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A New Intuitionistic Fuzzy Implication

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Abstract*: *A new intuitionistic fuzzy implication is constructed. Its basic properties, as its relations with Modus Ponens, with intuitionistic logic axioms and G e o r g K l i r and B o Y u a n's axioms are studied.

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1. Introduction

The concept of *"intuitionistic fuzzy propositional calculus"* was introduced about 20 years ago (see, e.g. [1, 2]). Initially, it contained only one form of conjunction, disjunction and two forms of implication. In a series of papers a lot of new implications were defined in the frame of the intuitionistic fuzzy logic – see, e.g. [3-8].

Here, we shall study a new implication and its basic properties.

In intuitionistic fuzzy propositional calculus, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b\rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x.

Below we shall assume that for the three variables x, y and z the equalities: $V(x) = \langle a, b\rangle, V(y) = \langle c, d\rangle, V(z) = \langle e, f\rangle$ $(a, b, c, d, e, f, a+b, c+d, e+f \in [0, 1])$ hold.

For the needs of the discussion below, following the definition from [1], we shall define the notion of Intuitionistic Fuzzy Tautology (IFT) by:

x is an IFT, if and only if for $V(x) = \langle a, b\rangle$ holds: $a \geq b$, while x will be a tautology iff $a = 1$ and $b = 0$. As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0\rangle$.

For two variables x and y the operations "conjunction" ($\&$) and "disjunction" ($\vee$) are defined (see [1]) by:

$$V(x\&y) = \langle \min(a, c), \max(b, d)\rangle,$$

$$V(x \vee y) = \langle \max(a, c), \min(b, d)\rangle.$$

2. Main results

In the intuitionistic fuzzy sets theory operation $@$ is defined over two IFSs

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$$

and

$$B = \{\langle x, \mu_B(x), \nu_B(x)\rangle | x \in E\}$$

by

$$A@B = \{\langle x, \frac{(\mu_A(x) + \mu_B(x))}{2}, \frac{(\nu_A(x) + \nu_B(x))}{2}\rangle | x \in E\}.$$

Here, we will introduce this operation for the case of intuitionistic fuzzy logic for a first time:

$$V(x \to_@ y) = \langle \frac{b + c}{2}, \frac{a + d}{2}\rangle.$$

The new implication generates the following negation:

$$V(\neg_@ x) = \langle \frac{b}{2}, \frac{a + 1}{2}\rangle$$

that does not have analogues among the other intuitionistic fuzzy negations.

Theorem 1. Implication $\to_@$

(a) does not satisfy Modus Ponens in the case of tautology,

(b) satisfies Modus Ponens in the IFT-case.

P r o o f: (b) Let x and $x \to_@ y$ be IFTs. Then

$$a \geq b$$

and

$$\frac{b + c}{2} \geq \frac{a + d}{2},$$

i.e.,

$$c - d \geq a - b \geq 0.$$

Therefore, y is an IFT.

For the new intuitionistic fuzzy implication and negation none of the following three properties is valid:

Property 1. $A \to_@ \neg_@\neg_@ A$.

Property 2. $\neg_@\neg_@ A \rightarrow_@ A$.

Property 3. $\neg_@\neg_@\neg_@ A = \neg_@ A$.

Now, the question about the form of expression $\neg_@\neg_@ ... \neg_@ A$ is interesting. Let us define:

$$\neg^1_@ A = \neg_@ A,$$

$$\neg^{n+1}_@ A = \neg_@\neg^n_@ A.$$

Theorem 2. Let $n \geq 0$ be an natural number. Then

$$\neg^{2n+1}_@ \langle a, b\rangle = \langle \frac{b}{2^{2n+1}} + \frac{2}{3}.\frac{4^n - 1}{2^{2n+1}}, \frac{a}{2^{2n+1}} + \frac{1}{3}.\frac{4^{n+1} - 1}{2^{2n+1}}\rangle,$$

$$\neg^{2n+2}_@ \langle a, b\rangle = \langle \frac{a}{2^{2n+2}} + \frac{1}{3}.\frac{4^{n+1} - 1}{2^{2n+2}}, \frac{b}{2^{2n+2}} + \frac{2}{3}.\frac{4^{n+1} - 1}{2^{2n+2}}\rangle.$$

Corollary.

$$\lim_{n\to\infty} \neg^n_@ \langle a, b\rangle = \langle \frac{1}{3}, \frac{2}{3}\rangle.$$

Following [10], we introduced the list of axioms for propositional intuitionistic logic:

(a) $A \rightarrow A$,
(b) $A \rightarrow (B \rightarrow A)$,
(c) $A \rightarrow (B \rightarrow (A\&B))$,
(d) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,
(e) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
(f) $A \rightarrow \neg\neg A$,
(g) $\neg(A\&\neg A)$,
(h) $(\neg A \vee B) \rightarrow (A \rightarrow B)$,
(i) $\neg(A \vee B) \rightarrow (\neg A\&\neg B)$,
(j) $(\neg A\&\neg B) \rightarrow \neg(A \vee B)$,
(k) $(\neg A \vee \neg B) \rightarrow \neg(A\&B)$,
(l) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$,
(m) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$,
(n) $\neg\neg\neg A \rightarrow \neg A$,
(o) $\neg A \rightarrow \neg\neg\neg A$,
(p) $\neg\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)$,
(q) $(C \rightarrow A) \rightarrow ((C \rightarrow (A \rightarrow B)) \rightarrow (C \rightarrow B))$ and proved

Theorem 3. (a) Axioms (a), (i), (j), (k) of the propositional intuitionistic logic are IFTs for $\rightarrow_@$.

(b) No axiom is a tautology for $\rightarrow_@$.

Some variants of fuzzy implications (marked by $I(x, y)$) are described in book [9] by George Klir and Bo Yuan and the following nine axioms are discussed, where

$$I(x, y) \equiv x \rightarrow y$$

and

$$N(x) \equiv I(x, 0).$$

Axiom 1. $(\forall x, y)(x \leq y \rightarrow (\forall z)(I(x, z) \geq I(y, z))$.

Axiom 2. $(\forall x, y)(x \leq y \to (\forall z)(I(z, x) \leq I(z, y))$.
Axiom 3. $(\forall y)(I(\overline{0}, y) = \overline{1})$.
Axiom 4. $(\forall y)(I(\overline{1}, y) = y)$.
Axiom 5. $(\forall x)(I(x, x) = \overline{1})$.
Axiom 6. $(\forall x, y, z)(I(x, I(y, z)) = I(y, I(x, z)))$.
Axiom 7. $(\forall x, y)(I(x, y) = \overline{1}$ iff $x \leq y)$.
Axiom 8. $(\forall x, y)(I(x, y) = I(N(y), N(x)))$.
Axiom 9. I is a continuous function,
where

$$V(\overline{0}) = \langle 0, 1 \rangle,$$

$$V(\overline{1}) = \langle 1, 0 \rangle.$$

Theorem 4. Implication $\to_{@}$ satisfies Axioms 1, 2, 8 and 9.
P r o o f: Let $x \leq y$, i.e. $a \leq c$ and $b \geq d$. Then for Axiom 1 we obtain that

$$V(I(x, z)) = \langle \frac{b+e}{2}, \frac{a+f}{2} \rangle,$$

$$V(I(y, z)) = \langle \frac{d+e}{2}, \frac{c+f}{2} \rangle.$$

From

$$\frac{b+e}{2} \geq \frac{d+e}{2}$$

and

$$\frac{a+f}{2} \geq \frac{c+f}{2}$$

we obtain that $V(I(x, z)) \geq V(I(y, z))$.
Now, we shall modify two of the above Axioms:
Axiom 3′. $(\forall y)(I(0, y)$ is an IFT$)$.
Axiom 5′. $(\forall x)(I(x, x)$ is an IFT$)$.
Theorem 5. Implication $\to_{@}$ satisfies Axioms 3′ and 5′.
The proofs are analogous to the above ones.

3. Conclusion

The new implication and negation have some unique properties with respect to the other already existing intuitionistic fuzzy logic implications and negations. One of these new properties is mentioned in Theorem 2 and its Corollary.

There exists an interesting relation between the new negation and one of the intuitionistic fuzzy modal like operators – operator $\boxplus$ (see [2]).

In a next research intuitionistic fuzzy set analogues of the new intuitionistic fuzzy logic implication and negation will be discussed.

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