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New Modifications of an Intuitionistic Fuzzy Implication from Kleene-Dienes Type

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Abstract: *New intuitonistic fuzzy implications are constructed. Their relations with Modus Ponens and intuitionistic logic axioms are studied.*

Keywords: *Intuitonistic fuzzy implications, Modus Ponens, Intuitionistic Fuzzy Tautology.*

1. Introduction

The concept of "intuitionistic fuzzy propositional calculus" was introduced about 20 years ago (see, e.g., [1, 2]). Initially, it contained only one form of conjunction, disjunction and two forms of implication. In a series of papers a lot of new implications are defined in the frame of the intuitionistic fuzzy logic – see, e.g., [1, 3-8].

Here, we shall introduce new intuitionistic fuzzy implications and will study their basic properties.

In intuitionistic fuzzy propositional calculus, if x is a variable then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x.

Below we shall assume that for the three variables x, y and z the equalities:

$$V(x) = \langle a, b \rangle,$$
$$V(y) = \langle c, d \rangle,$$
$$V(z) = \langle e, f \rangle$$

($a, b, c, d, e, f, a+b, c+d, e+f \in [0, 1]$) hold.

For the needs of the discussion below we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see [1]) by:
x is an IFT if and only if (iff) for $V(x) = <a, b>$ holds: $a \geq b$,
while

$$x \text{ is a tautology iff } a = 1 \text{ and } b = 0,$$

i.e., as in the case of ordinary logic, x is a tautology, if $V(x) = <1, 0>$.

For two variables x and y the operations "conjunction" (&) and "disjunction" ($\vee$) are defined (see [1]) by:

$$V(x \,\&\, y) = <\min(a, c), \max(b, d)>,$$
$$V(x \vee y) = < \max(a, c), \min(b, d)>.$$

In [7, 8] the implications $\rightarrow_{100}$, $\rightarrow_{101}$ and $\rightarrow_{102}$ are introduced by

$$V(x \rightarrow_{100} y) = < \max(b.\mathrm{sg}(a), c), \min(a.\mathrm{sg}(b), d)>,$$
$$V(x \rightarrow_{101} y) = <\max(b.\mathrm{sg}(a), c.\mathrm{sg}(d)), \min(a.\mathrm{sg}(b), d.\mathrm{sg}(c))>,$$
$$V(x \rightarrow_{102} y) = <\max(b, c.\mathrm{sg}(d)), \min(a, d.\mathrm{sg}(c))>,$$

and it is shown that their definitions are correct. There, it is discussed that these implications are modifications of Kleene-Dienes's implication (see [4, 5]). For the new implications are valid (see [7, 8]):

Theorem 1. Implication $\rightarrow_{100}$

(a) satisfies Modus Ponens in the case of tautology,

(b) does not satisfy Modus Ponens in the IFT-case.

Theorem 2. Implications $\rightarrow_{101}$ and $\rightarrow_{102}$

(a) do not satisfy Modus Ponens in the case of tautology,

(b) do not satisfy Modus Ponens in the IFT-case.

On the basis of each above operation "implication" an operation "negation" can be constructed, as follows:

$$\neg <a, b> = <a, b> \rightarrow <0, 1>.$$

The new implications are

$$V(\neg' x) = < b.\mathrm{sg}(a), a.\mathrm{sg}(b)>,$$
$$V(\neg'' x) = <b.\mathrm{sg}(a), 0>,$$
$$V(\neg''' x) = <b, 0>.$$

For these three implication and their negations in [7,8] are checked the following three properties:

Property P1: $A \rightarrow \neg\neg A$,

Property P2: $\neg\neg A \rightarrow A$,

Property P3: $\neg\neg\neg A = \neg A$.

Theorem 3 [7, 8]. (a) The new implications and negations satisfy Property 1 in its IFT-form.

(b) The new implications and negations satisfy Property 2 in its IFT-form.

(c) The first new negation satisfies Property 3.

(d) The second new negation does not satisfy Property 3.

Following [10] we introduce the list of axioms for propositional intuitionistic logic:

(a) $A \rightarrow A$,

(b) $A \rightarrow (B \rightarrow A)$,

(c) $A \to (B \to (A \& B))$,
(d) $(A \to (B \to C)) \to (B \to (A \to C))$,
(e) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$,
(f) $A \to \neg \neg A$,
(g) $\neg (A \& \neg A)$,
(h) $(\neg A \vee B) \to (A \to B)$,
(i) $\neg (A \vee B) \to (\neg A \quad \neg B)$,
(j) $(\neg A \backslash\& \neg B \to \neg (A \vee B)$,
(k) $(\neg A \vee \neg B \to \neg (A \& B)$,
(l) $(A \to B) \to (\neg B \to \neg A)$,
(m) $(A \to \neg B) \to (B \to \neg A)$,
(n) $\neg \neg \neg A \to \neg A$,
(o) $\neg A \to \neg \neg \neg A$,
(p) $\neg \neg (A \to B) \to (A \to \neg \neg B)$,
(q) $(C \to A) \to ((C \to (A \to B)) \to (C \to B))$.

Theorem 4 [7, 8]. All axioms of propositional intuitionistic logic are IFTs for $\to_{100}$, $\to_{101}$, and $\to_{102}$.

2. Main results

In [9] an implication $\to^*$ is introduced on the base of another, already defined implication $\to$ by

(*) $$x \to^* y = \Box x \to \Diamond y.$$

Here, using (*) and definitions of intuitionistic fuzzy modal operators (see, e.g., [2])

$$V(\Box x) = \langle a, 1-a \rangle,$$
$$V(\Diamond x) = \langle 1-b, b \rangle,$$

we shall construct three new implications and will study their properties that are analogous of the above ones.

2.1. The new implications have the forms:

$$V(x \to_{103} y) = V(\Box x \to_{100} \Diamond y) = \langle \max((1-a).\mathrm{sg}(a), 1-d), \min(a.\mathrm{sg}(1-a), d) \rangle,$$
$$V(x \to_{104} y) = V(\Box x \to_{101} \Diamond y) = \langle \max((1-a).\mathrm{sg}(a), (1-d).\mathrm{sg}(d)), \min(a.\mathrm{sg}(1-a), d.\mathrm{sg}(1-d)) \rangle,$$
$$V(x \to_{105} y) = V(\Box x \to_{102} \Diamond y) = \langle \max(1-a, (1-d).\mathrm{sg}(d)), \min(a, d.\mathrm{sg}(1-d)) \rangle.$$

Therefore, the three implications $\to_{100}$, $\to_{101}$, and $\to_{102}$ generate three different new implications ($\to_{103}$, $\to_{104}$ and $\to_{105}$). Similarly to [7, 8] we can check that the new three implications do not satisfy Modus Ponens in the case of tautologies, as well as in the case of IFTs.

2.2. The new implications will generate three negations with the forms:

$$V(\neg^{iv} x) = \langle (1-a).\mathrm{sg}(a), a.\mathrm{sg}(1-a) \rangle,$$

$V(\neg^{v} x) = \langle (1-a).\mathrm{sg}(a), 0\rangle,$

$V(\neg_{vi}\, x) = \langle 1-a, 0\rangle.$

Theorem 5. (a) The new implications and negations satisfy Property 1 in its IFT-form.

(b) The new implications and negations satisfy Property 2 in its IFT-form.

(c) The new negations satisfy Property 3.

P r o o f**.** We shall prove (a) for implication $\to_{103}$ and negation $\neg^{iv}$.

$V(A \to_{103} \neg^{iv} \neg^{iv} A)$

$= \langle a,b\rangle \to_{103} \neg^{iv} \neg^{iv} \langle a,b\rangle$

$= \langle a,b\rangle \to_{103} \neg^{iv} \langle (1-a).\mathrm{sg}(a), a.\mathrm{sg}(1-a)\rangle$

$= \langle a,b\rangle \to_{103} \langle (1-(1-a).\mathrm{sg}(a)).(1-a).\mathrm{sg}(a), (1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a))\rangle$

$= \langle \max((1-a).\mathrm{sg}(a), 1-(1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a))),$

$\min(a.\mathrm{sg}(1-a), d(1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a)))\rangle.$

Now, we see that

$\max((1-a).\mathrm{sg}(a), 1-(1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a))) - \min(a.\mathrm{sg}(1-a), (1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a))) \geq (1-a).\mathrm{sg}(a) - (1-a).\mathrm{sg}(1-a).\mathrm{sg}((1-a).\mathrm{sg}(a)) \geq 0.$

Therefore, $A \to_{103} \neg^{iv} \neg^{iv} A$ is an IFT.

The rest assertions in this Theorem and the next Theorem are proved similarly.

2.3. Finally, we formulate

Theorem 6. (a) The axioms of propositional intuitionistic logic without (e), (l), (p) and (q) are IFTs for $\to_{103}$.

(b) The axioms of propositional intuitionistic logic without (e) and (q) are IFTs for $\to_{104}$ and $\to_{105}$.

3. Conclusion

The present results are a part of the research devoted to defining and studying of new implications and negations over IFSs that have non-classical behaviour. The new implications and negations illustrate again the intuitionistic character of the IFSs. These implications will be object of a next research in near future.

The research over the implications and their properties is connected strongly to the development of new tools for decision making. For example, on the basis of the research in the last section and from Theorems 1 and 2 we see that implication $\to_{101}$ is the most suitable one for use in standard decision making procedures.

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