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Three Criteria to Rank *x*-Fuzzy-Rational Generalized Lotteries of I Type

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Summary: *The paper analyzes the problem of ranking fuzzy-rational generalized lotteries of I type, where uncertainty is partially quantified by x-ribbon distribution functions. The latter are built using interval quantile estimates of fuzzy-rational decision makers, whose preferences are partially non-transitive. The Wald, Hurwicz and maximax criteria under strict uncertainty are applied to approximate x-ribbon distribution functions into classical ones. In that way, x-fuzzy rational generalized lotteries of I type are approximated by classical ones and are ranked according to the proposed Wald, Hurwicz*$_\alpha$ *and maximax expected utility criteria.*

Keywords: *fuzzy rationality, interval estimates, generalized lotteries of I type, x-ribbon distribution functions, strict uncertainty, expected utility.*

1. Introduction

Problems under uncertainty are solved using utility theory (v o n N e u m a n n, M o r g e n s t e r n [23]). It assumes that alternatives should be modeled as lotteries, each being a full disjoint set of events associated with a prize. If the set of prizes and lotteries are discrete, then ordinary lotteries are defined. If sets are continuous, then generalized lotteries of I, II or III type apply (T e n e k e d j i e v [17]). In the case of generalized lotteries of I type (further denoted GL-I) prizes are actually a random variable, described probabilistically by the cumulative distribution function (CDF). A utility function $u(.)$ should be constructed over the set of prizes, which increases with the increase of preferences of the decision maker (DM) (K e e n e y, R a i f f a [7]). The values of $u(.)$, weighed by their probabilities define the expected utility of a lottery. The higher this index the more preferred the lottery.

If a problem under strict uncertainty arises, the DM must define which of the states are possible. The choice of an alternative can then be made using the Wald (W a l d [24]), Hurwicz$_\alpha$ (R a p o p o r t [14]), Savage (S a v a g e [16]), Laplace

(R a p o p o r t [14]), or the maximax criteria (G r o e b n e r, et al. [7]). The Wald criterion in particular reflects extreme pessimism and assumes that alternatives should be ranked in descending order of their worst outcome. The maximax criterion, on the other hand, reflects extreme optimism and assumes that alternatives should be ranked in descending order of their best prizes. The Hurwicz$_\alpha$ criterion uses the pessimism index $\alpha\in[0; 1]$, that measures the pessimism of the DM. Then alternatives are ranked in descending order of the weighted value of the worst and best prizes by the index α. As discussed in (F r e n c h [6]), none of the criteria under strict uncertainty obeys the minimal requirements of rational choice.

In the condition of risk, the DM must define unique probability estimates (B e r n s t e i n [4]) and then alternatives are modeled as classical-risky lotteries. In N i k o l o v a et al. [13], real DMs are called fuzzy-rational, since their subjective estimates of probabilities and utilities are in an interval form. Then fuzzy-rational lotteries are defined, which cannot be directly ranked according to expected utility (T e n e k e d j i e v et al. [20]). It is necessary to transform interval probabilities into point estimates of certain quality.

The problem of ranking alternatives with partially quantified uncertainty has been discussed in many works (T r o f f a e s [21]; A u g u s t i n [3]). Combination of techniques under risk and under strict uncertainty is often proposed (U t k i n, A u g u s t i n [22]). The problem of constructing probability distributions using probability confidence intervals is also discussed (K o z i n e, U t k i n [9]). In N i k o l o v a [12] and T e n e k e d j i e v [19], the Laplace and Hurwicz$_\alpha$ expected utility criteria are proposed to rank ordinary fuzzy-rational lotteries. Those criteria are based on the homology between probabilities and degrees of membership. In T e n e k e d j i e v et al. [20] the Wald expected utility criterion is proposed to rank generalized lotteries of I type. The interval probabilities are transformed into point estimates using three intuitionistic operators that transform intuitionistic fuzzy degrees of membership into classical fuzzy ones (A t a n a s s o v [1]; A t a n a s s o v [2]).

The work T e n e k e d j i e v et al. [20] introduces one-dimensional (1D) ribbon distribution functions, originating from the interval estimates of quantiles/quantile indices of fuzzy-rational DMs. These functions are either x-ribbon or p-ribbon depending on the type of uncertainty (on the quantile or on the quantile index). Fuzzy-rational GL-I are defined on that basis. The ribbon distribution functions are approximated by classical ones using the Laplace criterion under strict uncertainty. Then the fuzzy-rational GL-I are approximated by classical GL-I and are ranked according to the Laplace expected utility criterion.

This paper proposes algorithms that use the Wald, Hurwicz$_\alpha$ and maximax criteria under strict uncertainty to approximate one-dimensional x-ribbon distribution functions by classical ones. In that way, fuzzy-rational GL-I are approximated by classical GL-I and the Wald, Hurwicz$_\alpha$ and maximax expected utility criteria are defined to rank those.

2. One-dimensional CDFs and the imperfection of elicitation

The linear interpolation of classical CDF function by elicited knots is presented in the Appendix. Constructing CDF may be performed, for example, if several inner quantile indices F_2, F_3, ..., F_{z-1} are selected, randomly distributed in the interval [0; 1] and their corresponding quantiles $\hat{x}_2$, $\hat{x}_3$, ..., $\hat{x}_{z-1}$ are assessed (T e n e k e d j i e v et al.

[20]). For each knot, the DM should solve the preferential equation $l_1(X \le \hat{x}_l) \sim l_2(m, n)$, where $l_1(X \le \hat{x}_l)$ is a lottery that gives a huge prize if X takes values lower or equal to $\hat{x}_l$, and $l_2(m,n)$ is a lottery, giving the same prize if a white ball is drawn from a urn of n balls, of which m are white. For the ideal DM, the preferential equation $l_1(X \le \hat{x}_l) \sim l_2(m, n)$ has an unique solution $\hat{x}_l = x^*$, and $m/n = F_l$ (~ is the binary relation „equally preferred to" or „indifference"). For the real DM, there exist $x_b > x_a$, such that $l_1(X \le x_a) \sim l_2(m, n)$, $l_1(X \le x_b) \sim l_2(m, n)$, $l_1(X \le x_b) \succ l_1(X \le x_a)$ ($\succ$ is the binary relation "more preferred to", or "strict preference"). That is why it is necessary to find the highest $x = \hat{x}^{\text{down}}$, for which $l_2(m, n) \succ l_1(X \le \hat{x}^{\text{down}})$, and the smallest $x = \hat{x}^{\text{up}}$, for which $l_1(X \le \hat{x}^{\text{up}}) \succ l_2(m, n)$. Then the uncertainty interval of the root is $x^* \in (\hat{x}^{\text{down}}; \hat{x}^{\text{up}})$ and the belief of the DM regarding the quantiles takes the form

$$\hat{x}_l \in [\hat{x}_l^d; \hat{x}_l^u], \quad l = 2, 3, \ldots, z-1. \tag{1}$$

The extreme quantile indices $F_1=0$ and $F_z=1$ correspond to quantiles $\hat{x}_1 = \hat{x}_1^d = \hat{x}_1^u$ and $\hat{x}_z = \hat{x}_z^d = \hat{x}_z^u$. It is obvious then that the real DM's preference do not obey transitivity of ~ and the mutual transitivity of $\succ$ and ~, which are key assumptions for rationality (S a v a g e [15]; D e G r o o t [5]; T e n e k e d j i e v [18]). That is why real DMs are called fuzzy-rational (N i k o l o v a et al. [13]). A 1D CDF, linearly interpolated on knots with quantile uncertainty intervals, is presented in the Appendix. The former is referred hereafter as 1D x-ribbon CDF (T e n e k e d j i e v et al. [20]).

3. Ranking fuzzy-rational GL-I

Some alternatives with fully quantified uncertainty can be modeled by 1D classical-risky GL-I, which are presented in the Appendix. However, when the uncertainty is partially quantified with 1D x-ribbon CDF – $F_i^{xR}(.)$ then the resulting 1D GL-I takes the form (2) and shall be referred as x-fuzzy-rational

$$g_i^{xfr} = < F_i^{xR}(x); x >, \quad i = 1, 2, \ldots, q. \tag{2}$$

In T e n e k e d j i e v et al. [20], ranking 1D x-fuzzy-rational GL-I is performed in two stages:

1) Using a criterion under strict uncertainty Q, $F_i^{xR}(.)$ is piece-wise linearly interpolated by a 1D classical-risky CDF – $F_i^{xQ}(.)$, with knots:

$$\{(x_l^{Q,(i)}; F_l^{(i)}) \mid l = 1, 2, \ldots, z_i\}, \tag{3}$$

where

$$\begin{gathered} x_1^{Q,(i)} \le x_2^{Q,(i)} \le \ldots \le x_{z_i}^{Q,(i)}, \\ x_l^{d,(i)} \le x_l^{Q,(i)} \le x_l^{u,(i)}, \quad l = 2, 3, \ldots, z_i - 1, \\ x_1^{Q,(i)} = x_1^{d,(i)} = x_1^{u,(i)} \text{ and } x_{z_i}^{Q,(i)} = x_{z_i}^{d,(i)} = x_{z_i}^{u,(i)}. \end{gathered} \tag{4}$$

Then

$$F_i^{xQ}(x) = \begin{cases} 0 & \text{at } x < x_1^{Q,(i)}, \\ F_l^{(i)} & \text{at } x_l^{Q,(i)} = x < x_{l+1}^{Q,(i)}, \ l = 1, 2, \ldots, z_i - 1, \\ F_l^{(i)} + \dfrac{(x - x_l^{Q,(i)})(F_{l+1}^{(i)} - F_l^{(i)})}{x_{l+1}^{Q,(i)} - x_l^{Q,(i)}} & \text{at } x_l^{Q,(i)} < x < x_{l+1}^{Q,(i)}, \ l = 1, 2, \ldots, z_i - 1, \\ 1 & \text{at } x_{z_i}^{Q,(i)} \le x. \end{cases} \tag{5}$$

The 1D x-fuzzy rational GL-I (2) is approximated by a 1D classical-risky GL-I, referred as xQ-generalized (1D xQ-GL-I):

(6) $$g_i^{xQ}=<F_i^{xQ}(x);\ x>,\quad i=1, 2, \ldots, q;$$

2) The Q-expected utility of the 1D x-fuzzy-rational GL-I (2) is calculated as the expected utility of the 1D xQ-GL-I (6), according to (A15):

(7) $$E_i^{xQ}(u|F_i^{xR})=\int_{x_1^{Q,(i)}}^{x_{z_i}^{Q,(i)}} u(x)\mathrm{d}F_i^{xQ}(x)=$$

$$=\sum_{\substack{l=1\\ x_{l+1}^{Q,(i)}>x_l^{Q,(i)}}}^{z_i-1} \frac{F_{l+1}^{(i)}-F_l^{(i)}}{x_{l+1}^{Q,(i)}-x_l^{Q,(i)}} \int_{x_l^{Q,(i)}}^{x_{l+1}^{Q,(i)}} u(x)\mathrm{d}x+\sum_{\substack{l=1\\ x_{l+1}^{Q,(i)}>x_l^{Q,(i)}}}^{z_i-1} (F_{l+1}^{(i)}-F_l^{(i)})u(x_l^{Q,(i)}).$$

The resulting Q-expected utility of the 1D x-fuzzy-rational GL-I shall be referred as xQ-expected utility. Some decision criteria under strict uncertainty essentially use the one-dimensional utility function $u(.)$ in the approximation of $F_i^{xR}(.)$ by F_i^{xQ} (.).

4. Approximation of 1D x-ribbon CDF

The following task has to be solved in order to approximate a 1D x-ribbon CDF.

a) Setup:

– criterion under strict uncertainty: Q;

– one-dimensional utility function: $u(.)$;

– number of knots for approximation: $z_i>1$;

– quantile indices: $F_l^{(i)}$, for l=1, 2,…, z_i, such that

(8) $$0=F_1^{(i)}\leq F_2^{(i)}\leq\ldots\leq F_{z_i-1}^{(i)}\leq F_{z_i}^{(i)}=1;$$

– lower limits of the quantiles: $x_l^{d,(i)}$, for l=1, 2,…, z_i , such that

(9) $$x_1^{d,(i)}\leq x_2^{d,(i)}\leq\ldots\leq x_{z_i-1}^{d,(i)}\leq x_{z_i}^{d,(i)};$$

– upper limits of the quantiles: $x_l^{u,(i)}$, for l=1, 2,…, z_i , such that

(10) $$x_1^{d,(i)}=x_1^{u,(i)}\leq x_2^{u,(i)}\leq\ldots\leq x_{z_i-1}^{u,(i)}\leq x_{z_i}^{u,(i)}=x_{z_i}^{d,(i)},$$
$$x_l^{d,(i)}\leq x_l^{u,(i)},\ \text{for } l=2, 3,\ldots, z_i-1;$$

– end quantiles

(11) $$x_1^{Q,(i)}=x_1^{d,(i)}=x_1^{u,(i)},$$
$$x_{z_i}^{Q,(i)}=x_{z_i}^{d,(i)}=x_{z_i}^{u,(i)}.$$

b) Find:

– Inner quantiles $x_l^{Q,(i)}$, for l=2, 3,…, z_i–1, such that

(12) $$x_1^{Q,(i)}\leq x_2^{Q,(i)}\leq\ldots\leq x_{z_i-1}^{Q,(i)}\leq x_{z_i}^{Q,(i)},$$
$$x_l^{d,(i)}\leq x_l^{Q,(i)}\leq x_l^{u,(i)}.$$

4.1. Approximation of 1D *x*-ribbon CDF using the Wald criterion (Q=W)

4.1.1. General case

The Wald decision criterion under strict uncertainty assumes that the worst outcome always occurs. The application of this idea in the case of 1D *x*-fuzzy-rational GL-I would assume to choose the quantiles $x_l^{W,(i)}$, l=2, 3, ..., z_i–1, so that to minimize the *xW*-expected utility of the lottery:

$$E_i^{xW}(u\,|\,F_i^{xR}) = \int_{x_1^{W,(i)}}^{x_{z_i}^{W,(i)}} u(x)dF_i^{xW}(x) = \sum_{l=1}^{z_i-1}\int_{x_l^{W,(i)}}^{x_{l+1}^{W,(i)}} u(x)dF_i^{xW}(x) =$$

$$(13)\qquad = \sum_{\substack{l=1\\ x_{l+1}^{W,(i)}>x_l^{W,(i)}}}^{z_i-1} \frac{F_{l+1}^{(i)}-F_l^{(i)}}{x_{l+1}^{W,(i)}-x_l^{W,(i)}}\int_{x_l^{W,(i)}}^{x_{l+1}^{W,(i)}} u(x)dx + \sum_{\substack{l=1\\ x_{l+1}^{W,(i)}>x_l^{W,(i)}}}^{z_i-1}(F_{l+1}^{(i)}-F_l^{(i)})u(x_l^{W,(i)}) =$$

$$= \sum_{l=1}^{z_i-1}(F_{l+1}^{(i)}-F_l^{(i)})I_l^{xW,(i)},$$

where

$$(14)\qquad I_l^{xW,(i)} = \begin{cases} \dfrac{1}{x_{l+1}^{W,(i)}-x_l^{W,(i)}}\displaystyle\int_{x_l^{W,(i)}}^{x_{l+1}^{W,(i)}} u(x)dx & \text{at } x_{l+1}^{W,(i)} > x_l^{W,(i)} \\ u(x_l^{W,(i)}) & \text{at } x_{l+1}^{W,(i)} = x_l^{W,(i)} \end{cases},\quad l=1, 2, 3, \dots, z_i-1.$$

The variables $I_l^{xW,(i)}$ physically coincide with the expected utilities of hypothetical 1D classical-risky GL-I – $g_l^{h,xW,(i)}$=<$F_l^{h,xW,(i)}(x)$; x>, where the 1D classical CDF – $F_l^{h,xW,(i)}(.)$, are linearly interpolated on two knots ($x_l^{W,(i)}$; 0) and ($x_{l+1}^{W,(i)}$; 1). The 1D classical CDF – $F_l^{h,xW,(i)}(.)$, the hypothetical 1D classical-risky GL-I – $g_l^{h,xW,(i)}$ and their expected utilities $I_l^{xW,(i)}$ are unknown until the quantiles $x_l^{W,(i)}$, l=2, 3, ..., z_i–1, are defined, which obey the conditions

$$(15)\qquad \begin{aligned} &x_l^{d,(i)} - x_l^{W,(i)} \le 0,\ l=2, 3, ..., z_i-1,\\ &x_l^{W,(i)} - x_l^{u,(i)} \le 0,\ l=2, 3, ..., z_i-1,\\ &x_l^{W,(i)} - x_{l+1}^{W,(i)} \le 0,\ l=2, 3, ..., z_i-2,\\ &x_1^{d,(i)} - x_2^{W,(i)} \le 0,\\ &x_{z_i-1}^{W,(i)} - x_{z_i}^{d,(i)} \le 0. \end{aligned}$$

The so-defined (z_i–2)-dimensional non-linear optimization task with $3z_i$–5 linear constraints can be redefined in a task of lower dimension, using the following properties:

a) since the weight coefficients ($F_{l+1}^{(i)}$–$F_l^{(i)}$) of the variables $I_l^{xW,(i)}$ in the function $E_i^{xW}(u|F_i^{xR})$ are known and nonnegative, then the required quantile estimates should be found so that to minimize the quantities $I_l^{xW,(i)}$;

b) let all quantiles, but the l-th, be fixed at a certain level, where $l\in\{2, 3, ..., z_i-1\}$, and assume that for the l-th quantile

(16) $$x_l^{W,(i)} \in [\max\{x_l^{d,(i)},\ x_{l-1}^{W,(i)}\};\ \min\{x_l^{u,(i)},\ x_{l+1}^{W,(i)}\}],$$

then the change in $x_l^{W,(i)}$ only influences $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$;

c) let for some $l\in\{2, 3, \ldots, z_i-1\}$ the utility $u(.)$ be
– monotonically increasing in the interval $x\in[x_l^{d,(i)};\ x_l^{u,(i)}]$,
– limited from above by $u(x_l^{d,(i)})$ in the interval $x\in[x_{l-1}^{d,(i)};\ x_l^{d,(i)}]$,
– limited from below by $u(x_l^{u,(i)})$ in the interval $x\in[x_l^{u,(i)};\ x_{l+1}^{u,(i)}]$,

then $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically increasing functions of $x_l^{W,(i)}$;

d) let for some $l\in\{2, 3, \ldots, z_i-1\}$, the utility function $u(.)$ be
– monotonically decreasing in the interval $x\in[x_l^{d,(i)};\ x_l^{u,(i)}]$,
– limited from below by $u(x_l^{d,(i)})$ in the interval $x\in[x_{l-1}^{d,(i)};\ x_l^{d,(i)}]$,
– limited from above by $u(x_l^{u,(i)})$ in the interval $x\in[x_l^{u,(i)};\ x_{l+1}^{u,(i)}]$,

then $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically decreasing functions of $x_l^{W,(i)}$;

e) let for some $l\in\{2, 3, \ldots, z_i-1\}$ the utility function $u(.)$ be a constant in the interval $x\in[x_{l-1}^{d,(i)};\ x_{l+1}^{u,(i)}]$. Then $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ do not depend on changes of $x_l^{W,(i)}$.

When reducing the dimensionality of the optimization task, it is convenient to assign the quantiles $x_l^{W,(i)}$, $l=1, 2, \ldots, z_i$, to 5 disjoint sets: "known", "arbitrary", "left prone", "right prone" and "optimizing", according to Algorithm 1.

Algorithm 1

Step 1. All quantiles are marked as "optimizing".

Step 2. From left to right ($l=2, 3, \ldots, z_i-1$) all "optimizing" quantiles, whose lower and upper limits coincide (i.e. $x_l^{d,(i)}=x_l^{u,(i)}$), are marked as "known". The following assignments are made: $x_l^{W,(i)}= x_l^{d,(i)}$.

Step 3. From left to right ($l=2, 3, \ldots, z_i-1$) all "optimizing" quantiles that obey "property e" (i.e. $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ do not depend on the change of $x_l^{W,(i)}$), are marked as "arbitrary".

Step 4. From left to right ($l=2, 3, \ldots, z_i-1$) all "optimizing" quantiles that obey "property c" (i.e. $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically increasing functions on $x_l^{W,(i)}$), are marked as "left prone".

Step 5. From right to left ($l=z_i-1, z_i-2, \ldots, 3, 2$), all "optimizing" quantiles that obey "property d" (i.e. $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically decreasing functions of $x_l^{W,(i)}$), are marked as "right prone".

Step 6. From left to right ($l=2, 3, \ldots, z_i-1$) all "arbitrary" quantiles, whose left neighbor is "known", "left prone" or "optimizing", is marked as "left prone".

Step 7. From left to right ($l=2, 3, \ldots, z_i-1$) all "arbitrary" quantiles, whose right neighbor is "known", "right prone" or "optimizing", are marked as "right prone".

Step 8. From left to right ($l=2, 3, \ldots, z_i-1$) all "arbitrary" quantiles, whose left neighbor ll and right neighbor lr do not overlap (i.e. $x_{ll}^{u,(i)}\le x_{lr}^{d,(i)}$), are marked as "left prone", and both neighbors are marked as "known". The following assignments are made: $x_{ll}^{W,(i)}=x_{ll}^{u,(i)}$, $x_{ll}^{d,(i)}=x_{ll}^{u,(i)}$, $x_t^{d,(i)}=\max\{x_t^{d,(i)}; x_{ll}^{W,(i)}\}$, $t=ll+1, ll+2, \ldots, z_i-1$, $x_{lr}^{W,(i)}=x_{lr}^{d,(i)}$, $x_{lr}^{u,(i)}=x_{lr}^{d,(i)}$ and $x_t^{u,(i)}=\min\{x_t^{u,(i)}; x_{lr}^{W,(i)}\}$, $t=2, 3, \ldots, lr-1$.

Step 9. From left to right ($l=2, 3, \ldots, z_i-1$) all quantiles, whose lower and upper limit coincide (i.e. $x_l^{d,(i)}=x_l^{u,(i)}$), are marked as "known". The following assignment are made: $x_l^{W,(i)}=x_l^{d,(i)}$.

Step 10. From left to right ($l=2, 3, \ldots, z_i-1$) the first quantiles from the group of "arbitrary" quantiles, whose left neighbor ll and right neighbor lr overlap (i.e. $x_{ll}^{u,(i)}>x_{lr}^{d,(i)}$), are marked as "optimizing", and the other quantiles in the group are marked as "left prone".

Step 11. From left to right ($l=2, 3, \ldots, z_i-1$) all "right prone" quantiles, which do not overlap with their right "left prone" neighbor (i.e. $x_l^{u,(i)}\leq x_{l+1}^{d,(i)}$), are marked as "known", together with their right neighbors, and it is defined that $x_l^{W,(i)}=x_l^{u,(i)}$, $x_l^{d,(i)}=x_l^{u,(i)}$, $x_{l+1}^{W,(i)}=x_{l+1}^{d,(i)}$ and $x_{l+1}^{u,(i)}=x_{l+1}^{d,(i)}$.

Step 12. From left to right ($l=2, 3, \ldots, z_i-1$) all "right prone" quantiles, which do not overlap with their right "left prone" neighbor (i.e. $x_l^{u,(i)}> x_{l+1}^{d,(i)}$), are marked as "optimizing".

Step 13. From left to right ($l=2, 3, \ldots, z_i-1$) all "left prone" quantiles, whose left neighbor is "known" or which do not overlap with their left neighbor (i.e. $x_l^{d,(i)}\geq x_{l-1}^{u,(i)}$), are marked as "known". The following assumptions are made: $x_l^{W,(i)}=x_l^{d,(i)}$ and $x_l^{u,(i)}=x_l^{d,(i)}$.

Step 14. From right to left ($l=z_i-1, z_i-2, \ldots, 3, 2$) all "right prone" quantiles, whose right neighbor is "known", and which do not overlap with their right neighbor (i.e. $x_l^{u,(i)}\leq x_{l+1}^{d,(i)}$), are marked as "known". The following assignment are made: $x_l^{W,(i)}=x_l^{u,(i)}$ and $x_l^{d,(i)}= x_l^{u,(i)}$.

Step 15. If at least one quantile has been marked as "known" in steps 13 and 14, then go to step 13.

Step 16. If at least one "optimizing" quantile has been marked as "arbitrary", "left prone" or "right prone" in Steps 3 to 5, then go to Step 3, otherwise – the end.

After applying Algorithm 1:

- there are no "arbitrary" quantiles;
- if there are no "optimizing" quantiles, then there are only "known" quantiles;
- if there are "optimizing" quantiles, then there are no "right prone" quantiles with right "left prone" neighbors;
- the lower and upper limits of all "known" quantiles coincide with a fixed value;
- all quantile limits obey the initial conditions.

Let N be the cardinality of the set of "optimizing" quantiles. If $N=0$, then all quantiles have been found and the task is solved. If $N>0$, then Algorithm 2 should be applied. It uses only arbitrary permissible (ones which obey the linear constraints) values of the "optimizing" quantiles to calculate the function $E_i^{xW}(u|F_i^{xR})$.

Algorithm 2

Step 1. From left to right ($l=2, 3, \ldots, z_i-1$), all "optimizing" quantiles are set to coincide with the chosen values of the "optimizing" quantiles $x_l^{W,(i)}$, such that $x_l^{d,(i)}\leq x_l^{W,(i)}\leq x_l^{u,(i)}$.

Step 2. From left to right (l=2, 3, …, z_i–1) all "left prone" quantiles are set to coincide with the greater of the lower limit and the left neighbor.

Step 3. From right to left (l=z_i–1, z_i–2, …, 3, 2) all "right prone" quantiles are set to coincide with the smaller of their upper limit and the right neighbor: $x_l^{W,(i)}=\min\{x_l^{u,(i)};\ x_{l+1}^{W,(i)}\}$.

Step 4. The values $I_l^{xW,(i)}$, l=1, 2, 3,…, z_i–1, are calculated using the defined $x_l^{W,(i)}$.

Step 5. $E_i^{xW}(u/F_i^{xR})$ is calculated using the defined $I_l^{xW,(i)}$.

It is again necessary to optimize $E_i^{xW}(u|F_i^{xR})$, but the dimensionality N of this task does not exceed and is usually smaller than z_i–2. From the $3z_i$–5 number of linear constraints only those that include an "optimizing" quantile are analyzed.

Algorithm 2 is realized in a numerical procedure, where a set of 10 values are defined for each "optimizing" quantile, uniformly distributed in its uncertainty interval. Then the permissible out of all possible quantile value combinations are identified, which obey the initial conditions. The final estimates are the permissible combination, which minimizes $I_l^{xW,(i)}$. Once the "optimizing" quantiles have been defined, it is possible to find the values of the "left prone" and "right prone" quantiles according to Steps 2 and 3 of Algorithm 2, and calculate $I_l^{xW,(i)}$ (l=1, 2, 3,…, z_i–1) and $E_i^{xW}(u|F_i^{xR})$.

4.1.2. Special case of monotonically increasing utility function

Let the utility function $u(.)$ be monotonically increasing in the interval $[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$:

(17) if $x_a>x_b$, then $u(x_a)\geq u(x_b)$ for $x_a\in[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$, $x_b\in[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$.

Then all unknown quantiles $x_l^{W,(i)}$, l=2, 3, …, z_i–1, would obey "property c" in the general case (i.e. $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically increasing functions on $x_l^{W,(i)}$). Then, in order to minimize $I_l^{xW,(i)}$, all quantiles will be set to their lower limits, which are the smallest values that obey the linear constraints

(18) $$x_l^{W,(i)}= x_l^{d,(i)},\ \ l=2, 3, \dots, z_i-1.$$

4.1.3. Special case of monotonically decreasing utility function

Let the utility function $u(.)$ be monotonically decreasing in the interval $[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$:

(19) if $x_a > x_b$, then $u(x_a)\leq u(x_b)$, for $x_a\in[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$, $x_b\in[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$.

Then all unknown quantiles $x_l^{W,(i)}$, l=2, 3, …, z_i–1, obey "property d" in the general case (i.e. $I_{l-1}^{xW,(i)}$ and $I_l^{xW,(i)}$ are monotonically decreasing functions of $x_l^{W,(i)}$). Then, in order to minimize $I_l^{xW,(i)}$, the quantiles should be set to their upper limits, which are the highest possible values that obey the linear constraints

(20) $$x_l^{W,(i)} = x_l^{u,(i)},\ \text{for } l=2, 3, \dots, z_i-1.$$

4.2. Approximation of x-ribbon CDF using the maximax criterion ($Q=\neg W$)

4.2.1. General case

The maximax criterion is the opposite of the Wald criterion and the quantile values might be found using the algorithms from Section 4.1 with the substitution

(21) $$u(x)=-u(x), \text{ for } x\in(-\infty;+).$$

The maximax decision criterion under strict uncertainty assumes that the best outcome always occurs. The application of this idea in the case of 1D x-fuzzy-rational GL-I would assume to choose the quantiles $x_l^{\neg W,(i)}$, l=2, 3, …, z_i–1, so that to maximize the $x\neg W$-expected utility of the lottery:

(22)
$$E_i^{x\neg W}(u\,|\,F_i^{xR})=\int_{x_1^{\neg W,(i)}}^{x_{z_i}^{\neg W,(i)}}u(x)dF_i^{x\neg W}(x)=\sum_{l=1}^{z_i-1}\int_{x_l^{\neg W,(i)}}^{x_{l+1}^{\neg W,(i)}}u(x)dF_i^{x\neg W}(x)=$$
$$=\sum_{\substack{l=1\\ x_{l+1}^{\neg W,(i)}>x_l^{\neg W,(i)}}}^{z_i-1}\frac{F_{l+1}^{(i)}-F_l^{(i)}}{x_{l+1}^{\neg W,(i)}-x_l^{\neg W,(i)}}\int_{x_l^{\neg W,(i)}}^{x_{l+1}^{\neg W,(i)}}u(x)dx+\sum_{\substack{l=1\\ x_{l+1}^{\neg W,(i)}>x_l^{\neg W,(i)}}}^{z_i-1}(F_{l+1}^{(i)}-F_l^{(i)})u(x_l^{\neg W,(i)})=$$
$$=\sum_{l=1}^{z_i-1}(F_{l+1}^{(i)}-F_l^{(i)})I_l^{x\neg W,(i)},$$

where

(23) $$I_l^{x\neg W,(i)}=\begin{cases}\dfrac{1}{x_{l+1}^{\neg W,(i)}-x_l^{\neg W,(i)}}\displaystyle\int_{x_l^{\neg W,(i)}}^{x_{l+1}^{\neg W,(i)}}u(x)dx & \text{at } x_{l+1}^{\neg W,(i)}>x_l^{\neg W,(i)}\\ u(x_l^{\neg W,(i)}) & \text{at } x_{l+1}^{\neg W,(i)}=x_l^{\neg W,(i)}\end{cases}, \quad l=1, 2, 3, \dots, z_i-1.$$

The variables $I_l^{x\neg W,(i)}$ physically coincide with the expected utilities of hypothetical 1D classical-risky GL-I – $g_l^{h,x\neg W,(i)}=\langle F_l^{h,x\neg W,(i)}(x); x\rangle$, where the 1D classical CDF – $F_l^{h,x\neg W,(i)}(.)$ are linearly interpolated on two knots ($x_l^{\neg W,(i)}$; 0) and ($x_{l+1}^{\neg W,(i)}$; 1). The 1D classical CDF – $F_l^{h,x\neg W,(i)}(.)$, the hypothetical 1D classical-risky GL-I – $g_l^{h,x\neg W,(i)}$ and their expected utilities $I_l^{x\neg W,(i)}$ are unknown until the quantiles $x_l^{\neg W,(i)}$, l=2, 3, ..., z_i–1, are defined, which obey the conditions

(24)
$$x_l^{d,(i)}-x_l^{\neg W,(i)}\le 0,\ \ l=2, 3, \dots, z_i-1,$$
$$x_l^{\neg W,(i)}-x_l^{u,(i)}\le 0,\ \ l=2, 3, \dots, z_i-1,$$
$$x_l^{\neg W,(i)}-x_{l+1}^{\neg W,(i)}\le 0,\ \ l=2, 3, \dots, z_i-2,$$
$$x_1^{d,(i)}-x_2^{\neg W,(i)}\le 0,$$
$$x_{z_i-1}^{\neg W,(i)}-x_{z_i}^{d,(i)}\le 0.$$

The so-defined (z_i–2)-dimensional non-linear optimization task with $3z_i$–5 linear constraints can be redefined in a task of lower dimension, using the following properties:

a) since the weight coefficients $(-F_l^{(i)})$ of the variables $I_l^{x\neg W,(i)}$ in the function $E_i^{x\neg W}(u|F_i^{xR})$ are known and nonnegative, then the required quantile estimates should be set so that to maximize the quantities $I_l^{x\neg W,(i)}$;

b) let all quantiles, but the l-th, be fixed at a certain level, where $l\in\{2, 3, ..., z_i-1\}$, and assume that for the l-th quantile

$$x_l^{\neg W,(i)}\in[\max\{x_l^{d,(i)},\ x_{l-1}^{\neg W,(i)}\};\ \min\{x_l^{u(i)},\ x_{l+1}^{\neg W,(i)}\}], \tag{25}$$

then the change in $x_l^{\neg W,(i)}$ only influences $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$;

c) let for some $l\in\{2, 3, \ldots, z_i-1\}$ the utility $u(.)$ be

– monotonically increasing in the interval $x\in[x_l^{d,(i)};\ x_l^{u,(i)}]$,

– limited from above by $u(x_l^{d,(i)})$ in the interval $x\in[x_{l-1}^{d,(i)};\ x_l^{d,(i)}]$,

– limited from below by $u(x_l^{u,(i)})$ in the interval $x\in[x_l^{u,(i)};\ x_{l+1}^{u,(i)}]$,

then $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically increasing functions of $x_l^{\neg W,(i)}$;

d) let for some $l\in\{2, 3, \ldots, z_i-1\}$ the utility function $u(.)$ be

– monotonically decreasing in the interval $x\in[x_l^{d,(i)};\ x_l^{u,(i)}]$,

– limited from below by $u(x_l^{d,(i)})$ in the interval $x\in[x_{l-1}^{d,(i)};\ x_l^{d,(i)}]$,

– limited from above by $u(x_l^{u,(i)})$ in the interval $x\in[x_l^{u,(i)};\ x_{l+1}^{u,(i)}]$,

then $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically decreasing functions of $x_l^{\neg W,(i)}$;

e) let for some $l\in\{2, 3, ..., z_i-1\}$ the utility function $u(.)$ be a constant in the interval $x\in[x_{l-1}^{d,(i)};\ x_{l+1}^{u,(i)}]$; then $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ do not depend on changes of $x_l^{\neg W,(i)}$.

When reducing the dimensionality of the optimization task, it is convenient to assign the quantiles $x_l^{\neg W,(i)}$, $l=1, 2, \ldots, z_i$, to 5 disjoint sets: “known”, “arbitrary”, “left prone”, “right prone” and “optimizing”, according to Algorithm 3.

Algorithm 3

Step 1. All quantiles are marked as “optimizing”.

Step 2. From left to right ($l=2, 3, \ldots, z_i-1$) all “optimizing” quantiles, whose lower and upper limits coincide (i.e. $x_l^{d,(i)}=x_l^{u,(i)}$), are marked as “known”. The following assignments are made: $x_l^{\neg W,(i)}=x_l^{d,(i)}$.

Step 3. From left to right ($l=2, 3, \ldots, z_i-1$) all “optimizing” quantiles that obey “property e” (i.e. $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ do not depend on the change of $x_l^{\neg W,(i)}$), are marked as “arbitrary”.

Step 4. From left to right ($l=2, 3, \ldots, z_i-1$) all “optimizing” quantiles that obey “property c” (i.e. $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically increasing functions on $x_l^{\neg W,(i)}$), are marked as “right prone”.

Step 5. From right to left ($l=z_i-1, z_i-2, \ldots, 3, 2$), all “optimizing” quantiles that obey “property d” (i.e. $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically decreasing functions of $x_l^{\neg W,(i)}$), are marked as “left prone”.

Step 6. From left to right ($l=2, 3, \ldots, z_i-1$) all “arbitrary” quantiles, whose left neighbor is “known”, “left prone” or “optimizing”, is marked as “left prone”.

Step 7. From left to right (l=2, 3, …, z_i–1) all “arbitrary” quantiles, whose right neighbor is “known”, “right prone” or “optimizing”, are marked as “right prone”.

Step 8. From left to right (l=2, 3, …, z_i–1) all “arbitrary” quantiles, whose left neighbor ll and right neighbor lr do not overlap (i.e. $x_{ll}^{u,(i)} \leq x_{lr}^{d,(i)}$), are marked as “left prone”, and both neighbors are marked as “known”. The following assignments are made: $x_{ll}^{\neg W,(i)}=x_{ll}^{u,(i)}$, $x_{ll}^{d,(i)}= x_{ll}^{u,(i)}$, $x_t^{d,(i)}=\max\{x_t^{d,(i)}; x_{ll}^{\neg W,(i)}\}$, for t=ll+1, ll+2, ..., z_i–1, $x_{lr}^{\neg W,(i)}=x_{lr}^{d,(i)}$, $x_{lr}^{u,(i)}=x_{lr}^{d,(i)}$ and $x_t^{u,(i)}=\min\{x_t^{u,(i)}; x_{lr}^{\neg W,(i)}\}$, t=2, 3, …, lr–1.

Step 9. From left to right (l=2, 3, …, z_i–1) all quantiles, whose lower and upper limit coincide (i.e. $x_l^{d,(i)}=x_l^{u,(i)}$), are marked as “known”. The following assignment are made: $x_l^{\neg W,(i)}=x_l^{d,(i)}$.

Step 10. From left to right (l=2, 3, …, z_i–1) the first quantiles from the group of “arbitrary” quantiles, whose left neighbor ll and right neighbor lr overlap (i.e. $x_{ll}^{u,(i)}>x_{lr}^{d,(i)}$), are marked as “optimizing”, and the other quantiles in the group are marked as “left prone”.

Step 11. From left to right (l=2, 3, …, z_i–1) all “right prone” quantiles, which do not overlap with their right “left prone” neighbor (i.e. $x_l^{u,(i)} \leq\ x_{l+1}^{d,(i)}$), are marked as “known”, together with their right neighbors, and it is defined that $x_l^{\neg W,(i)}=x_l^{u,(i)}$, $x_l^{d,(i)}=x_l^{u,(i)}$, $x_{l+1}^{\neg W,(i)} = x_{l+1}^{d,(i)}$ and $x_{l+1}^{u,(i)} = x_{l+1}^{d,(i)}$.

Step 12. From left to right (l=2, 3, …, z_i–1) all “right prone” quantiles, which do not overlap with their right “left prone” neighbor (i.e. $x_l^{u,(i)}> x_{l+1}^{d,(i)}$), are marked as “optimizing”.

Step 13. From left to right (l=2, 3, …, z_i–1) all “left prone” quantiles, whose left neighbor is “known” or which do not overlap with their left neighbor (i.e. $x_l^{d,(i)}\geq x_{l-1}^{u,(i)}$), are marked as “known”. The following assignments are made: $x_l^{\neg W,(i)}=x_l^{d,(i)}$, $x_l^{u,(i)}=x_l^{d,(i)}$.

Step 14. From right to left (l=z_i–1, z_i–2, …, 3, 2) all “right prone” quantiles, whose right neighbor is “known”, and which do not overlap with their right neighbor (i.e. $x_l^{u,(i)}\leq$), are marked as “known”. The following assignments are made: $x_l^{\neg W,(i)}=x_l^{u,(i)}$, $x_l^{d,(i)}=x_l^{u,(i)}$.

Step 15. If at least one quantile has been marked as “known” in steps 13 and 14, then go to Step 13.

Step 16. If at least one “optimizing” quantile has been marked as “arbitrary”, “left prone” or “right prone” in Steps 3 to 5, then go to Step 3, otherwise – the end.

After applying Algorithm 3:

- there are no “arbitrary” quantiles;
- if there are no “optimizing” quantiles, then there are only “known” quantiles;
- if there are “optimizing” quantiles, then there are no “right prone” quantiles with right “left prone” neighbors.
- the lower and upper limits of all “known” quantiles coincide with a fixed value;
- all quantile limits obey the initial conditions.

Let N be the cardinality of the set of “optimizing” quantiles. If N=0, then all quantiles have been found and the task is solved. If N>0, then Algorithm 4 should be applied. It again uses only arbitrary permissible values of the “optimizing” quantiles to calculate $E_i^{x\neg W}(u|F_i^{xR})$.

Algorithm 4

Step 1. From left to right (l=2, 3, …, z_i–1), all "optimizing" quantiles are set to coincide with the chosen values of the "optimizing" quantiles $x_l^{\neg W,(i)}$, such that $x_l^{d,(i)} \leq x_l^{\neg W,(i)} \leq x_l^{u,(i)}$.

Step 2. From left to right (l=2, 3, …, z_i–1) all "left prone" quantiles are set to coincide with the greater of the lower limit and the left neighbor: $x_l^{\neg W,(i)}=\max\{x_l^{d,(i)};\ x_{l-1}^{\neg W,(i)}\}$.

Step 3. From right to left (l=z_i–1, z_i–2, …, 3, 2) all "right prone" quantiles are set to coincide with the smaller of their upper limit and the right neighbor: $x_l^{\neg W,(i)}=\min\{x_l^{u,(i)};\ x_{l+1}^{\neg W,(i)}\}$.

Step 5. The values $I_l^{x\neg W,(i)}$, for l=1, 2, 3,…, z_i–1, are calculated using the defined $x_l^{\neg W,(i)}$.

Step 6. $E_i^{x\neg W}(u|F_i^{xR})$ is calculated using the defined $I_l^{x\neg W,(i)}$.

It is again necessary to optimize $E_i^{x\neg W}(u|F_i^{xR})$, but the dimensionality N of this task does not exceed and is usually smaller than z_i–2. From the 3z_i–5 number of linear constraints only those that include an "optimizing" quantile are analyzed.

Algorithm 2 is realized in a numerical procedure, similar to the one of Algorithm 2. The only difference is that the final estimate of the quantiles are the permissible combination, which maximizes $I_l^{x\neg W,(i)}$.

4.2.2. Special case of monotonically increasing utility function

Let the utility function u(.) be monotonically increasing in the interval $[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$:

(26) if $x_a > x_b$, then $u(x_a) \geq u(x_b)$, for $x_a \in [x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$, $x_b \in [x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$.

Then all unknown quantiles $x_l^{\neg W,(i)}$, l=2, 3, ..., z_i–1, would obey "property c" in the general case from section 4.2.1 (i.e. $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically increasing functions on $x_l^{\neg W,(i)}$). Then, in order to maximize $I_l^{x\neg W,(i)}$, all quantiles will be set to their upper limits, which are the highest values that obey the linear constraints, i.e.

(27) $$x_l^{\neg W,(i)}=x_1^{u,(i)},\ l=2, 3, ..., z_i–1.$$

4.2.3. Special case of monotonically decreasing utility function

Let the utility function u(.) be monotonically decreasing in the interval $[x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$:

(28) if $x_a > x_b$, then $u(x_a) \leq u(x_b)$, for $x_a \in [x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$, $x_b \in [x_1^{d,(i)};\ x_{z_i}^{u,(i)}]$.

Then all unknown quantiles $x_l^{\neg W,(i)}$, l=2, 3, ..., z_i–1, obey "property d" in the general case from Section 4.2.1 (i.e. $I_{l-1}^{x\neg W,(i)}$ and $I_l^{x\neg W,(i)}$ are monotonically decreasing functions of $x_l^{\neg W,(i)}$). Then, in order to maximize $I_l^{x\neg W,(i)}$, the quantiles will be set to their lower limits, which are the smallest possible values that obey the linear constraints, i.e.

(29) $$x_l^{\neg W,(i)}= x_l^{d,(i)},\ \text{for } l=2, 3, ..., z_i–1.$$

4.3. Approximation of 1D x-ribbon CDF using the Hurwicz$_\alpha$ criterion (Q=H_α)

The Hurwicz$_\alpha$ decision criterion under strict uncertainty assumes that the choice of an alternative must be guided by a numerical index, which is a weighed sum of the worst and best that may occur. Implementing this idea for 1D x-fuzzy-rational GL-I means to choose the quantiles $x_l^{H_\alpha,(i)}$, l=2, 3, …, z_i–1, as weighed values of the quantiles $x_l^{W,(i)}$ and $x_l^{\neg W,(i)}$, l=2, 3, …, z_i–1:

(30) $$x_l^{H_\alpha,(i)}=\alpha x_l^{W,(i)}+(1-\alpha)x_l^{\neg W,(i)},\ l=2, 3, \ldots, z_i-1.$$

Here, $\alpha\in[0; 1]$ is the pessimism index and measure the pessimism of the DM. The H_α-expected utility can be calculated by (7), using the substitution Q=H_α

5. Numerical example

The 1D random variable X is defined, which takes values in the interval [30; 42]. A 1D x-ribbon CDF $F^{xR}(.)$ is defined over the values of X by linear interpolation on the end points of uncertainty intervals (assessed by a fuzzy-rational DM) of z=11 knots: (x_1^d=x_1^u=30; F_1=0), (x_2^d=31; x_2^u=32; F_2=0.1), (x_3^d=32; x_3^u=33; F_3=0.2), (x_4^d=32.8; x_4^u=33.5; F_4=0.3), (x_5^d=33; x_5^u=34.5; F_5=0.4), (x_6^d=34; x_6^u=36; F_6=0.5), (x_7^d=35.5; x_7^u=37; F_7=0.6), (x_8^d=36; x_8^u=37.5; F_8=0.7), (x_9^d=37; x_9^u=40; F_9=0.8), (x_{10}^d=40; x_{10}^u=41; F_{10}=0.9) and (x_{11}^d= x_{11}^u=42; F_{11}=1). The conditions in (A8) apply for the elicited knots. A 1D x-fuzzy-rational GL-I – g^{xfr}=<$F^{xR}(x)$; x >, is defined on that basis.

The utility function is non-monotonic in the interval [30; 42], with a maximum extremum. It was constructed using techniques from N i k o l o v a et al. [11]. The following results were obtained (represented by their point estimates): $u(30)$=0, $u(31)$=0.06, $u(32)$=0.09, $u(33)$=0.15, $u(34)$=0.3, $u(35)$=0.55, $u(36)$=0.7, $u(37)$=0.6, $u(38)$=0.4, $u(39)$=0.2, $u(40)$=0.15, $u(41)$=0.1, $u(42)$=0 (see Figs. 1–3). The task is to approximate $F^{xR}(x)$ using Wald, maximax and Hurwicz$_\alpha$ criteria, and then calculate the Wald, maximax and Hurwicz$_\alpha$ expected utilities of g^{xfr}.

a) Approximation of $F^{xR}(x)$ using the Wald criterion and calculation of the W-expected utility of g^{xfr}

According to (11), x_1^W=30, x_{11}^W=42. After applying Algorithm 1, six quantiles are marked as "known" with values x_2^W=31, x_3^W=32, x_4^W=32.8, x_5^W=33, x_6^W=34 and x_{10}^W=41. The other three quantiles are marked as "optimizing" and their estimates should be found in the intervals $35.5\le x_7^W\le 37$, $36\le x_8^W\le 37.5$ and $37\le x_9^W\le 40$.

Following the comments to Algorithm 2, the permissible combinations of the quantile estimates are identified on the possible values of the "optimizing" quantiles. Those combinations should obey the conditions in (15), and I_l^{xW}, l=1, 2, ..., 10, and $E^{xW}(u|F^{xR})$ are calculated for each.

For example, for the permissible combination x_7^W=36.5, x_8^W=37 and x_9^W=39, according to (14), I_1^{xW}=0.03, I_2^{xW}=0.075, I_3^{xW}=0.114, I_4^{xW}=0.144, I_5^{xW}=0.225, I_6^{xW}=0.525, I_7^{xW}=0.675, I_8^{xW}=0.425, I_9^{xW}=0.15 and I_{10}^{xW}=0.05. According to (13), the expected utility $E^{xW}(u|F^{xR})$=0.2413. The minimal possible value of the expected utility $E^{xW}(u|F^{xR})$=0.2147 is calculated for the following values of the "optimizing" quantiles: x_7^W=35.5, x_8^W=37.5 and x_9^W=40, where I_1^{xW}=0.03, I_2^{xW}=0.075, I_3^{xW}=0.114, I_4^{xW}=0.144, I_5^{xW}=0.225, I_6^{xW}=0.4708, I_7^{xW}=0.6281, I_8^{xW}=0.285, I_9^{xW}=0.125 and I_{10}^{xW}=0.05.

$F^{xR}(.)$ now can be approximated by $F^{xW}(.)$ using the knots (x_1^W=30; F_1=0), (x_2^W=31;

F_2=0.1), (x_3^W=32; F_3=0.2), (x_4^W=32.8; F_4=0.3), (x_5^W=33; F_5=0.4), (x_6^W=34; F_6=0.5), (x_7^W=35.5; F_7=0.6), (x_8^W=37.5; F_8=0.7), (x_9^W=40; F_9=0.8), (x_{10}^W=41; F_{10}=0.9) and (x_{11}^W=42; F_{11}=1). Then g^{xfr} is approximated using the 1D xW-GL-I – g^{xW}= <$F^{xW}(x)$; x>. Graphical representation of $F^{xW}(x)$ and its corresponding density (PDF) are given on Fig. 1.

b) Approximation of $F^{xR}(x)$ using the maximax criterion and calculation of the $\neg W$ -expected utility of g^{xfr}

According to (11), $x_1^{\neg W}$=30, $x_{11}^{\neg W}$=42. After applying Algorithm 3, two quantiles are marked as "known" with estimates $x_2^{\neg W}$=32 and x_{10}^W=40. Four of the remaining quantiles are marked as "right prone" and their estimates must be found in the intervals $32 \le x_3^{\neg W} \le 33$, $32.8 \le x_4^{\neg W} \le 33.5$, $33 \le x_5^{\neg W} \le 34.5$ and $34 \le x_6^{\neg W} \le 36$. The other three quantiles are marked as "optimizing" and their estimates must be found in the intervals $35.5 \le x_7^{\neg W} \le 37$, $33 \le x_8^{\neg W} \le 37.5$ and $37 \le x_9^{\neg W} \le 40$.

Following the comments to Algorithm 4, the permissible combinations of the quantile estimates are identified on the possible values of the "optimizing" quantiles. Those combinations should obey the conditions in (24), and $I_l^{x\neg W}$, l=1, 2, 10, and $E^{x\neg W}(u|F^{xR})$ are calculated for each.

For example, at the permissible combination $x_7^{\neg W}$=36, $x_8^{\neg W}$=36.5 and $x_9^{\neg W}$=38, the "right prone" quantiles are set as follows: $x_3^{\neg W}$=33, $x_4^{\neg W}$=33.5, $x_5^{\neg W}$=34.5 and $x_6^{\neg W}$=36. According to (23), $I_1^{x\neg W}$=0.0525, $I_2^{x\neg W}$=0.12, $I_3^{x\neg W}$=0.1875, $I_4^{x\neg W}$=0.3125, $I_5^{x\neg W}$=0.5458, $I_6^{x\neg W}$=0.6, $I_7^{x\neg W}$=0.625, $I_8^{x\neg W}$=0.5917, $I_9^{x\neg W}$=0.2375 and $I_{10}^{x\neg W}$=0.0875. According to (22), the expected utility $E^{x\neg W}(u|F^{xR})$ =0.336. The maximal possible value $E^{x\neg W}(u|F^{xR})$ =0.3697 is calculated for the following values of the "optimizing" quantiles: $x_7^{\neg W}$=37, $x_8^{\neg W}$=37 and $x_9^{\neg W}$=37. Here, the estimates of the "right prone" quantiles are again $x_3^{\neg W}$=33, $x_4^{\neg W}$=33.5, $x_5^{\neg W}$=34.5 and $x_6^{\neg W}$=36, whereas according to (23), $I_1^{x\neg W}$=0.0525, $I_2^{x\neg W}$=0.12, $I_3^{x\neg W}$=0.1875, $I_4^{x\neg W}$=0.3125, $I_5^{x\neg W}$=0.5458, $I_6^{x\neg W}$=0.65, $I_7^{x\neg W}$=0.7, $I_8^{x\neg W}$=0.7, $I_9^{x\neg W}$=0.3417 and $I_{10}^{x\neg W}$=0.0875.

Then $F^{xR}(.)$ is approximated by $F^{x\neg W}(.)$ on the knots ($x_1^{\neg W}$=30; F_1=0), ($x_2^{\neg W}$=32; F_2=0.1), ($x_3^{\neg W}$=33; F_3=0.2), ($x_4^{\neg W}$=33.5; F_4=0.3), ($x_5^{\neg W}$=34.5; F_5=0.4), ($x_6^{\neg W}$=36; F_6=0.5), ($x_7^{\neg W}$=37; F_7=0.6), ($x_8^{\neg W}$=37; F_8=0.7), ($x_9^{\neg W}$=37; F_9=0.8), ($x_{10}^{\neg W}$=40; F_{10}=0.9) and ($x_{11}^{\neg W}$=42; F_{11}=1). Then g^{xfr} is approximated with the 1D $x\neg W$-GL-I – $g^{x\neg W}$ =<$F^{x\neg W}(x)$; x>. Graphics of $F^{x\neg W}(.)$ and its corresponding density are presented on Fig. 2.

c) Approximation of $F^{xR}(x)$ using the Hurwicz$_\alpha$ criterion and calculation of the H_α-expected utility of g^{xfr}

Let α=0.7. According to (11), $x_1^{H_{0.7}}$ =30 and $x_{11}^{H_{0.7}}$ =42. The values of the quantiles $x_l^{H_{0.7}}$, l=2, 3, ..., 10, may be calculated according to (30), using the estimates of x_l^W and $x_l^{\neg W}$, l=2, 3, ..., 10, as follows $x_2^{H_{0.7}}$ =0.7x_2^W+(1–0.7)$x_2^{\neg W}$=0.7×31+0.3×32=31.3, $x_3^{H_{0.7}}$ =32.3, $x_4^{H_{0.7}}$ =33.01, $x_5^{H_{0.7}}$ =33.45, $x_6^{H_{0.7}}$ =34.6, $x_7^{H_{0.7}}$ =35.95, $x_8^{H_{0.7}}$ =37.35, $x_9^{H_{0.7}}$ =39.1 and $x_{10}^{H_{0.7}}$ =40.7. The graphics of $F^{xH_{0.7}}(.)$ and its corresponding density are given on Fig. 3. The $H_{0.7}$-expected utility calculated according to (7) is $E^{xH_{0.7}}(u\,|\,F^{xR})$ =0.2542.

6. Conclusions

The paper proposed methods and procedures to approximate *x*-ribbon CDF, constructed on interval estimates of quantiles of a fuzzy-rational DM, and to rank *x*-fuzzy-rational GL-I. The partially linear interpolation of the *x*-ribbon CDF by classical ones was performed on the basis of three decision criteria under strict uncertainty – Wald, Hurwicz$_\alpha$ and maximax criterion. The Wald approximation was performed according to two algorithms – the first reduced the dimensionality of the optimization task, and the second defined permissible values of the unknown quantiles so that to minimize $I_l^{xW,(i)}$. A similar set of algorithms was developed for the maximax approximation. The Hurwicz$_\alpha$ approximation took into account the Wald and maximax quantiles, which were weighed by the pessimism index $\alpha\in[0;1]$. Once the *x*-ribbon CDF were approximated by classical ones, it was possible to approximate the *x*-fuzzy-rational GL-I by classical-risky GL-I. The latter obey the assumptions of the expected utility rule. The Wald, Hurwicz$_\alpha$ and maximax expected utility criteria were proposed to rank one-dimensional fuzzy-rational GL-I according to the preferences of the DM. The expected utility of the *x*-fuzzy-rational GL-I in the numerical example significantly differs in the pessimistic (Wald) and the optimistic (maximax) case. The choice of α=0.7 assumes a pessimistic DM, and justifies results similar to those of the Wald expected utility. The opposite effect would be observed if α<0.5 (i.e. if the DM is an optimist).

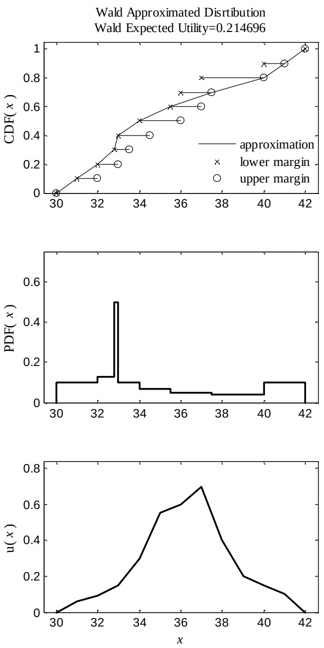

Fig. 1. Graphics of $F^{xW}(.)$, PDF and utility function of the DM over values of *x* in the interval [30; 42]

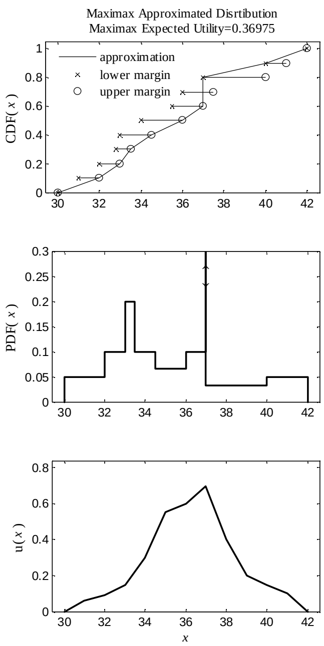

Fig. 2. Graphics of $F^{x\neg W}(.)$, PDF and utility function of the DM over values of *x* in the interval [30; 42] (the double arrow in the PDF graphics means infinity

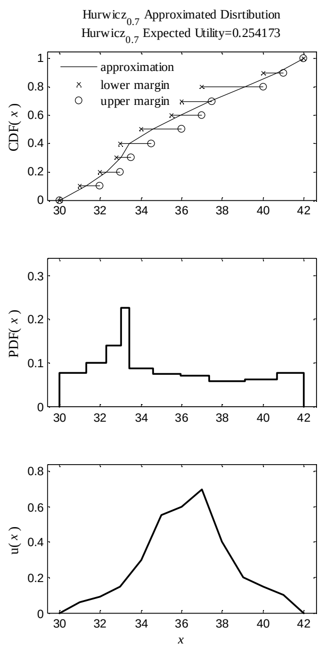

Fig. 3. Graphics of $F^{xH_{0.7}}(.)$, (PDF) and utility function of the DM over values of *x* in the interval [30; 42]

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Appendix

Each 1D random variable X describes a 1D object with a single attribute – the realization of the random variable. A 1D distribution function (CDF), that entirely describes the uncertainty in X shall be referred as classical, and will be denoted as $F(.)$:

(A1) $$F(x)=P(X\leq x),\ x\in(-\infty;+\infty).$$

The function (A1) is usually defined by linear interpolation on a set of $z>1$ points from its graphics, where x_l is an α-quantile of X, and $\alpha=F_l$:

(A2) $$\{(x_l;\ F_l)\mid l=1,2,\ldots,z\},$$

where

(A3) $$x_1\leq x_2\leq\ldots\leq x_z,$$
$$0=F_1\leq F_2\leq\ldots\leq F_z=1;$$

(A4)

A 1D CDF, that partially describes the uncertainty in X shall be referred as ribbon, and will be denoted as $F^R(.)$. It entirely lies between two 1D classical CDFs, called lower and upper border functions – $F^d(.)$ and $F^u(.)$, where the following conditions hold (T e n e k e d j i e v et al. [20]):

(A5) $$F^d(x)\leq F^R(x)\leq F^u(x),\ x\in(-\infty;+\infty),$$

(A6) $$F^d(x)\leq F^u(x),\ x\in(-\infty;+\infty).$$

The 1D x-ribbon CDF is a special case of a 1D ribbon CDF, when probability distributions are interpolated on knots with uncertainty interval for the quantile. Let $F^{xR}(.)$, $F^{xd}(.)$ and $F^{xu}(.)$ are respectively a 1D x-ribbon CDF and its lower and upper x-border functions. The 1D x-ribbon CDF is defined by linear interpolation on the end points of the uncertainty intervals of $z>1$ quantiles:

(A7) $$\{(x_l^d;\ x_l^u;\ F_l)\mid l=1,2,\ldots,z\},$$

where

(A8) $$x_1^d\leq x_2^d\leq\ldots\leq x_z^d,$$
$$x_1^u\leq x_2^u\leq\ldots\leq x_z^u,$$
$$x_l^d\leq x_l^u,\ l=2,3,\ldots,z-1,$$
$$x_1^d=x_1^u,\ x_z^d=x_z^u,$$
$$0=F_1\leq F_2\leq\ldots\leq F_z=1;$$

(A9) $$F^{xd}(x)=\begin{cases}0 & \text{at } x<x_1^d,\\ F_l & \text{at } x_l^d=x<x_{l+1}^d,\ l=1,2,\dots,z-1,\\ F_l+\dfrac{(x-x_l^d)(F_{l+1}-F_l)}{x_{l+1}^d-x_l^d} & \text{at } x_l^d<x<x_{l+1}^d,\ l=1,2,\dots,z-1,\\ 1 & \text{at } x_z^d\le x;\end{cases}$$

(A10) $$F^{xu}(x)=\begin{cases}0 & \text{at } x<x_1^u,\\ F_l & \text{at } x_l^u=x<x_{l+1}^u,\ l=1,2,\dots,z-1,\\ F_l+\dfrac{(x-x_l^u)(F_{l+1}-F_l)}{x_{l+1}^u-x_l^u} & \text{at } x_l^u<x<x_{l+1}^u,\ l=1,2,\dots,z-1,\\ 1 & \text{at } x_z^u\le x,\end{cases}$$

(A11) $$F^{xd}(x)\le F^{xR}(x)\le F^{xu}(x),\ \ x\in(-\infty;+\infty).$$

A 1D GL-I is defined as alternative that gives 1D prizes x from an almost everywhere continuous 1D set X, according to continuous or mixed 1D probability law (T e n e k e d j i e v et al. [20]). An 1D GL-I with 1D classical CDF – $F_i(.)$ – shall be referred as classical-risky and takes the form (A12). Its expected utility (A13) is calculated using Stieltjes integral (K r a m e r [10]) with respect to the function $F_i(.)$:

(A12) $$g_i=<F_i(x);\ x>,\quad i=1,\ 2,\ \dots,\ q\ ,$$

(A13) $$E_i(u/F_i)=\int_{-\infty}^{+\infty}u(x)\mathrm{d}F_i(x)\ .$$

If $F_i(.)$ is a piece-wise linear 1D classical CDF with knots as in (A14), then its expected utility is calculated by (A15):

(A14) $$\{(x_l^{(i)};\ F_l^{(i)})\mid l=1,\ 2,\ \dots,\ z_i\},\ \text{where}$$
$$x_1^{(i)}\le x_2^{(i)}\le\dots\le x_{z_i}^{(i)}\ ,$$
$$0=F_1^{(i)}\le F_2^{(i)}\le\dots\le F_{z_i}^{(i)}=1.$$

(A15) $$E_i(u/F_i)=\int_{x_1^{(i)}}^{x_{z_i}^{(i)}}u(x)\mathrm{d}F_i(x)=\sum_{\substack{l=1\\ x_{l+1}^{(i)}>x_l^{(i)}}}^{z_i-1}\frac{F_{l+1}^{(i)}-F_l^{(i)}}{x_{l+1}^{(i)}-x_l^{(i)}}\int_{x_l^{(i)}}^{x_{l+1}^{(i)}}u(x)\mathrm{d}x+\sum_{\substack{l=1\\ x_{l+1}^{(i)}=x_l^{(i)}}}^{z_i-1}(F_{l+1}^{(i)}-F_l^{(i)})u(x_l).$$