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Upper and Lower Bounds for the Minimum of a Linear Function over the Efficient Set

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Abstract. *The minimization of a linear function on the efficient set of a MOLP problem is considered. A procedure for obtaining upper and lower bounds for the needed value is proposed. The core technique in this procedure is the reference point method for multiobjective optimization. An illustrative example is given.*

Keywords: *Multiple objective optimization, efficient set, reference point.*

1. Introduction

The problems for optimization of a function over the efficient set E of a multiobjective linear programming (MOLP) problem have been considered in many papers . The survey of Y a m a m o t o (2002) proposes a classification of the existing algorithms for optimization over the efficient set. This classification contains seven classes: adjacent vertex search algorithms, nonadjacent vertex search algorithms, face search algorithms, branch and bound search algorithms, Lagrangean relaxation based algorithms, dual approach, bisection algorithms. In Yamamoto's paper each class is presented with one typical algorithm and these algorithms are compared with respect to the computational requirements.

A b b a s and C h a a b a n e (2006) consider a linear functions optimization on an integer efficient set. The problem for minimization of a convex function over the weakly efficient set is considered in the paper of Y a m a d a, T a n i n o, I n u i-g u c h i (2001). To find an approximate solution they propose a method that uses a branch and bound procedure. H o r s t and T h o a i (1999) consider a similar problem. After one transformation they obtain a new global optimization problem.

For the last problem they propose an algorithm of branch and bound type. A penalty function approach to maximize a function over the efficient set is proposed by W h i t e (1986). The paper of J o r g e (2005) contains a description of an algorithm that gives an exact solution of the problem for optimization of a linear function over the efficient set. The author proposes "a simplified disjoint bilinear program" that can be solved by the use of some special methods of nonconvex optimization.

The here proposed paper contains a description of a procedure that is based on a multiobjective optimization technique.

2. Some preliminaries

2.1. The formulation of the multiobjective linear programming problem is as follows:

$$
\begin{aligned}
&\max\ f_1(x)\\
&\max\ f_2(x)\\
&\quad\dots\\
&\quad\dots\\
&\quad\dots\\
&\max\ f_m(x)\\
&\text{s. t.}\\
&x \in S \subseteq R^n.
\end{aligned}
\tag{1}
$$

All $f_i(x)$, $i = 1, 2,\dots, m$, are linear functions. The vector x is the argument vector. The vector $f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in R^m$ is the criteria vector. The set S is the feasible set in R^n. It is defined as follows:

$$S = \{ x \in R^n \mid \sum_{j=1}^{n} a_{ij}.x_j + \mathrm{SL}_i = c_i;\ i = 1, 2,\dots, p\}. \tag{2}$$

Here $\mathrm{SL}_i \geq 0$ $(\forall i)$ and c_i $(\forall i)$ are unrestricted in sign. The restrictions $x_j \geq 0$ $(\forall j)$ are included in the above shown constraints. The set S is not empty and is bounded. The set

$$Z = Z(S) = \{z \in R^m \mid z = f(x),\ x \in S\}$$

is the attainable set in the criteria space. The point $z^1 = f(x^1) \in Z$, $x^1 \in S$, is called a nondominated (Pareto) point, if there **does not** exist another point $x^2 \in S$, $x^2 \neq x^1$, such that the following conditions are fulfilled simultaneously :

$$f_i(x^2) \geq f_i(x^1) \text{ for all } i,\ i = 1, 2, \dots, m,$$

$$f_j(x^2) > f_j(x^1) \text{ for one } j \text{ at least.}$$

If the point $z^1 = f(x^1)$, $z^1 \in Z$, is nondominated, then the point $x^1 \in S$ is an efficient point. The set $P \subseteq Z$, containing all nondominated points from Z, is called a nondominated (Pareto) set. The set $E \subseteq S$, containing all efficient points from S, is called the efficient set. For each MOLP problem the set E is closed and connected.

We suppose that we have given the linear function $\varphi(x)$ ($x \in R^n$). Having in mind problem (1), our purpose in this paper will be to obtain upper and lower bounds for the minimum of $\varphi(x)$ over the set E, i. e. $\min\{\varphi(x) \mid x \in E\}$. It must be noted that the set E is not convex .

2.2. Now we will introduce the notion of *a wall of the set* S. In problem (1) the set S is described by the constraints (2). In addition, we suppose that the list of constraints does not contain redundant constraints. Let us consider the sets W_j where

(3) $$W_j = \{x \in S \mid \mathrm{SL}_j = 0\}, \quad j = 1, 2, \dots, p.$$

Each one of these sets is called *a wall of the set* S. (Here we include all walls described by the restriction $x_j = 0$ (not redundant)). There is a more general notion of *a facet* (S t e u e r (1986)). For clarity: each wall is a facet, but there can be a facet that is not a wall.

2.3. Besides the set S, defined in §1, we will refer to the set S_1, defined in the following way:

(4) $$S_1 = \{y \in R^n \mid \sum_{j=1}^{n} a_{ij} y_j + \mathrm{SLP}_i = c_i + \delta;\ i = 1, 2, \dots, p\}.$$

Here δ is a small positive number. It is clear that all points of S belong to S_1, too. The walls of S_1 will be denoted by W'_j. Having in mind that S_1 depends on δ we will use sometimes the symbol $S_1(\delta)$.

2.4. We will suppose that all efficient points of S belong to the frontier of this set, i.e. each efficient point of S belongs to one of its walls, at least.

2.5. To find nondominated points of problem (1) we will use the reference point method, proposed by W i e r z b i c k i (1980, 1986). For such purposes the reference point method recommends to solve the following linear programming problem

(5) $$\begin{aligned} &\min\ D \\ &\text{s.t.} \\ &D \geq b_i(r_i - f_i(x)) - \rho \sum_{j=1}^{m} f_j(x),\ i = 1, \dots, m. \\ &x \in S. \end{aligned}$$

Here the set S and the functions $f_i(x)$ are defined as in problem (1), we have $b_i > 0\ \ \forall i$, and ρ is a small positive number. The variable D is unrestricted in sign. A very important property of problem (5) is that for any **arbitrary** reference point $r \in R^m$ the obtained solution determines an **efficient point** $x \in E \subseteq S$ for problem (1) as well as a corresponding **nondominated point** $f(x) \in P \subseteq Z$.

On the other hand problem (5) can be used for checking whether a given point belongs to the set $P \subseteq Z$.

2.6. An augmented problem

Let us consider the following augmented problem :

$$
\begin{aligned}
&\min \; = \; D \\
&\text{s.t.} \\
&D \geq b_i(r_i - f_i(x)) - \rho \sum_{j=1}^{m} f_j(x), \quad i = 1, 2, \dots, m, \\
(6) \qquad &x \in S, \\
&y \in S_1 \subset R^n, \\
&r_i = f_i(y), \qquad i = 1, 2, \dots, m.
\end{aligned}
$$

Here the vector y is analogous to the vector x, but $y \in S_1$. The functions $f_i(y)$ are obtained from the functions $f_i(x)$, substituting x by y. The set S_1 is described by the constraints containing the variables SLP_i (as in (4)).

We will use this problem under an additional requirement : the vector r must belong to one of the walls of the set $Z(S_1)$. If we have the wall W'_q chosen this requirement is satisfied when the constraint $\mathrm{SLP}_q = 0$ is added to the other constraints describing S_1. So we have $y \in W'_q$ and r belongs to the corresponding wall of $Z(S_1)$. And, in addition, if $\delta > 0$, then the following assertion is true: the set containing r is bounded and does not have common points with $Z(S)$.

An important property of problem (6). Suppose that the wall W_q of the set S contains some efficient points. Then the corresponding wall of $Z(S)$ contains Pareto points. If we solve problem (6) setting $\mathrm{SLP}_q = 0$, then the vector y belongs to the wall W'_q of S_1 and the vector $r = f(y)$ belongs to the corresponding wall of $Z(S_1)$. The walls W_q and W'_q are parallel, the same is true for the corresponding walls of $Z(S)$ and $Z(S_1)$.

Assertion. If the wall W_q of S contains points from $E \subset S$, and $\mathrm{SLP}_q = 0$ in the formulation of the problem (6), then the solution of this problem determines one of these efficient points and the corresponding point from the wall W'_q.

This assertion is based on the fact that all points of S that do not belong to W_q are on one site of this wall and all points of W'_q are on the other site.

2.7. Suppose that d is the value of min $\{\varphi(x) \mid x \in E\}$. Suppose in addition that the minimal value of φ over S is b and $b < d$. In such a case we can choose a number u such that $b < u < d$. Denote:

$$V = \{x \in S \mid \varphi(x) < u\}.$$

Theorem. All efficient points of the set V belong to the wall

$$V_u = \{x \in S \mid \varphi(x) = u\}.$$

P r o o f: The set V does not contain efficient points from S by definition. Suppose that we have fixed an efficient point from S and we remove some other points from S. Then the fixed efficient point remains efficient in the rest of the set S. This means that the set V can have an efficient point x' with $\varphi(x') \neq u$ if and only if x' is an efficient point of S. But the set V does not contain such points ■

3. A procedure for obtaining upper and lower bounds for the value of min $\{\varphi(x) \mid x \in E\}$

1. For the set S solve all problems: max $f_i(x)$, min $f_i(x)$, max x_i, min x_i, max φ, min φ on the whole set S. This step is not obligatory but the obtained results help to better estimate the results of the next steps.

2. For all j, $j = 1, 2, \dots, p$ solve the problems

$$\begin{aligned} &\min \varphi(y), \\ &\text{s.t.} \\ &y \in S_1, \\ &r_i = f_i(y), \ i = 1, 2, \dots, m, \\ &\mathrm{SLP}_j = 0. \end{aligned} \tag{7}$$

The obtained points y^j have "small" values $\varphi(y^j)$, do not belong to S and are close to the walls W_j of S. The points $r^j = f(y^j)$ have a similar property: they do not belong to $Z(S)$ but are close to it.

3. Consider problem (5). Solve this problem for each r^q, $q = 1, 2, \dots, p$, obtained above. This means that we use the reference point method and for each wall of $Z(S)$ we have a corresponding reference point r^q that does not belong to $Z(S)$ but is close to this set. Having the results for all q we have found various efficient points, the corresponding Pareto points and the corresponding values of φ. The sense is that if for a fixed q the wall W_q contains some efficient points one such point will be obtained and the corresponding value of φ will be "small". The minimal among these obtained values of φ is an upper bound for min $\{\varphi \mid x \in E\}$. In addition these computations show the walls of S containing efficient points.

4. Checking for a lower bound. We choose a number u supposing that it is a lower bound, i.e. min $\{\varphi \mid x \in E\} > u$. We consider problem (6) and we add the constraint $\varphi(x) \leq u$ to the other constraints describing S. So we obtain a description of the set V. We solve the so obtained problem under the additional constraint $\mathrm{SLP}_i = 0$ consecutively for all i, in other words for all walls of S_1.

If for all obtained points x (as solutions) we have $\varphi(x) = u$, we conclude that all efficient points of V belong to the wall $\varphi(x) = u$. Therefore the number u is a lower bound.

4. Example

For illustrative purposes the above described procedure was used for the following problem. The set S is determined by the following constraints:

$$2x_4 + \mathrm{SL}_1 = 28;$$
$$3x_1 + 1x_2 + 2x_3 + \mathrm{SL}_2 = 30;$$

$$1\,x_1 - x_2 + 6\,x_3 + 5\,x_4 + \mathrm{SL}_3 = 44;$$
$$2\,x_4 + 4\,x_5 + \mathrm{SL}_4 = 42;$$
$$3\,x_5 + \mathrm{SL}_5 = 30;$$
$$x_1 + 1.7\,x_2 + 1.9\,x_3 + 2.1\,x_4 + 1.5\,x_5 - \mathrm{SL}_6 = 4;$$
$$x_i \ge 0 \quad \forall i.$$

The criteria $f_i(x)$ are given below:

$$f_1 = 2\,x_2 + 2\,x_3 - 2\,x_4 + x_5;$$
$$f_2 = -x_1 - 2\,x_2 + x_3 + 2\,x_4 + 2\,x_5;$$
$$f_3 = +x_1 - x_3 + 5\,x_4 + x_5;$$
$$f_4 = 4\,x_2 - 2\,x_5.$$

The function φ is as follows

$$\varphi = 2\,x_1 + 10\,x_2 - 2\,x_3 + 6\,x_4 - 2\,x_5.$$

For the computations the demo software of LINGO 11 was used. The obtained values of the needed upper and lower bounds are given here:

$$-33.9 \le \min\{\varphi \mid x \in E\} \le -33.60918.$$

5. Conclusion

A procedure for estimating the minimal value of a linear function over the efficient set of a MOLP problem is given. The procedure is based on the usage of linear programming problems only.

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