

Selection of Alternatives, Modeled as Multi-Dimensional Generalized Lotteries of I Type

*Natalia D. Nikolova**, *Snejana Ivanova***, *Kiril Tenekedjiev**,

** Technical University – Varna, 1 Studentska Str., 9010 Varna, Bulgaria, tel: +359 52 383670,
E-mail: {natalia,kiril}@dilogos.com*

*** Technical University – Varna, 1 Studentska Str., 9010 Varna, Bulgaria, tel: +359 52 383670,
E-mail: hotelmak@abv.bg*

Summary: *Problems, where there are several alternatives, each generating an infinite number of consequences (prizes), are modeled as generalized lotteries of I type. The paper focuses on such models for the case when prizes are modeled as multi-dimensional vectors of attributes. The description of partially measured uncertainty is handled by multi-dimensional ribbon distributions that embody the idea of p-boxes. That leads to new models of alternatives, called fuzzy rational multi-dimensional generalized lotteries of I type. These alternatives are modeled according to the Q-expected utility (where Q is a criterion under strict uncertainty), which is calculated using Stieltjes integrals.*

Keywords: *multi-dimensional prizes, generalized lotteries of I type, p-boxes, ribbon distributions, fuzzy rational lotteries, Q-expected utility.*

1. Introduction

Lotteries are a typical model of risky alternatives and represent the connection between random events (states) and prizes (consequences) (von Neumann, Morgenstern [20]). The classical version of lotteries (classical risky lotteries) suggests uncertainty is modeled by classical probability distributions (CDF, DPF) (Nikolova et al. [10]), whereas preferences are measured by a utility function. If either the set of prizes X or the set of lotteries L is continuous, then generalized

lotteries of I, II or III type (GL-I, GL-II, GL-III) apply (Tenekedjiev, Nikolova [15]; Tenekedjiev [13]).

In the course of the analysis, the decision maker (DM) is requested to make subjective estimates of utilities and probabilities using iterative techniques (McCord, De Neufville [7]; Abdellaoui et al. [1]) based on triple dichotomy (Tenekedjiev [14]). The classical elicitation assumption suggests DMs make their choices according to expected utility (which is the measure of utilities of a consequence weighted by their probabilities). This rule holds for the ideal DM that elicits unique quantitative measures of utilities/probabilities. The real DM disobeys this rule and elicits interval estimates that lead to partially non-transitive preferences (due to contradiction with the axioms of rational preferences (French [4]). In Nikolova et al. [9] such DMs are called fuzzy rational. The new data format requires new models of alternatives, called fuzzy rational lotteries, i.e. lotteries where the uncertainty associated with the prizes is only partially measured by the so-called ribbon distributions (Nikolova et al. [10]), the latter being special cases of the p -boxes (Ferson, Hajagos [3]). A two-stage procedure is proposed to rank fuzzy rational lotteries (Tenekedjiev et al. [18]). The first stage suggests approximating the ribbon distributions by classical ones using a Q criterion under strict uncertainty. Approximation techniques for different types of ribbon distributions are presented in (Tenekedjiev, Nikolova [15]; Nikolova [8]). At the second stage, the resulting Q lotteries (which are classical risky ones) are ranked according to expected utility. This two-step procedure is called Q -expected utility. Laplace (Rapoport [11]), Wald (Fabrycky et al. [2]), maximax (Hackett, Luffrum [5]) and Hurwicz _{α} (Yager [21]) criteria under strict uncertainty usually serve as Q criteria. Ranking lotteries with discrete prizes (ordinary lotteries) according to Q -expected utility has been discussed in (Tenekedjiev, Nikolova [16]). The Laplace expected utility criterion was applied in (Tenekedjiev et al. [18]) to rank fuzzy rational one-dimensional GL-I (i. e. GL-I with one-dimensional prizes), whereas the work (Tenekedjiev et al. [17]) introduced procedures to rank fuzzy rational GL-II.

This paper discusses multi-dimensional GL-I, i.e. GL-I whose prizes are multi-dimensional vectors of attributes. It uses the concept of p -boxes to construct probability functions that interpret partially measured uncertainty. Following that a model of fuzzy rational multi-dimensional GL-I is elaborated in its general form and special cases. It is suggested to rank these new alternatives according to the Q -expected utility. The latter is calculated with the help of new algorithms that use Stieltjes integrals.

In what follows, Section 2 introduces the basis of multi-dimensional utility theory and multi-dimensional classical distributions, and then ends up discussing multi-dimensional ribbon CDF and their special cases. Section 3 is entirely devoted to classical risky multi-dimensional GL-I and their special cases. Section 4 has the same purpose for the case of multi-dimensional fuzzy rational GL-I. Appendices to the paper contain description of theoretical principles needed to clarify the main text.

2. Measuring preferences and uncertainty over multi-dimensional prizes

2.1. Multi-dimensional utility functions

Assume that the elements of the set of prizes X are d -dimensional vectors that describe aspects of the DM's choice of importance for her, called *attributes*. Let the j -th attribute of the prizes in X be a random variable X_j and x_j be a possible realization. The multi-dimensional set of prizes X is a multiplication of the one-dimensional sets of attributes X_1, X_2, \dots, X_d . Each prize in X takes the form of a d -dimensional vector $\bar{x}=(x_1, x_2, \dots, x_d)$ with arbitrary fixed values. It is often convenient to group the attributes X_1, X_2, \dots, X_d in $n \in \{2, 3, \dots, d\}$ base vector attributes Y_1, Y_2, \dots, Y_n , each containing at least one attribute. Furthermore, each attribute should belong to exactly one base vector attribute. In that way each base vector attribute $Y_j, j=1, 2, \dots, n$, is a system of d_j random variables $(X_1^{Y_j}, X_2^{Y_j}, \dots, X_{d_j}^{Y_j})$ with realizations $\bar{y}_j = (x_1^{y_j}, x_2^{y_j}, \dots, x_{d_j}^{y_j})$. The prizes can be presented as $\bar{x}=(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)$:

$$\begin{aligned}
 (1) \quad & X=(X_1, X_2, \dots, X_d)=(Y_1, Y_2, \dots, Y_n), \\
 (2) \quad & 2 \leq n \leq d, \\
 & Y_j \cap Y_k = \emptyset, j=1, 2, \dots, n, k=1, 2, \dots, j-1, j+1, \dots, n, \\
 & Y_j \neq \emptyset, j=1, 2, \dots, n, \\
 & Y_j = (X_1^{Y_j}, X_2^{Y_j}, \dots, X_{d_j}^{Y_j}), j=1, 2, \dots, n, \\
 & d_1+d_2+\dots+d_n=d.
 \end{aligned}$$

Preferences over multi-dimensional prizes can be measured by a multi-dimensional utility function $u(\cdot)$, whose domain are all possible multi-dimensional prizes $\bar{x} \in \square^d$:

$$(3) \quad u=u(\bar{x})=u(x_1, x_2, \dots, x_d).$$

An algorithm for that purpose is proposed in Keeney, Raiffa [6], but it is inapplicable if $d>3$. A better approach is to decompose (3) to base utility functions over Y_1, Y_2, \dots, Y_n :

$$(4) \quad u(\bar{x})=u(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)=f[u_1(\bar{y}_1), u_2(\bar{y}_2), \dots, u_n(\bar{y}_n)].$$

Here, $f(\cdot)$ is a real-valued function of n real variables. The adequacy of (4) depends on conditions for independence of preferences over the attributes. Preferential independence is the weakest, but most common condition. The strongest condition is additive independence, but also very difficult to establish. Of greatest importance is utility independence, as it allows to construct the utility function $u(\cdot)$ over n arguments using n normalized base utility functions $u_i(\cdot)$ ($i=1, 2, \dots, n$) (Keeney, Raiffa [6]). Appendix 2 presents the forms that $u(\cdot)$ can take in the case of mutual utility independence of the base vector attributes. Those depend on the values of scaling constants that measure the relative significance of each base vector attribute for the preferences of the DM. Two types

of preferences are identified depending on the sum of the scaling constants of the base vector attributes.

2.2. Multi-dimensional classical CDF

The uncertainty associated with multi-dimensional prizes can be measured by multi-dimensional Cumulative Distribution Functions – CDF. Assume the uncertainty, associated with a d -dimensional system of random variables (X_1, X_2, \dots, X_d) is entirely measured by a known d -dimensional CDF, called *classical CDF* (*multi-dimensional classical CDF*) (see Appendix 1). If $F(\cdot)$ is a d -dimensional classical CDF of (X_1, X_2, \dots, X_d) , whereas $\bar{\mathbf{x}}=(x_1, x_2, \dots, x_d)$ is a d -dimensional vector with arbitrary fixed values, then for all $\bar{\mathbf{x}} \in \square^d$

$$(5) \quad F(\bar{\mathbf{x}})=F(x_1, x_2, \dots, x_d)=P(X_1 \leq x_1 \cap X_2 \leq x_2 \cap \dots \cap X_d \leq x_d).$$

The multi-dimensional classical CDF has a special case related to independence of the base vector attributes (classical VICDF) (see Appendix 1).

2.3. Multi-dimensional ribbon CDF

2.3.1. General case of multi-dimensional ribbon CDF

Assume the uncertainty in a d -dimensional system of random variables (X_1, X_2, \dots, X_d) is partially measured by a d -dimensional CDF that is only known to lie between two d -dimensional classical CDF, called *lower* and *upper bounds*. This representation is in compliance with the discussion in Utkin [19]. Such a multi-dimensional CDF shall be called *multi-dimensional ribbon CDF*. If $F^R(\cdot)$ is a d -dimensional ribbon CDF of (X_1, X_2, \dots, X_d) , then $F^d(\cdot)$ and $F^u(\cdot)$ are the lower and upper bounds of F^R , and $\bar{\mathbf{x}}=(x_1, x_2, \dots, x_d)$ is a d -dimensional vector with arbitrary fixed values, then for all $\bar{\mathbf{x}} \in \square^d$

$$(6) \quad F^d(\bar{\mathbf{x}}) \leq F^R(\bar{\mathbf{x}}) \leq F^u(\bar{\mathbf{x}}).$$

Here $F^d(\cdot)$ and $F^u(\cdot)$ obey the condition

$$(7) \quad F^d(\bar{\mathbf{x}}) \leq F^u(\bar{\mathbf{x}}).$$

If X is a one-dimensional random variable, then the uncertainty associated with it might be partially measured by a one-dimensional ribbon CDF (Tenekedjiev et al. [18]).

2.3.2. Ribbon vector argument CDF

Assume the DM has grouped the attributes X_1, X_2, \dots, X_d in the greatest possible number of base vector attributes Y_1, Y_2, \dots, Y_n following (1) and (2). Then a special case of the multi-dimensional ribbon CDF is when it is represented as a product of d_j -dimensional ribbon CDF of the base vector attributes. Such a multi-dimensional ribbon CDF shall be called *vector argument CDF* (*ribbon VACDF*). Let $F^R(\cdot)$ be a ribbon VACDF. If the events $A_j, j=1, 2, \dots, n-1$, are defined as in (A1.5), then

$$(8) \quad F^R(\bar{\mathbf{x}}) = F_{y,1}^R(\bar{\mathbf{y}}_1) F_{y,2}^R(\bar{\mathbf{y}}_2 | A_1) \dots F_{y,n}^R(\bar{\mathbf{y}}_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

Here, $F_{y,1}^R(\cdot)$ is a marginal d_1 -dimensional ribbon CDF of Y_1 , whereas $F_{y,j}^R(\cdot | A_1 \cap A_2 \cap \dots \cap A_{j-1})$ is a conditional d_j -dimensional ribbon CDF of Y_j , if the

event $A_1 \cap A_2 \cap \dots \cap A_{j-1}$ has occurred (i.e. if the intersection of the events A_1, A_2, \dots, A_{j-1} has occurred). Then for all $\bar{\mathbf{y}}_1 \in \square^{d_1}$ and for all $\bar{\mathbf{x}} \in \square^d$:

$$(9) \quad F_{y,1}^d(\bar{\mathbf{y}}_1) \leq F_{y,1}^R(\bar{\mathbf{y}}_1) \leq F_{y,1}^u(\bar{\mathbf{y}}_1),$$

$$(10) \quad \begin{aligned} F_{y,j}^d(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}) &\leq F_{y,j}^R(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}) \leq \\ &\leq F_{y,j}^u(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), \quad j = 2, 3, \dots, d. \end{aligned}$$

Here $F_{y,j}^d(\cdot)$ and $F_{y,j}^u(\cdot)$ are known d_j -dimensional classical CDF of the base vector attributes, that obey the following condition for all $\bar{\mathbf{y}}_1 \in \square^{d_1}$ and for $\bar{\mathbf{x}} \in \square^d$:

$$(11) \quad F_{y,1}^d(\bar{\mathbf{y}}_1) \leq F_{y,1}^u(\bar{\mathbf{y}}_1),$$

$$(12) \quad F_{y,j}^d(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}) \leq F_{y,j}^u(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), \quad j=2, 3, \dots, n.$$

Then the lower and upper bounds will be multi-dimensional classical CDF that are the product of n number of d_j -dimensional classical CDF of the base vector attributes:

$$(13) \quad F^d(\bar{\mathbf{x}}) = F_{y,1}^d(\bar{\mathbf{y}}_1) F_{y,2}^d(\bar{\mathbf{y}}_2 | A_1) \dots F_{y,n}^d(\bar{\mathbf{y}}_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}),$$

$$(14) \quad F^u(\bar{\mathbf{x}}) = F_{y,1}^u(\bar{\mathbf{y}}_1) F_{y,2}^u(\bar{\mathbf{y}}_2 | A_1) \dots F_{y,n}^u(\bar{\mathbf{y}}_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

2.3.3. Ribbon vector independent CDF

A special case of a ribbon VACDF is when the base vector attributes are probabilistically independent. Such a ribbon VACDF is called *vector independent* (ribbon VICDF). Let $F^R(\cdot)$ be a ribbon VICDF. Then

$$(15) \quad F^R(\bar{\mathbf{x}}) = F_{y,1}^R(\bar{\mathbf{y}}_1) F_{y,2}^R(\bar{\mathbf{y}}_2) \dots F_{y,n}^R(\bar{\mathbf{y}}_n).$$

Here, $F_{y,j}^R(\cdot)$ is the marginal d_j -dimensional ribbon CDF of Y_j , and for all $\bar{\mathbf{y}}_j \in \square^{d_j}$

$$(16) \quad F_{y,j}^d(\bar{\mathbf{y}}_j) \leq F_{y,j}^R(\bar{\mathbf{y}}_j) \leq F_{y,j}^u(\bar{\mathbf{y}}_j), \quad j=1, 2, \dots, n,$$

$$(17) \quad F_{y,j}^d(\bar{\mathbf{y}}_j) = F_{y,j}^d(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), \quad j=2, 3, \dots, n,$$

$$(18) \quad F_{y,j}^u(\bar{\mathbf{y}}_j) = F_{y,j}^u(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), \quad j=2, 3, \dots, n,$$

$$(19) \quad F_{y,j}^R(\bar{\mathbf{y}}_j) = F_{y,j}^R(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), \quad j=2, 3, \dots, n.$$

The last three formulae, where the events $A_j, j=1, 2, \dots, n-1$, are defined as in (A1.5), follow from the probability independence of Y_1, Y_2, \dots, Y_n .

In that case, the lower and upper bounds will be classical VICDF that are the product of n number of d_j -dimensional classical CDF of the base vector attributes:

$$(20) \quad F^d(\bar{\mathbf{x}}) = F_{y,1}^d(\bar{\mathbf{y}}_1) F_{y,2}^d(\bar{\mathbf{y}}_2) \dots F_{y,n}^d(\bar{\mathbf{y}}_n),$$

$$(21) \quad F^u(\bar{\mathbf{x}}) = F_{y,1}^u(\bar{\mathbf{y}}_1) F_{y,2}^u(\bar{\mathbf{y}}_2) \dots F_{y,n}^u(\bar{\mathbf{y}}_n).$$

3. Multi-dimensional GL-I

3.1. General case of multi-dimensional classical risky GL-I

Let's compare according to preference q alternatives that give multi-dimensional prizes $\bar{\mathbf{x}}$ from a piece-wise continuous d -dimensional set X according to continuous or mixed multi-dimensional probability laws. Such alternatives can be modeled as multi-dimensional GL-I. The multi-dimensional GL-I with a multi-dimensional classical CDF shall be called multi-dimensional classical risky:

$$(22) \quad g_i^{\text{cr}} = \langle F_i(\bar{\mathbf{x}}); \bar{\mathbf{x}} \rangle, i=1, 2, \dots, q.$$

Here, $F_i(\cdot)$ is a d -dimensional classical CDF.

The expected utility of the multi-dimensional classical risky GL-I can be calculated as a multi-dimensional Stieltjes integral (Stroock [12]) with an integrating function $F_i(\cdot)$:

$$(23) \quad E_i(u|F_i) = \int \dots \int_{\square^d} u(\bar{\mathbf{x}}) \partial^d F_i(\bar{\mathbf{x}}).$$

3.2. Special cases of classical risky GL-I

3.2.1. Classical risky GL-I with independent distributions

A special case of a multi-dimensional classical risky GL-I is when the uncertainty in the prizes is described by a classical VICDF (A1.8). If the events $A_j, j=1, 2, \dots, n-1$, are defined as in (A1.5), then the following holds:

$$(24) \quad F_i(\bar{\mathbf{x}}) = F_{y,1}^{(i)}(\bar{\mathbf{y}}_1) F_{y,2}^{(i)}(\bar{\mathbf{y}}_2) \dots F_{y,n}^{(i)}(\bar{\mathbf{y}}_n),$$

$$(25) \quad F_{y,j}^{(i)}(\bar{\mathbf{y}}_j) = F_{y,j}^{(i)}(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}), j=2, 3, \dots, n.$$

When calculating the expected utility of such a lottery, the d -dimensional Stieltjes integral is brought down to n number of consecutive d_j -dimensional Stieltjes integrals (one per each base vector attribute):

$$(26) \quad E_i(u|F_i) = \int \dots \int_{\square^{d_n}} \dots \int \dots \int_{\square^{d_2}} \dots \int \dots \int_{\square^{d_1}} u(\bar{\mathbf{x}}) \partial^{d_1} F_{y,1}^{(i)}(\bar{\mathbf{y}}_1) \partial^{d_2} F_{y,2}^{(i)}(\bar{\mathbf{y}}_2) \dots \partial^{d_n} F_{y,n}^{(i)}(\bar{\mathbf{y}}_n).$$

3.2.2. Mutual non-additive utility independence in classical risky GL-I with independent distributions

A special case of a classical risky GL-I with classical VICDF (A1.8) is when the preferences of the DM over the base vector attributes Y_1, Y_2, \dots, Y_n are MUVI (see (A2.5)). The expected utility of such a lottery can be calculated as follows:

$$\begin{aligned}
(27) \quad E_i(u | F_i) &= \int_{\square^{d_n}} \dots \int_{\square^{d_2}} \dots \int_{\square^{d_1}} \dots \int \frac{1}{K_y} \times \\
&\times \prod_{j=1}^n [K_y k_{y,j} u_{y,j}(\bar{\mathbf{y}}_j) + 1] \partial^{d_1} F_{y,1}^{(i)}(\bar{\mathbf{y}}_1) \partial^{d_2} F_{y,2}^{(i)}(\bar{\mathbf{y}}_2) \dots \partial^{d_n} F_{y,n}^{(i)}(\bar{\mathbf{y}}_n) - \frac{1}{K_y} = \\
&= \frac{1}{K_y} \prod_{j=1}^n [K_y k_{y,j} \int_{\square^{d_j}} \dots \int u_{y,j}(\bar{\mathbf{y}}_j) \partial^{d_j} F_{y,j}^{(i)}(\bar{\mathbf{y}}_j) + 1] - \frac{1}{K_y} = \\
&= \frac{1}{K_y} \prod_{j=1}^n [K_y k_{y,j} E_{y,j}^{(i)}(u | F_{y,j}^{(i)}) + 1] - \frac{1}{K_y}.
\end{aligned}$$

Here, the d_j -dimensional Stieltjes integral under the multiplication symbol stands for the expected utility of a hypothetical d_j -dimensional classical risky GL-I that gives the j -th base vector attribute according to a d_j -dimensional classical CDF $F_{y,j}^{(i)}(\cdot)$:

$$(28) \quad g_{y,j}^{cr,(i)} = \langle F_{y,j}^{(i)}(\bar{\mathbf{y}}_j); \bar{\mathbf{y}}_j \rangle, \quad j=1, 2, \dots, n.$$

Then it follows that

$$(29) \quad E_{y,j}^{(i)}(u | F_{y,j}^{(i)}) = \int_{\square^{d_j}} \dots \int u_{y,j}(\bar{\mathbf{y}}_j) \partial^{d_j} F_{y,j}^{(i)}(\bar{\mathbf{y}}_j), \quad j=1, 2, \dots, n.$$

The multi-dimensional classical risky GL-I that gives the j -th base vector attribute with a d_j -dimensional classical CDF shall be called *base fictitious (classical risky BF-GL-I)*. In that way the expected utility of the classical risky GL-I with a classical VICDF in the case of MUVI preferences takes a multiplicative form, and its multipliers depend on the expected utilities of the classical risky BF-GL-I.

3.2.3. Mutual additive independence in classical risky GL-I with independent distributions

A special case of a classical risky GL-I with a classical VICDF (A1.8) is when the preferences of the DM over the base vector attributes Y_1, Y_2, \dots, Y_n are MAVI (A2.4). The expected utility of such a lottery can be calculated as follows:

$$\begin{aligned}
(30) \quad E_i(u | F_i) &= \\
&= \int_{\square^{d_n}} \dots \int_{\square^{d_2}} \dots \int_{\square^{d_1}} \dots \int \sum_{j=1}^n k_{y,j} u_{y,j}(\bar{\mathbf{y}}_j) \partial^{d_1} F_{y,1}^{(i)}(\bar{\mathbf{y}}_1) \partial^{d_2} F_{y,2}^{(i)}(\bar{\mathbf{y}}_2) \dots \partial^{d_n} F_{y,n}^{(i)}(\bar{\mathbf{y}}_n) = \\
&= \sum_{j=1}^n k_{y,j} \int_{\square^{d_j}} \dots \int u_{y,j}(\bar{\mathbf{y}}_j) \partial^{d_j} F_{y,j}^{(i)}(\bar{\mathbf{y}}_j) = \sum_{j=1}^n k_{y,j} E_{y,j}^{(i)}(u | F_{y,j}^{(i)}).
\end{aligned}$$

In this formula, the integral under the multiplication symbol is a d_j -dimensional Stieltjes integral. In this way, the expected utility of the classical risky GL-I with classical VICDF in the case of MAVI preferences takes an additive form, and its addends depend on the expected utilities of the classical risky BF-GL-I.

4. Multi-dimensional fuzzy rational GL-I

4.1. General case of multi-dimensional fuzzy rational GL-I

As demonstrated in Section 2, partially measured uncertainty can only be interpreted in terms of ribbon distributions. For that reason the models of alternatives from Section 3 do not apply, as they envisage the presence of classical distributions. Therefore, alternatives where uncertainty is only partially measured require a modified representation.

A multi-dimensional GL-I with a multi-dimensional ribbon CDF shall be called *multi-dimensional fuzzy rational GL-I* and it takes the form

$$(31) \quad g_i^{\text{fr}} = \langle F_i^R(\bar{\mathbf{x}}); \bar{\mathbf{x}} \rangle, i=1, 2, \dots, q.$$

Here $F_i^R(\cdot)$ is a multi-dimensional ribbon CDF with lower and upper bounds $F_i^d(\cdot)$ and $F_i^u(\cdot)$.

The multi-dimensional fuzzy rational GL-I may be ranked in two stages:

1) using a Q criterion under strict uncertainty, each multi-dimensional ribbon distribution function $F_i^R(\cdot)$ is approximated by a multi-dimensional classical CDF $F_i^Q(\cdot)$, which obeys the following condition for all $\bar{\mathbf{x}} \in \square^d$:

$$(32) \quad F_i^d(\bar{\mathbf{x}}) \leq F_i^Q(\bar{\mathbf{x}}) \leq F_i^u(\bar{\mathbf{x}}).$$

In that way, the multi-dimensional fuzzy rational GL-I are approximated by multi-dimensional classical risky GL-I, called *Q-generalized (multi-dimensional Q-GL-I)*,

$$(33) \quad g_i^Q = \langle F_i^Q(\bar{\mathbf{x}}); \bar{\mathbf{x}} \rangle;$$

2) the alternatives are ranked in descending order of the expected utilities of the multi-dimensional Q-GL-I:

$$(34) \quad E_i^Q(u | F_i^R) = \iint_{\square^d} \dots \int u(\bar{\mathbf{x}}) \partial^d F_i^Q(\bar{\mathbf{x}});$$

this formula uses a multi-dimensional Stieltjes integral with an integrating function $F_i^Q(\cdot)$.

The resulting criterion to rank multi-dimensional fuzzy rational GL-I shall be called *Q-expected utility*.

4.2. Special cases of fuzzy rational GL-I

4.2.1. Mutual non-additive utility independence in fuzzy rational GL-I with vector independent distributions

Assume the DM has grouped the attributes X_1, X_2, \dots, X_d in the largest possible number of base vector attributes Y_1, Y_2, \dots, Y_n , according to (1) and (2). A special case of a multi-dimensional fuzzy rational GL-I is when the uncertainty associated with the prizes is described by a ribbon VICDF, and the preferences of the DM over

the base vector attributes Y_1, Y_2, \dots, Y_n are MUVI. Let $F_i^R(\cdot)$ be a ribbon VICDF with lower and upper bounds $F_i^d(\cdot)$ and $F_i^u(\cdot)$ that are classical VICDF, as follows:

$$(35) \quad F_i^R(\bar{\mathbf{x}}) = F_{y,1}^{R,(i)}(\bar{\mathbf{y}}_1)F_{y,2}^{R,(i)}(\bar{\mathbf{y}}_2)\dots F_{y,n}^{R,(i)}(\bar{\mathbf{y}}_n),$$

$$(36) \quad F_i^d(\bar{\mathbf{x}}) = F_{y,1}^{d,(i)}(\bar{\mathbf{y}}_1)F_{y,2}^{d,(i)}(\bar{\mathbf{y}}_2)\dots F_{y,n}^{d,(i)}(\bar{\mathbf{y}}_n),$$

$$(37) \quad F_i^u(\bar{\mathbf{x}}) = F_{y,1}^{u,(i)}(\bar{\mathbf{y}}_1)F_{y,2}^{u,(i)}(\bar{\mathbf{y}}_2)\dots F_{y,n}^{u,(i)}(\bar{\mathbf{y}}_n).$$

Here, $F_{y,j}^{R,(i)}(\cdot)$ is a marginal d_j -dimensional ribbon CDF of Y_j such that for all $\bar{\mathbf{y}}_j \in \square^{d_j}$

$$(38) \quad F_{y,j}^{d,(i)}(\bar{\mathbf{y}}_j) \leq F_{y,j}^{R,(i)}(\bar{\mathbf{y}}_j) \leq F_{y,j}^{u,(i)}(\bar{\mathbf{y}}_j), \quad j=1, 2, \dots, n.$$

Each fuzzy rational GL-I with ribbon VICDF in the case of MUVI preferences may be decomposed to n number of fictitious d_j -dimensional fuzzy rational GL-I:

$$(39) \quad g_{y,j}^{fr,(i)} = \langle F_{y,j}^{R,(i)}(\bar{\mathbf{y}}_j); \bar{\mathbf{y}}_j \rangle, \quad j=1, 2, \dots, n.$$

A multi-dimensional fuzzy rational GL-I that gives the j -th base vector attribute according to a d_j -dimensional ribbon CDF shall be called *base fictitious (fuzzy rational BF-GL-I)*.

Calculating the Q -expected utility of the fuzzy rational GL-I with a ribbon VICDF in the case of MUVI preferences may be simplified to the following three steps:

1) using a Q criterion under strict uncertainty each marginal d_j -dimensional ribbon CDF $F_{y,j}^{R,(i)}(\cdot)$ is approximated by a marginal d_j -dimensional classical CDF

$F_{y,j}^{Q,(i)}(\cdot)$ that obeys the following condition for all $\bar{\mathbf{y}}_j \in \square^{d_j}$,

$$(40) \quad F_{y,j}^{d,(i)}(\bar{\mathbf{y}}_j) \leq F_{y,j}^{Q,(i)}(\bar{\mathbf{y}}_j) \leq F_{y,j}^{u,(i)}(\bar{\mathbf{y}}_j);$$

in that way, any hypothetical d_j -dimensional fuzzy rational BF-GL-I is approximated by a d_j -dimensional Q-GL-I, called *base fictitious (d_j -dimensional BF-Q-GL-I)*,

$$(41) \quad g_{y,j}^{Q,(i)} = \langle F_{y,j}^{Q,(i)}(\bar{\mathbf{y}}_j); \bar{\mathbf{y}}_j \rangle;$$

2) the Q -expected utility of the d_j -dimensional fuzzy rational BF-GL-I can be calculated as the expected utility of the respective d_j -dimensional BF-Q-GL-I,

$$(42) \quad E_{y,j}^{Q,(i)}(u | F_{y,j}^{R,(i)}) = \int_{\square^{d_j}} \dots \int u_{y,j}(\bar{\mathbf{y}}_j) \partial^{d_j} F_{y,j}^{Q,(i)}(\bar{\mathbf{y}}_j);$$

the integral in the formula is a d_j -dimensional Stieltjes integral with an integrating function $F_{y,j}^{Q,(i)}(\cdot)$;

3) the Q -expected utility of the fuzzy rational GL-I with a ribbon VICDF in the case of MUVI preferences takes a multiplicative form, and its multipliers depend on the Q -expected utilities of the fuzzy rational BF-GL-I,

$$(43) \quad E_i^Q(u | F_i^R) = \frac{1}{K_y} \prod_{j=1}^n [K_y k_{y,j} E_{y,j}^{Q,(i)}(u | F_{y,j}^{R,(i)}) + 1] - \frac{1}{K_y}.$$

4.2.2. Mutual additive independence in fuzzy rational GL-I with vector independent distributions

A special case of a multi-dimensional fuzzy rational GL-I is when the uncertainty associated with the prizes is described by a ribbon VICDF, and the preferences of the DM for the base vector attributes Y_1, Y_2, \dots, Y_n are MAVI.

Calculating the Q -expected utility of a fuzzy rational GL-I with a ribbon VICDF in the case of MAVI preferences differs from the case with MUVI preferences, described by formulae (40)-(43), only in the third step, which transforms into the following:

3') the Q -expected utility of the fuzzy rational GL-I with ribbon VICDF in the case of MAVI preferences takes an additive form, and its addends depend on the Q -expected utilities of the fuzzy rational BF-GL-I,

$$(44) \quad E_i^Q(u | F_i^R) = \sum_{j=1}^n k_{y,j} E_{y,j}^{Q,(i)}(u | F_{y,j}^{R,(i)}).$$

5. Conclusions

The paper focused on procedures to rank multi-dimensional GL-I, where prizes were described as multi-dimensional vectors of attributes. The models strongly depended on the type of distributions that measure the uncertainty associated with the prizes, and on the type of independence between the base vector attributes.

A key contribution of the paper is that it embodied the concept of p-boxes into multi-dimensional cumulative probability functions that apply to problems with continuous consequences and describe states of nature whose uncertainty is only partially measured. Formulae for ribbon vector argument and ribbon vector independent CDF have been derived. On that ground, fuzzy rational alternatives were modeled as fuzzy rational GL-I in the general case and in the special cases of mutual non-additive utility independence with vector independent distributions and of additive utility independence with vector independent distributions. Algorithms that calculate the Q -expected utility using Stieltjes integrals were elaborated, and so the multi-dimensional fuzzy rational GL-I were ranked in descending order of their expected utility.

References

1. Abdellaoui, M., C. Barrios, P. Wakker. Reconciling Introspective Utility with Revealed Preference: Experimental Arguments Based on Prospect Theory. – *Journal of Econometrics*, **138**, 2007, 356-378.
2. Fabrycky, W. J., G. J. Thuesen, D. Verma. *Economic Decision Analysis*. Third Edition. Prentice Hall, 1998.
3. Ferson, S., J. G. Hajagos. Varying Correlation Coefficients can Underestimate Uncertainty in Probabilistic Models. – *Reliability Engineering and Systems Safety*, **9**, 2006, 1461-1467.

4. French, S. Decision Theory: an Introduction to the Mathematics of Rationality. Ellis Horwood, 1993.
5. Hackett, G., P. Luffrum. Business Decision Analysis – An Active Learning Approach. Blackwell, 1999.
6. Keeney, R. L., H. Raiffa. Decisions with Multiple Objectives: Preference and Value Tradeoffs. Cambridge University Press, 1993, 282-344.
7. McCord, M., R. De Neufville. Lottery Equivalents’: Reduction of the Certainty Effect Problem in Utility Assessment. – Management Science, **32**, 1986, 56-60.
8. Nikolova, N. Three Criteria to Rank x -Fuzzy-Rational Generalized Lotteries of I Type. – Cybernetics and Information Technologies, Vol. **7**, 2007, No 1, 3-20.
9. Nikolova, N. D., A. Shulus, D. Toneva, K. Tenekedjiev. Fuzzy Rationality in Quantitative Decision Analysis. – Journal of Advanced Computational Intelligence and Intelligent Informatics, Vol. **9**, 2005, No 1, 65-69.
10. Nikolova, N., D. Toneva, K. Tenekedjiev. Definition and Ranking of Fuzzy Rational Semi-Generalized Lotteries of I Type. – Cybernetics and Information Technologies, Vol. **7**, 2007, No 3, 25-40.
11. Rapoport, A. Decision Theory and Decision Behavior – Normative and Descriptive Approaches. Kluwer Academic Publishers, 1989.
12. Strock, D. W. A Concise Introduction to the Theory of Integration. Birkhauser. Third Edition. 1998.
13. Tenekedjiev, K. Decision Problems and Their Place Among Operational Research. – Automatica and Informatics. Year **XXXVIII**, 2004, No 1, 6-9.
14. Tenekedjiev, K. Triple Bisection Algorithms for 1-D Utility Elicitation in the Case of Monotonic Preferences. – Advanced Studies in Contemporary Mathematics, Vol. **14**, 2007, No 2, 259-281.
15. Tenekedjiev, K., N. D. Nikolova. Decision Theory – Subjectivity, Reality and Fuzzy Rationality. CIELA, Bulgaria, 2007.
16. Tenekedjiev, K., N. D. Nikolova. Ranking Discrete Outcome Alternatives with Partially Quantified Uncertainty. – International Journal of General Systems, Vol. **37**, 2008, No 2, 249-274.
17. Tenekedjiev, K., N. D. Nikolova, C. Kobashikawa, K. Hirota. Fuzzy-Rational Betting on Sport Games with Interval Probabilities, Studies in Fuzziness and Soft Computing. – In: IEEE Post-Conference Volume, Springer-Verlag, 2008 (in print).
18. Tenekedjiev, K., N. D. Nikolova, D. Toneva. Laplace Expected Utility Criterion for Ranking Fuzzy Rational Generalized Lotteries of I Type. – Cybernetics and Information Technologies, Vol. **6**, 2006, No 3, 93-109.
19. Utkin, L.V. Risk Analysis under Partial Prior Information and Non-Monotone Utility Functions. – International Journal of Information Technology and Decision Making, Vol. **6**, 2007, No 4, 625-647.
20. Von Neumann, J., O. Morgenstern. Theory of Games and Economic Behaviour. Second Edition. Princeton University Press, 1947.
21. Yager, R. R. Generalizing Variance to Allow the Inclusion of Decision Attitude in Decision Making under Uncertainty. – International Journal of Approximate Reasoning, **42**, 2006, 137-158.

Appendix 1. Multi-dimensional classical CDF

Assume the uncertainty, associated with a d -dimensional system of random variables (X_1, X_2, \dots, X_d) is entirely measured by a known d -dimensional CDF, called *classical CDF (multi-dimensional classical CDF)*. If $F(\cdot)$ is a d -dimensional classical CDF of (X_1, X_2, \dots, X_d) , whereas $\vec{x}=(x_1, x_2, \dots, x_d)$ is a d -dimensional vector with arbitrary fixed values, then for all $\vec{x} \in \square^d$

$$(A1.1) \quad F(\bar{\mathbf{x}}) = F(x_1, x_2, \dots, x_d) = P(X_1 \leq x_1 \cap X_2 \leq x_2 \cap \dots \cap X_d \leq x_d).$$

The multi-dimensional classical CDF must be increasing on each of its arguments and bounded in the interval [0; 1]:

$$(A1.2) \quad \text{if } x_{j,1} > x_{j,2}, \text{ then}$$

$$F(x_1, x_2, \dots, x_{j-1}, x_{j,1}, x_{j+1}, \dots, x_d) \geq F(x_1, x_2, \dots, x_{j-1}, x_{j,2}, x_{j+1}, \dots, x_d),$$

$$(A1.3) \quad \lim_{x_j \rightarrow -\infty} F(x_1, x_2, \dots, x_j, \dots, x_d) = 0, \quad j=1, 2, \dots, d,$$

$$(A1.4) \quad \lim_{\substack{x_1 \rightarrow +\infty \\ x_2 \rightarrow +\infty \\ \vdots \\ x_d \rightarrow +\infty}} F(x_1, x_2, \dots, x_d) = 1.$$

It is sometimes convenient to group the attributes X_1, X_2, \dots, X_d in several base vector attributes Y_1, Y_2, \dots, Y_n , according to (1) and (2). Let's define $n-1$ auxiliary events:

$$(A1.5) \quad A_j = X_1^{Y_j} \leq x_1^{y_j} \cap X_2^{Y_j} \leq x_2^{y_j} \cap \dots \cap X_{d_j}^{Y_j} \leq x_{d_j}^{y_j}, \quad j=1, 2, \dots, n-1,$$

The multi-dimensional classical CDF can always be presented as a product of d_j -dimensional CDF over the base vector attributes:

$$(A1.6) \quad F(\bar{\mathbf{x}}) = F_{y,1}(\bar{\mathbf{y}}_1) F_{y,2}(\bar{\mathbf{y}}_2 | A_1) \dots F_{y,n}(\bar{\mathbf{y}}_n | A_1 \cap A_2 \cap \dots \cap A_{n-1}).$$

Here, $F_{y,1}(\cdot)$ is a marginal d_1 -dimensional classical CDF of Y_1 , whereas $F_{y,j}(\cdot | A_1 \cap A_2 \cap \dots \cap A_{j-1})$ is a conditional d_j -dimensional classical CDF of Y_j , if the event $A_1 \cap A_2 \cap \dots \cap A_{j-1}$ has occurred.

If X is a one-dimensional random variable, then the uncertainty associated with it is measured in terms of a *one-dimensional classical CDF* (Tenekedjiev et al. [18]).

A special case of a multi-dimensional classical CDF is when the base vector attributes Y_1, Y_2, \dots, Y_n are probabilistically independent. It follows that

$$(A1.7) \quad F_{y,j}(\bar{\mathbf{y}}_j | A_1 \cap A_2 \cap \dots \cap A_{j-1}) = F_{y,j}(\bar{\mathbf{y}}_j), \quad j=2, 3, \dots, n.$$

$$(A1.8) \quad F(\bar{\mathbf{x}}) = F_{y,1}(\bar{\mathbf{y}}_1) F_{y,2}(\bar{\mathbf{y}}_2) \dots F_{y,n}(\bar{\mathbf{y}}_n).$$

Such a multi-dimensional classical CDF shall be called *vector independent (classical VICDF)*.

Appendix 2. Types of preferences over attributes of multi-dimensional prizes

Assume the DM has grouped the attributes X_1, X_2, \dots, X_d in the largest possible number of mutually utility independent base vector attributes Y_1, Y_2, \dots, Y_n according to (1) and (2). Mutual utility independence implies the following:

1) there exist and it is possible to construct n number of d_j -dimensional bounded utility functions $u_{y,j}(\cdot)$, defined over all possible values $\bar{\mathbf{y}}_j$ of Y_j :

$$(A2.1) \quad u_{y,j} = u_{y,j}(\bar{\mathbf{y}}_j) = u(x_1^{y_j}, x_2^{y_j}, \dots, x_{d_j}^{y_j}), \quad \bar{\mathbf{y}}_j \in \square^{d_j},$$

$$(A2.2) \quad u_{y,j}(\bar{\mathbf{y}}_{j,best}) = 1, \quad j=1, 2, \dots, n,$$

$$(A2.3) \quad u_{y,j}(\bar{\mathbf{y}}_{j,worst}) = 0, \quad j=1, 2, \dots, n;$$

2) there exist and it is possible to identify n number of scaling constants $k_{y,j} \in [0; 1]$ that indicate the relative significance of the base vector attributes for the preferences of the DM over the multi-dimensional prizes;

3) if $k_{y,1} + k_{y,2} + \dots + k_{y,n} = 1$, then the multi-dimensional utility function takes an additive form:

$$(A2.4) \quad u = u(\bar{\mathbf{x}}) = u(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \dots, \bar{\mathbf{y}}_n) = \sum_{j=1}^n k_{y,j} u_{y,j}(\bar{\mathbf{y}}_j);$$

the multi-dimensional utility function is additive if and only if Y_1, Y_2, \dots, Y_n are mutually additively independent according to the DM. Preferences that are mutually additively independent regarding the base vector attributes shall be called *MAVI preferences*;

4) if $k_{y,1} + k_{y,2} + \dots + k_{y,n} \neq 1$, then the multi-dimensional utility function takes a multiplicative form

$$(A2.5) \quad u = u(\bar{\mathbf{x}}) = u(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2, \dots, \bar{\mathbf{y}}_n) = \frac{1}{K_y} \prod_{j=1}^n [K_y k_{y,j} u_{y,j}(\bar{\mathbf{y}}_j) + 1] - \frac{1}{K_y};$$

here, K_y is a general constant that depends on $k_{y,j}$, finding K_y requires to introduce the one-dimensional real function $\varphi(\cdot)$ of the independent real variable η defined over the real numbers greater than -1 :

$$\text{Domain: } \eta \in (-1; \infty),$$

$$(A2.6) \quad \varphi(\eta) = 1 + \eta - \prod_{j=1}^n (\eta k_{y,j} + 1).$$

The value of K_y is the zero of $\varphi(\cdot)$, which is a polynomial of η of the n -th power:

$$(A2.7) \quad \begin{aligned} &\text{if } \sum_{j=1}^n k_{y,j} < 1, \text{ then } K_y \text{ is the only zero of } \varphi(\cdot) \text{ in the interval } (0; \infty), \\ &\text{if } \sum_{j=1}^n k_{y,j} > 1, \text{ then } K_y \text{ is the only zero of } \varphi(\cdot) \text{ in the interval } (-1; 0). \end{aligned}$$

The multi-dimensional utility function is multiplicative if and only if Y_1, Y_2, \dots, Y_n are mutually utility independent, but are not mutually additively independent according to the DM. Preferences that are mutually utility independent but are not mutually additively independent regarding the base vector attributes shall be called *MUVI preferences*.

The utility function, constructed according to (A2.4) or (A2.5) is normalized so that to equal to either 1 or 0 respectively for the most preferred $\bar{\mathbf{x}}_{best}$ and the least preferred value $\bar{\mathbf{x}}_{worst}$ of the d -dimensional consequence in X :

$$(A2.8) \quad u(\bar{\mathbf{x}}_{best}) = u(\bar{\mathbf{y}}_{1,best}, \bar{\mathbf{y}}_{2,best}, \dots, \bar{\mathbf{y}}_{n,best}) = 1,$$

$$(A2.9) \quad u(\bar{\mathbf{x}}_{worst}) = u(\bar{\mathbf{y}}_{1,worst}, \bar{\mathbf{y}}_{2,worst}, \dots, \bar{\mathbf{y}}_{n,worst}) = 0.$$