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On a Minimum Ratio Cancelling Algorithm of the Problem of a Minimal Network Flow with One Side Constraint

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Abstract: *An approach is proposed for searching an ε-optimal solution of the problem of a minimal network flow with one side constraint. The paper suggests to change the flow iterativel, cancelling the structure, consisting of two cycles, which have a minimal ratio.*

Keywords: *Flow with side constraints, bicriteria network flow, one side constraint, min ratio canceling.*

1. Introduction

Usually in the ordinary minimum cost flow problem the purpose is to find such a flow on the arcs of the network, which satisfies the capacity constraints on those arcs and which is the cheapest one. In the paper presented a flow is investigated, for which a capacity constraint is added to the constraint set of the minimum cost flow problem. This constraint is a linear combination of the arc flows, constrained by a rational number. Such a flow is named here "a flow with one side constraint (Flosc)".

The problem for minimal Flosc may arise in practical and theoretical generalizations of the minimum cost flow problem. It arises also in the scalarization problem of the bicriteria network flow problem. [1, 9].

The classical optimal flow problems are phenomena in the field of linear programming in view of the fact, that their unimodular structure of the constraint matrix ensures the existence of strongly polynomial combinatorial algorithms for finding the optimal solution. And moreover, for these kinds of problems the optimal flow is integer for integer costs and capacities of arcs. It turns out that even the smallest “violations” of the unimodular property of the constraint matrix make these algorithms inappropriate for searching the optimal solution.

Having in mind the fact that such problems, like the optimal generalized flow, the submodular flow and the flow with side constraints are linear programming problems, they can be solved, using adapted versions of linear programming algorithms. But usually these algorithms are not polynomial and do not ensure the integrality of the optimal solution.

Another interesting approach for solving this kind of problems is offered by Goldberg and Tarjan [2] – a minimum mean cycle cancelling algorithm, developed for the minimum cost flow problem. This idea is generalized by Wallacher [3] by means of cancelling the minimum ratio cycles. The ratio of the cycle is determined by the cost of the cycle, divided by the sum of the arcs' weights. An arc weight is represented as a reciprocal of its capacity. This approach is extended later and applied to the generalized minimum cost flow problem [4], to the minimum submodular flow problem [5] and even to the linear programming problem [6].

The purpose of this paper is to apply the minimum ratio cancelling technique in receiving a ε-optimal solution of Flosc.

The problem is formulated in Section 2. In Section 3 we propose an iterative algorithm, which reduces the gap between the next two feasible solutions x and y by a value, which is M times smaller than the gap between x and the optimal solution of the problem. M is a function of the input data of the problem.

2. Formulation of the problem

Let $G = \{N, U, c, b, u\}$ be a network, which has: a set of nodes $N = \{i, j, s, t, p, q, \ldots\}$, $|N| = n$; a set of arcs U, $U = N \times N$, $|U| = m$; a cost function c: $U \to R^+$, $c = \{c_{ij}, (i, j) \in U\}$; a constraint function b: $U \to R$, $b = \{b_{ij}, (i, j) \in U\}$; a capacity function u: $U \to R^+$, $u = \{u_{ij}, (i, j) \in U\}$. The flow function can be denoted by x, y, z; x: $U \to R^+$, $x = \{x_{ij}, (i, j) \in U\}$.

We add an arc (t, s) with $c_{ts} = b_{ts} = 0$ to the arc set U and formulate the problem for minimum Flosc like a MC problem for minimum circulation with one side constraint.

MC: $$\min c(x) = \sum_{(i,j)\in U} c_{ij} x_{ij}$$

subject to

(1) $$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = 0, \quad i \in N,$$

(2) $$b(x) = \sum_{(i,j)\in U} b_{ij} x_{ij} = b;$$

(3) $$0 \le x_{ij} \le u_{ij}, \quad (i,j) \in U.$$

We would like to develop an approximation algorithm in order to find an ε-optimal solution of Flosc, i.e. such an algorithm that makes at any iteration the gap between the current flow and the optimal flow near to zero. If the current flow and the optimal flow are denoted by x' and x^* respectively, then $(1 + \varepsilon)c(x') = c(x^*)$.

The optimal solution of the problem being investigated may be received by a specialization of the simplex algorithm, which exploits the embedded network (unimodular) structure in the constraint matrix. In [7] it is proved that the basic solution of this problem has a structure that corresponds to a spanning tree with one additional arc in the graph $\{N, U\}$of the network G. In [8] three different methods (primal, dual and Lagrangean) are described and their computational efficiency is compared. These methods handle the side constraint within the framework of a network code. But none of them is polynomial.

The combinatorial flow programming methods are interior point methods. They "work" with objects like paths, cycles, circulations. They could help in understanding the nature of the problem, and hence, the search of efficient algorithms for an optimal rational or integer flow.

3. Min ratio cancelling

Let y be a circulation in the network G. We denote by $G(y) = \{N, U_y, c(y), b(y), u(y)\}$ the residual network for y. This network is constructed in the following way:

– for each arc $(i, j) \in U$ an arc $(i, j) \in U(y)$ exists if $y_{ij} < u_{ij}$ and $c_{ij}(y) = c_{ij}$, $b_{ij}(y) = b_{ij}$,

$$u_{ij}(y) = u_{ij} - y_{ij};$$

– for each arc $(i, j) \in U$ an arc $(i, j) \in U(y)$ exists if $y_{ij} > 0$ and $c_{ji}(y) = -c_{ij}$, $b_{ji}(y) = -b_{ij}$, $u_{ji}(y) = x_{ij}$.

A circulation x in the residual network $G(y)$ is feasible, if it satisfies the conditions:

(4) $$\sum_{(i,j)\in U_y} b_{ij}(y)x_{ij} = 0;$$

(5) $$0 \le x_{ij} \le u_{ij}(y), \ (i, j) \in U_y.$$

Note that if x is a feasible circulation in the residual network $G(y)$, then the circulation $z = x + y$, where $z_{ij} = y_{ij} + x_{ij} - x_{ji}$, $(i, j) \in U$, is a feasible circulation in the network G.

Lemma 1. The circulation y is optimal if and only if there is not a feasible circulation in the network $G(y)$.

Proof: If there is another circulation z with a lower cost, then the flow z–y is a feasible one in the residual network and vise versa.

Let σ_i be a cycle and x – a feasible flow in $G(y)$. We denote by $x_i(\sigma)$ the flow on the cycle σ_i.

By $c_i(\sigma)$ and by $b_i(\sigma)$ the values $\sum_{(i,j)\in\sigma_i} c_{ij}$ and $\sum_{(i,j)\in\sigma_i} b_{ij}$ are denoted respectively.

We call $c_i(\sigma)$ a cost of the cycle and by $b_i(\sigma)$ – a b-cost of the cycle.

Because x is a circulation, we can decompose it into k, $k \le m$, cycles of a flow and thus obtain:

$$x = \sum_{i\in I_k} x_i(\sigma); \quad c(x) = \sum_{i\in I_k} c_i(\sigma)x_i(\sigma); \quad b(\sigma) = \sum_{i\in I_k} b_i(\sigma)x_i(\sigma).$$

Lemma 2. Let x be a feasible circulation in $G(y)$. Then x can be represented like a linear combination with positive coefficients of the flows on the cycles with positive b-costs.

Proof. For each cycle σ_i we denote its b-cost by $\overset{+}{b_i}(\sigma)$ if it is positive and by $\overset{-}{b_i}(\sigma)$, if it is negative. The positive b-cost cycles are numerated from 1 to k_1 and the negatives are numerated with the rest upto k.

Since x is feasible, it is obtained:

$$\sum_{i\in I_{k_1}} b_i^+(\sigma)x_i(\sigma) + \sum_{i\in I_{k_2}} b_i^-(\sigma)x_i(\sigma) = 0;$$
$$I_{k_1} = \{1, 2, \ldots, k_1\}, I_{k_2} = \{k_1, k_1+1, \ldots, k\}, k_1 + k_2 = k. \tag{6}$$

We apply an iterative procedure where at each iteration one $x_j(\sigma)$, $j\in I_{k_2}$, is determined as a linear combination of the positive terms of (6):

$$x_j(\sigma) = (-1/\overset{-}{b_i}(\sigma)) \sum_{i\in I_{k_1}} a_{ij} b_i^+(\sigma)x_i(\sigma). \tag{7}$$

Then

$$\begin{aligned} x &= \sum_{i\in I_k} x_i(\sigma) = \sum_{i\in I_{k_1}} x_i(\sigma) + \sum_{j\in I_{k_2}} x_j(\sigma) = \\ &= \sum_{i\in I_{k_1}} x_i(\sigma) + \sum_{j\in I_{k_2}} [1/(-b_j^-(\sigma))x_i(\sigma)] \sum_{i\in I_{k_1}} a_{ji} b_i^+(\sigma)x_i(\sigma) = \\ &= \sum_{i\in I_{k_1}} (1 + b_i^+(\sigma)) \sum_{j\in I_{k_2}} a_{ji}/(-b_j^-(\sigma))x_i(\sigma). \end{aligned} \tag{8}$$

If we define a Flosc for the residual network and substitute $x_j(\sigma)$, $j\in I_{k_2}$, by (8), we get a MCC problem:

MCC: $$\min \sum_{i\in I_{k_1}} (c_i(\sigma) - b_i^+(\sigma)) \sum_{j\in I_{k_2}} a_{ji} c_j(\sigma)/b_j^-(\sigma) x_i(\sigma)$$

subject to

$$\sum_{i\in I_{k_1}} b_i^+(\sigma)x_i(\sigma) \le B^{\max}.$$

We denote:

$$B^{\max} = \max \sum_{(i,j)\in U, b_{ij}>0} b_{ij}x_{ij} \text{ s.t. (1) and (3);}$$

$$B^{\min} = \min \sum_{(i,j)\in U, b_{ij}>0} b_{ij}x_{ij} \text{ s.t. (1) and (3).}$$

Following the already made notes, we will propose an algorithm MCCR with canceling a defined min ratio in the search for an approximate solution of the minimum Flosc problem, which is very close to an optimal point.

Let σ_i be one cycle with a positive b-cost, $\overset{+}{b_i}(\sigma)$ and σ_j be another with a negative b-cost $\overset{-}{b_i}(\sigma)$ in $G(y)$. We name “two-cycle” the set of arcs of these two cycles and denote it by σ_{ij}.

We define the cost $c_{ij}(\sigma)$ of the two-cycle as follows:

$$c_{ij}(\sigma) = (c_j(\sigma)\,\overset{-}{b_j}(\sigma) - c_j(\sigma)\overset{+}{b_i}(\sigma))/\overset{-}{b_j}(\sigma).$$

We define a MCRC problem:

MCRC: $$\min \sum_{i\in I_{k_1},\, j\in I_{k_2}} c_{ij}(\sigma)x_i(\sigma)$$

subject to

$$\sum_{i\in I_{k_1}} b_i^+(\sigma)x_i(\sigma) \le B^{\max}.$$

We denote:

$u_i(\sigma) = \min u_{pq}$, $(p, q) \in \sigma_i$, $u_j(\sigma) = \min u_{pq}$, $(p, q)\in (p, q) \in\sigma_j$.

The flow on the two-cycle σ_{ij} is denoted by $x_{ij}(\sigma)$ and defined as follows:

$x_{ij}(\sigma) = x_i(\sigma)$ on the arcs of σ_i, $x_i(\sigma) \le u_i(\sigma)$

$x_{ij}(\sigma) = x_j(\sigma)$ on the arcs of σ_j and $x_j(\sigma) \le u_j(\sigma)$, $x_j(\sigma)= \overset{+}{b_i}(\sigma)/(-\overset{-}{b_j}(\sigma))x_i(\sigma)$.

It is clear that the flow on a circulation may be represented as a sum of flows on the two-cycles of the network, because from (7) we can determine k_2k_1 two-cycle σ_{ij} with a flow $x_i(\sigma)$ and $x_j(\sigma) = (-1/\overset{-}{b_j}(\sigma))\,\overset{+}{b_i}(\sigma)\,a_{ji}\,x_i(\sigma)$.

Then it is obtained that $\overset{+}{b_i}(\sigma)x_i(\sigma) + \overset{-}{b_j}(\sigma)x_j(\sigma) = 0$, i.e. the flow $x_{ij}(\sigma)$ is a feasible flow in the residual network, it satisfies the capacity and the side constraint. In case at least one arc is saturated, we say that the two-cycle is cancelled.

A two-cycle is called augmenting if there is a nonzero flow z on it in $G(y)$. If one two-cycle is augmenting and has a negative cost, then the cost of the flow in $G(y)$ will be decreased by the value $c_{ij}(\sigma)z_i(\sigma)$.

We call the value $c_{ij}(\sigma)/\overset{+}{b_i}(\sigma)$ a two-cycle ratio.

The algorithm (MTCRA), described below, cancels the minimum two-cycle ratio in the residual network. In this way the gap between the current solution and the optimal solution will be decreased to zero.

The algorithm MTCRA:

Step 1. Let y be an initial feasible solution of the problem MC.

Step 2. Define the residual network $G(y)$. If there are not any negative augmenting two cycles, then y is an optimal flow, stop.

Step 3. Find a minimum two-cycle σ_{ij} ratio in $G(y)$. Determine the flow $x_{ij}(\sigma)$.

Step 4. The new flow y is $y := y + x_i(\sigma) + x_j(\sigma)$. Go to Step 2.

Theorem 1. Let x and z be the solutions received at iterations i_1 and i_1+1 of the algorithm MTCRA and x^{opt} – an optimal solution of the problem Flosc. Then

$$c(z-x) \le c(x^{\text{opt}} - x)/(B^{\max}/B^{\min}).$$

P r o o f. Since x and z are neighborhood solutions, $z-x$ is a flow on the two-cycle σ_{ij} and the ratio $c_{ij}(\sigma)/\overset{+}{b_i}(\sigma)$ is minimal in $G(x)$. On the other hand, the circulation $x^{\text{opt}} - x$ is feasible in $G(x)$ and it can be represented as a linear combination of the two-cycles:

$$x^{\text{opt}} - x = \sum_{r\in I_1,\, l\in I_2} x_{rl}(\sigma),\ I_1 \text{ and } I_2 \text{ are subsets of } I_{k_1} \text{ and } I_{k_2}.$$

Let σ_{pq} be a negative one, with the minimum ratio in this representation. Then:

$$c_{ij}(\sigma)/\overset{+}{b_i}(\sigma) \le c_{pq}(\sigma)/\overset{+}{b_p}(\sigma);$$

$$c_{ij}(\sigma)x_i(\sigma)/\overset{+}{b_i}(\sigma)x_i(\sigma) \le c_{pq}(\sigma)x_p(\sigma)/\overset{+}{b_p}(\sigma)x_p(\sigma) \le \sum_{r\in I_1,\, l\in I_2} c_{rl}(\sigma)x_r(\sigma) \,/\, \sum b_{rl}^{+}(\sigma)x_r(\sigma) =$$

$$= c(x^{\text{opt}} - x)/\sum_{r\in I_1,\, l\in I_2} b_{rl}^{+}(\sigma)x_r(\sigma),$$

which leads to

$$c(z-x)/B^{\min} = c_{ij}(\sigma)x_i(\sigma)/B^{\min} \le c_{ij}(\sigma)x_i(\sigma)/\overset{+}{b_i}(\sigma)x_i(\sigma) \le c(x^{\text{opt}} - x)/\sum_{r\in I_1,\, l\in I_2} b_{rl}^{+}(\sigma)x_r(\sigma) \le$$

$$\le c(x^{\text{opt}} - x)/B^{\max}.$$

4. Conclusion

The minimal two-cycle ratio can be determined as a sum of the min ratio $c_i(\sigma)/\overset{+}{b_i}(\sigma)$ of the cycles with positive b-costs and the min ratio $c_i(\sigma)/\overset{-}{b_i}(\sigma)$of the cycles with negative b-costs.

The statement of the theorem says that the amount of the objectives decrease at each iteration is always at least a $B^{\min}/B^{\max}$ fraction of the remaining to the optimal objective value amount.

In the paper presented the author has not investigated how to get an exact minimal solution of Flosc from the solution, received by MTCRA.

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