ by the equality

$$\tau(A) = (\mu_A \circ T, \nu_A \circ T).$$

3.2. Definition. $A \backslash B = A \odot ((1_\Omega, 0_\Omega) - B) = ((\mu_A - \mu_B) \vee 0, (\nu_A - \nu_B + 1) \wedge 1)$.

The following proposition is evident:

3.3. Proposition. $\tau(A \setminus B) = \tau(A) \setminus \tau(B), m(\tau(A) = m(A)$.

3.4. Theorem. Let $m : \mathcal{F} \to [0,1]$ be a strongly additive state. Then for any $A \in \mathcal{F}$ there holds

$$m(A \setminus \bigvee_{i=1}^{\infty} \tau^i(A)) = 0.$$

P r o o f: Put $B_j = \bigvee_{i=j}^{\infty} \tau^i(A), B = A \setminus \bigvee_{i=1}^{\infty} \tau^i(A) = (A - \bigvee_{i=1}^{\infty} \tau^i(A)) \vee (0_\Omega, 1_\Omega) = (A - B_1) \vee (0_\Omega, 1_\Omega)$. Then

$$\tau^n(B) = (\tau^n(A) - \bigvee_{i=n+1}^{\infty} \tau^i(A)) \vee 0 = (\tau^n(A) - B_{n+1}) \vee (0_\Omega, 1_\Omega).$$

We have

$$B + \tau(B) + \tau^2(B) + ... + \tau^n(B) =$$

$$= (A - B_1) \vee 0 + (\tau(A) - B_2) \vee 0 + (\tau^2(A) - B_3) \vee 0 + ... + (\tau^n(A) - B_{n+1}) \vee 0 \leq$$

$$\leq \bigvee_{j=1}^{n+1} \bigvee_{i=j+1}^{n+1} (\tau^j(A) - B_i) \vee 0.$$

Since

$$(\tau^i(A) - B_i) \vee (0, 1) \leq (\tau^i(A)) \leq (1, 0),$$

we obtain

$$B + \tau(B) + \tau^2(B) + ... + \tau^n(B) \leq (1, 0).$$

By the strong additivity

$$m(\bigvee_{i=0}^{\infty} \tau^i(B)) = \sum_{i=0}^{\infty} m(\tau^i(B)) = \sum_{i=1}^{\infty} m(B),$$

hence

$$m(B) = 0.$$

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