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Laplace Expected Utility Criterion for Ranking Fuzzy Rational Generalized Lotteries of I Type

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Abstract: *Real decision makers partially disobey axioms of rationality and are called fuzzy rational. Fuzzy rationality in probabilities leads to the construction of x-ribbon and p-ribbon distribution functions, whose quantiles/quantile indices are elicited as uncertainty intervals. As a result x-fuzzy rational and p-fuzzy rational generalized lotteries are introduced and a general approach to rank each type is developed, which approximates ribbon functions by classical partially linear ones and then applies expected utility. In this paper, approximation is made via Laplace criterion under strict uncertainty. Its use is formalized for both types of ribbon functions. Laplace expected utility is also introduced to rank p-fuzzy-rational and x-fuzzy-rational generalized lotteries of I type. An example is developed to show the importance of decision modeling and detailed analysis of available subjective information for the final decision.*

Keywords: *fuzzy rationality, subjective probability, strict uncertainty, generalized lotteries, ribbon functions, expected utility, Laplace strict uncertainty criterion.*

1. Introduction

Lotteries are proposed by utility theory as a model of uncertain alternatives (v o n N e u m a n n, M o r g e n s t e r n, [20]). They consist of a full set of disjoint events (called states) each associated with a holistic consequence for the decision maker (DM), called a prize. The cardinality of the set of lotteries and the set of prizes defines the type of lotteries. Ordinary lotteries apply to discrete sets of prizes. It is also possible for the set of prizes to take one or more continuous intervals, and the prize is actually a random variable that is probabilistically described by a distribution function $F(.)$. In that case, generalized lotteries apply of either I, II or III type (T e n e k e d j i e v, [15]). In problems with generalized lotteries of I type, in particular, the set of lotteries

is discrete, whereas the set of prizes is a continuous one. A set of axioms of utility define the conditions that should hold for the preferences of the rational DM. On that basis, a utility function $u(.)$ can be constructed over the prizes, such that the more preferred the consequence the higher the value of the utility function (K e e n e y, R a i f f a, [6]). The main paradigm of utility theory is that all kind of lotteries should be ranked in descending order of their expected utility, which is the utility of prizes, weighted by their probabilities.

Traditionally, decisions are made under risk or under strict uncertainty according to the information provided by the DM. In the first case, the DM has to assign unique probability measure (B e r n s t e i n, [4]) for the chance of receiving a prize (such a lottery will be referred here as a classical risky one). In the second case, the DM only has to identify the possible states of nature (such a lottery will be referred here as a strictly uncertain one). There exist decision criteria to rank strictly uncertain lotteries according to preferences, like Wald, Hurwic z_a, Savage and Laplace criteria, but none obeys the minimal rationality requirements of choice (R a p p a p o r t [12]).

Real DMs can define subjective probabilities, quantiles and quantile indices only in an interval form. As a result, utility theory assumptions are disobeyed, and partially non-transitive preferences are observed. For that reason, in (N i k o l o v a et al. [9]), real DMs are referred to as fuzzy rational. Then uncertain alternatives are modeled with fuzzy rational lotteries where the chance of receiving each prize is quantified by interval probability measure (T e n e k e d j i e v et al. [17]). Since fuzzy rational DMs only partially quantify uncertainty, ranking fuzzy rational lotteries is a problem of mixed type, and generalizes decisions under risk and under strict uncertainty.

In decision problems with partially quantified uncertainty, it is necessary to select a method to approximate fuzzy rational lotteries by classical risky ones, which can then be ranked according to the expected utility criterion. These ideas lead to the introduction of the Laplace expected utility criterion for the case of ordinary fuzzy rational lotteries in (N i k o l o v a [8], of the Hurwicz expected utility criterion for the case of ordinary fuzzy rational lotteries in (T e n e k e d j i e v [17]), and of the Wald expected utility criterion for the case of generalized lotteries of II type in (T e n e k e d j i e v et al [19]). These procedures benefit from the existing mathematical homology between the descriptions of the triples "event from a probability field – interval subjective probability – point estimate probability" and "object from an universe – degree of membership to an intuitionistic fuzzy set (A t a n a s s o v [3]) – degree of membership to a (classical) fuzzy set" (S z m i d t, K a c p r z y k [14]). That allows transforming interval probabilities into point estimates using the operators that transform an intuitionistic fuzzy degree of membership into classical fuzzy degree of membership (A t a n a s s o v [1]; A t a n a s s o v [2]).

In this paper the fuzzy rationality modeling is facilitated by the introduction of ribbon probability distributions. As a result of those, fuzzy-rational generalized lotteries of I type are constructed. If the problem is to be solved using expected utility it should be brought down to one under risk. For that purpose, the Laplace criterion under strict uncertainty is used to approximate ribbon distributions with classical ones. On the basis of that, alternatives are ranked by the Laplace expected utility criterion.

In what follows, Section 1 formalized classical and fuzzy rational probability distributions, the latter also called *ribbon distributions*. Section 3 discusses two

methods to build subjective distribution functions on elicited knots. Problem modeling via classical risky and fuzzy-rational generalized lotteries of I type is discussed in Section 4. Two procedures are proposed for the second task, which depend on the type of ribbon functions. Section 5 discusses the use of the Laplace criterion of strict uncertainty in the transformation of ribbon functions. It also introduces Laplace expected utility for both types of fuzzy-rational generalized lotteries of I type. An example for the use of all those techniques is developed in Section 6.

2. Classical and fuzzy rational probability distributions

2.1. One-dimensional classical distribution functions

Let the uncertainty associated with a one-dimensional (1D) random variable X be entirely quantified by a known 1D distribution function F (.), which will be called *classical function*. If x is an arbitrary fixed value of X, then

(1) $$F(x)=P(X \le x) \text{ for } x \in (-\infty; +\infty).$$

Classical distribution functions must be increasing and limited within the interval [0 ;1]:

(2) $$\text{if } x_1>x_2 \text{ , then } F(x_1) \le F(x_2),$$
$$\lim_{x\to-\infty} F(x) = 0, \text{ and } \lim_{x\to+\infty} F(x) = 1.$$

A very convenient and practically universal approach to define a classical distribution function is by linear interpolation using a set of $z>1$ defined points from its graphics:

(3) $$\{(x_l; F_l) \mid l=1, 2, \ldots, z\},$$

where

$$x_1 \le x_2 \le \ldots \le x_z,$$
$$0 = F_1 \le F_2 \le \ldots \le F_z = 1.$$

Each point (x_l, F_l) is called a *knot point*, where x_l is the α-quantile of the random variable X with $\alpha = F_l$. Then:

(4) $$F(x)=\begin{cases} 0 & \text{for } x < x_1 \\ F_l & \text{for } x_l = x < x_{l+1}, \quad l=1,2,...,z-1 \\ F_l + \dfrac{(x-x_l)(F_{l+1}-F_l)}{x_{l+1}-x_l} & \text{for } x_l < x < x_{l+1}, \quad l=1,2,...,z-1 \\ 1 & \text{for } x_z \le x \end{cases}.$$

2.2. 1D ribbon distribution functions

2.2.1. General case of 1D ribbon distribution function

Let the uncertainty in a 1D random variable X be partially quantified by a 1D distribution function $F^R(.)$. It is only known that it entirely lies between the so called

lower and *upper border functions* $F_d(.)$ and $F_u(.)$, i.e.

$$F_d(x) \le F^R(x)\ F_u(x) \quad \text{for } x \in (-\infty; +\infty). \tag{5}$$

Here, $F_d(.)$ and $F_u(.)$ are known classical distribution functions, which obey the condition

$$F_d(x) \le F_u(x), \text{ for } x \in (-\infty; +\infty). \tag{6}$$

A 1D distribution function $F^R(.)$ that obeys this definition shall be called *ribbon distribution function*.

2.2.2. 1D *x-ribbon* distribution functions

A common special case is to construct distributions (usually subjective) by interpolation on knots with an uncertainty interval for the quantile (error on the abscissa x). Then the fuzzy distribution function may be called *x-ribbon* $F^{xR}(.)$, whereas the border functions are respectively *lower* and *upper x-border functions* $F_{xd}(.)$ and $F_{xu}(.)$.

A convenient way to define x-border distribution functions is via linear interpolation on the margins of the set of $z>1$ defined uncertainty intervals for quantiles of the x-ribbon function:

$$\{(x_{d,l};\ x_{u,l}\ ;\ F_l)|\ l=1,\ 2,\ \ldots,\ z\}, \tag{7}$$

where

$$\begin{gathered} x_{d,1} \le\ x_{d,2} \le \ldots\ \le x_{d,z}, \\ x_{u,1} \le\ x_{u,2} \le \ldots\ \le x_{u,z}, \\ x_{d,l} \le x_{u,l}\ ,\ \text{for } l = 2,\ 3,\ \ldots,\ z-1, \\ x_{d,1} = x_{u,1},\ \ x_{d,z} = x_{u,z}\ , \\ 0 = F_1 \le\ F_2 \le\ \ldots \le F_z = 1. \end{gathered}$$

Then

$$F_{xd}(x)=\begin{cases} 0 & \text{for } x < x_{d,1} \\ F_l & \text{for } x_{d,l} = x < x_{d,l+1}, \quad l=1,2,\ldots,z-1 \\ F_l + \dfrac{(x - x_{d,l})(F_{l+1} - F)}{x_{d,l+1} - x_{d,l}} & \text{for } x_{d,l} < x < x_{d,l+1}, \quad l=1,2,\ldots,z-1 \\ 1 & \text{for } x_{d,z} \le x \end{cases}, \tag{8}$$

$$F_{xu}(x)=\begin{cases} 0 & \text{for } x < x_{u,1} \\ F_l & \text{for } x_{u,l} = x < x_{u,l+1}, \quad l=1,2,\ldots,z-1 \\ F_l + \dfrac{(x - x_{u,l})(F_{l+1} - F_l)}{x_{u,l+1} - x_{u,l}} & \text{for } x_{u,l} < x < x_{u,l+1}, \quad l=1,2,\ldots,z-1 \\ 1 & \text{for } x_{u,z} \le x \end{cases}, \tag{9}$$

$$F_{xd}(x) \le F^{xR}(x) \le F_{xu}(x), \text{ for } x \in (-\infty; +\infty). \tag{10}$$

2.2.3. 1D *p-ribbon* distribution functions

Another common approach is when (usually subjective) distributions are interpolated on knots with uncertainty interval for the quantile index (error on the ordinate, i.e. probability). Then the fuzzy distribution function may be called *p-ribbon* $F^{pR}(.)$, whereas the border functions are respectively *lower and upper p-border functions* $F_{pd}(.)$ and $F_{pu}(.)$.

A convenient way to define *p*-border distribution functions is by linear interpolation on the borders of the set of $z>1$ defined uncertainty intervals for quantile indices of the *p*-ribbon function:

(11) $$\{(x_l ; F_{d,l} ; F_{u,l}) \mid l = 1, 2, \ldots, z\},$$

where

$$x_1 \le x_2 \le \ldots \le x_z,$$
$$0 = F_{d,1} \le F_{d,2} \le \ldots \le F_{d,z} = 1,$$
$$0 = F_{u,1} \le F_{u,2} \le \ldots \le F_{u,z} = 1,$$
$$F_{d,l} \le F_{u,l}, \text{ for } l = 2, 3, \ldots, z-1.$$

Then

(12) $$F_{pd}(x)=\begin{cases} 0 & \text{for } x < x_1 \\ F_{d,l} & \text{for } x_l = x < x_{l+1},\ l=1,2,...,z-1 \\ F_{d,l} + \dfrac{(x - x_l)(F_{d,l+1} - F_{d,l})}{x_{l+1} - x_l} & \text{for } x_l < x < x_{l+1},\ l=1,2,...,z-1 \\ 1 & \text{for } x_z \le x \end{cases},$$

(13) $$F_{pu}(x)=\begin{cases} 0 & \text{for } x < x_1 \\ F_{u,l} & \text{for } x_l = x < x_{l+1},\ l=1,2,...,z\text{-}1 \\ F_{u,l} + \dfrac{(x - x_l)(F_{u,l+1} - F_{u,l})}{x_{l+1} - x_l} & \text{for } x_l < x < x_{l+1},\ l=1,2,...,z\text{-}1 \\ 1 & \text{for } x_z \le x \end{cases},$$

(14) $$F_{pd}(x) \le F^{pR}(x) \le F_{pu}(x), \text{ for } x \in (-\infty; +\infty).$$

3. Subjective distributions' elicitation and fuzzy rationality

Let's assume that the random variable X belongs to a closed interval $[\tilde{x}_1; \tilde{x}_z]$, assessed by the DM. Then the subjective elicitation of the probability distribution is brought down to the assessment of several knots of the CDF curve $\{(x_l ; F_l) \mid l = 2, 3, \ldots, z-1\}$. That could be accomplished in two ways, both using the binary relations of preference $\succ$ ("more proffered than", or "strict preference"), and ~ ("equally preferred to", or "indifference").

In the first method, several quantiles $x_2, x_3, \ldots, x_{z-1}$ are selected uniformly in the assessed interval, and their quantile indices $\hat{F}_2, \hat{F}_3, \ldots, \hat{F}_{z-1}$ are assessed. That coincides with the general case of subjective probability elicitation for the random events "the random variable X takes values equal or less than x_l (i.e. $X \le x_l$)" (P r a t t et al. [10]). For each knot, the DM has to solve, according to m, the preferential equation $l_1(X \le x_l) \sim l_2(m, n)$ between $l_1(X \le x_l)$, giving a huge award if $X \le x_l$, and $l_2(m, n)$, giving the same huge award if a white ball is drawn out of an urn with m white and $(n–m)$ black balls. Then, $\hat{F}_l = m^*/n$, where m^* is the root, assessed using classical dichotomy (P r e s s et al. [11]). This normative scheme holds only for ideal DMs. For real DMs, there could exist $m_2 > m_1$, such that $l_1(X \le x_l) \sim l_2(m_1, n)$, $l_1(X \le x_l) \sim l_2(m_2, n)$, and $l_2(m_2, n) \succ l_2(m_1, n)$. Thus, the DM must find the greatest possible $m = \hat{m}_{down}$, such that $l_1(X \le x_l) \succ l_2(\hat{m}_{\text{down}}, n)$, and the smallest possible $m = \hat{m}_{up}$, such that $l_2(\hat{m}_{\text{up}}, n) \succ l_1(X \le x_l)$. Then $m^* \in (\hat{m}_{\text{down}}; \hat{m}_{\text{up}})$, and $\hat{F}_l \in (\hat{m}_{\text{down}}/n; \hat{m}_{\text{up}}/n)$. That is the uncertainty interval of the quantile index, which could be elicited using triple dichotomy (T e n e k e d j i e v et al. [18]). Thus, the DM's degree of belief about the quantile indices takes the form

$$\hat{F}_l \in [\hat{F}_{d,l}; \hat{F}_{u,l}] \quad \text{for } l = 2, 3, \ldots, z–1. \tag{15}$$

The interval in (15) is closed so that to accommodate the cases when the quantile indeces are known. The extreme quantiles $x_1 = \hat{x}_1$ and $x_z = \hat{x}_z$ have quantile indices respectively $\hat{F}_1 = \hat{F}_{d,1} = \hat{F}_{u,1} = 0$ and $\hat{F}_z = \hat{F}_{d,z} = \hat{F}_{u,z} = 1$.

In the second method, several quantile indices $F_2, F_3, \ldots, F_{z-1}$ are selected uniformly in the interval [0;1] and their quantiles $\hat{x}_2, \hat{x}_3, \ldots, \hat{x}_{z-1}$ are assessed. For each knot, the DM has to solve, according to $\hat{x}_l$, the preferential equation $l_1(X \le \hat{x}_l) \sim l_2(m,n)$, where $m/n = F_l$. Then, $\hat{x}_l = x^*$, where x^* is the root, assessed using classical dichotomy. Again, this normative scheme holds only for ideal DMs. For real DMs, there could exist $x_2 > x_1$, such that $l_1(X \le x_1) \sim l_2(m, n)$, $l_1(X \le x_2) \sim l_2(m, n)$, and $l_1(X \le x_2) \succ l_1(X \le x_1)$. Thus, the DM must find the greatest possible $x = \hat{x}_{\text{down}}$, such that $l_2(m,n) \succ l_1(X \le \hat{x}_{\text{down}})$, and the smallest possible $x = \hat{x}_{\text{up}}$, such that $l_1(X \le \hat{x}_{\text{up}}) \succ l_2(m,n)$. Then $x^* \in (\hat{x}_{\text{down}}; \hat{x}_{\text{up}})$. That is the uncertainty interval of the quantile, which could be elicited using triple dichotomy. Thus, the DM's degree of belief about the quantiles takes the form

$$\hat{x}_l \in [\hat{x}_{d,l}; \hat{x}_{u,l}], \text{ for } l = 2, 3, \ldots, z–1. \tag{16}$$

The interval in (16) is closed in order to accommodate the cases when the quantiles are known. The extreme quantile indices $F_1 = 0$ and $F_z = 1$ correspond to quantiles respectively $\hat{x}_1 = \hat{x}_{d,1} = \hat{x}_{u,1}$ and $\hat{x}_z = \hat{x}_{d,z} = \hat{x}_{u,z}$.

There are several sets of requirements to the rationality of the DM (S a v a g e [13]; T e n e k e d j i e v [16]; D e G r o o t [5]). A common requirement in all is transitivity of DM's binary relations of preference. As it was demonstrated, real DMs partially disobey transitivity of indifference and the mutual transitivity between strict preference and indifference. Such DMs are denoted as fuzzy rational [N i k o l o v a et al. [9]).

If an ideal DM makes the elicitation by any of the described methods, then classical probability distributions shall be defined. On the contrary, a fuzzy rational DM shall elicit ribbon probability distribution functions. The *p*-ribbon version shall be formed using the first approach described above, whereas the *x*-ribbon version shall be formed using the second approach.

4. 1D generalized lotteries of I type

4.1. 1D classical risky generalized lotteries of I type

Let's compare q alternatives according to DM's preference, which give 1D prizes x with a utility function $u(.)$, defined for all possible prizes from all alternatives.

A 1D generalized lottery of I type with a classical distribution function $F_i(.)$ shall be called *classical risky generalized lottery of I type*:

(17) $$g_i = \langle F_i(x); x \rangle \text{ for } i = 1, 2, \ldots, q.$$

A theorem proves that such lotteries should be ranked in descending order of the expected utility, which is calculated as a Stiltes integral:

(18) $$E_i(u/F_i) = \int_{-\infty}^{+\infty} u(x) dF_i(x).$$

Let $F_i(.)$ be a partially linear distribution function with knots

(19) $$\{ (x_l^{(i)}; F_l^{(i)}) \mid l = 1, 2, \ldots z_i \},$$

where $x_1^{(i)} \le x_2^{(i)} \le \ldots \le x_{z_i}^{(i)}$, $0 = F_1^{(i)} \le F_2^{(i)} \le \ldots \le F_{z_i}^{(i)} = 1$.

Then the expected utility is brought down to

(20) $$E_i(u/F_i) = \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x) dF_i(x) = \sum_{\substack{l=1 \\ x_{l+1}^{(i)} > x_l^{(i)}}}^{z_i - 1} \frac{F_{l+1}^{(i)} - F_l^{(i)}}{x_{l+1}^{(i)} - x_l^{(i)}} \int_{x_l^{(i)}}^{x_{l+1}^{(i)}} u(x) dx + \sum_{\substack{l=1 \\ x_{l+1}^{(i)} = x_l^{(i)}}}^{z_i - 1} (F_{l+1}^{(i)} - F_l^{(i)}) u(x_l).$$

4.2. 1D fuzzy rational generalized lotteries of I type

4.2.1. *General case of 1D fuzzy-rational generalized lotteries of I type*

1D generalized lottery of I type with a ribbon distribution function shall be called *fuzzy-rational*:

(21) $$g_i^{fr} = \langle \, ; x \rangle, \text{ for } i = 1, 2, \ldots, q.$$

By analogy to fuzzy-rational ordinary lotteries, 1D fuzzy-rational generalized lotteries of I type may be ranked at two stages:

1) The ribbon distribution functions $F_i^R(x)$ are approximated by classical distribution functions using a criterion under strict uncertainty *S*. In that way the resulting alternatives are approximated by 1D classical-risky generalized lotteries of I type, which can be called *S-generalized*:

(22) $$g_i^S = \langle F_i^S(x); x \rangle \text{ for } i = 1, 2, \ldots, q.$$

2) Alternatives are ranked in descending order of the expected utility of the *S*-generalized lotteries. The resulting ranking criterion can be called *S-expected utility* of the fuzzy-rational generalized lotteries:

(23) $$E_i^S(u/F_i^R) = \int_{-\infty}^{+\infty} u(x)dF_i^S(x) .$$

4.2.2. *1D x-fuzzy-rational generalized lotteries of I type*

A special case of a 1D generalized lottery of I type with an *x*-ribbon distribution function $F_i^{xR}(x)$ shall be denoted *x-fuzzy-rational*:

(24) $$g_i^{xfr} = \langle F_i^{xR}(x) ; x \rangle, \text{ for } i=1, 2,\ldots, q.$$

Then the calculation of the *S*-expected utility of the *x*-fuzzy-rational lottery could be brought down to the following steps:

1) The *x*-ribbon distribution function $F_i^{xR}(x)$ is approximated by a classical, partially linear distribution function $F_i^{xS}(x)$ using a criterion under strict uncertainty *S*, with knots

(25) $$\{ (\hat{x}_l^{S,(i)}; F_l^{(i)}) \mid l=1, 2,\ldots, z_i \}, \text{ where}$$

$$\hat{x}_1^{S,(i)} \le \hat{x}_2^{S,(i)} \le \ldots \le \hat{x}_{z_i}^{S,(i)},$$

$$\hat{x}_{d,l}^{(i)} \le \hat{x}_l^{S,(i)} \le \hat{x}_{u,l}^{(i)} \text{ for } l = 2, 3,\ldots, z_i-1,$$

$$\hat{x}_1^{S,(i)} = \hat{x}_{d,1}^{(i)} = \hat{x}_{u,1}^{(i)} \text{ and } \hat{x}_{z_i}^{S,(i)} = \hat{x}_{d,z_i}^{(i)} = \hat{x}_{u,z_i}^{(i)} .$$

Then

(26) $$F_i^{xS}(x) = \begin{cases} 0 & \text{for } x < \hat{x}_1^{S,(i)} \\ F_l^{(i)} & \text{for } \hat{x}_l^{S,(i)} = x < \hat{x}_{l+1}^{S,(i)},\ l=1,2,...,z_i-1 \\ F_l^{(i)} + \dfrac{(x - \hat{x}_l^{S,(i)})(F_{l+1}^{(i)} - F_l^{(i)})}{\hat{x}_{l+1}^{S,(i)} - \hat{x}_l^{S,(i)}} & \text{for } \hat{x}_l^{S,(i)} < x < \hat{x}_{l+1}^{S,(i)},\ l=1,2,...,z_i-1 \\ 1 & \text{for } \hat{x}_{z_i}^{S,(i)} \le x \end{cases} .$$

The resulting alternatives are approximated by a 1D classical-risky generalized lottery of I type, which can be called *xS-generalized*:

(27) $$g_i^{xS} = \langle F_i^{xS}(x) ; x \rangle \text{ for } i=1, 2,\ldots, q.$$

2) Alternatives are ranked in descending order of the expected utility of the *xS*-generalized lotteries. The resulting ranking criterion may be called *xS-expected utility* of the fuzzy-rational generalized lotteries:

(28) $$E_i^{xS}(u/F_i^{xR}) = \int_{\hat{x}_1^{S,(i)}}^{\hat{x}_{z_i}^{S,(i)}} u(x)dF_i^{xS}(x) =$$

$$= \sum_{\substack{l=1 \\ \hat{x}_{l+1}^{S,(i)} > \hat{x}_l^{S,(i)}}}^{z_i-1} \frac{F_{l+1}^{(i)} - F_l^{(i)}}{\hat{x}_{l+1}^{S,(i)} - \hat{x}_l^{S,(i)}} \int_{\hat{x}_l^{S,(i)}}^{\hat{x}_{l+1}^{S,(i)}} u(x)dx + \sum_{\substack{l=1 \\ \hat{x}_{l+1}^{S,(i)} = \hat{x}_l^{S,(i)}}}^{z_i-1} (F_{l+1}^{(i)} - F_l^{(i)}) u(\hat{x}_l^{S,(i)}) .$$

4.2.3. *1D p-fuzzy-rational generalized lotteries of I type*

A special case of a 1D generalized lottery of I type with a *p*-ribbon distribution function $F_i^{pR}(x)$ shall be called *p-fuzzy-rational*:

(29) $$g_i^{pfr} = \langle F_i^{pR}(x); x\rangle, \text{ for } i = 1, 2,\ldots, q.$$

Then the calculation of the *S*-expected utility of the *p*-fuzzy-rational lottery may be brought down to the following steps:

1) The *p*-ribbon distribution function $F_i^{pR}(x)$ is approximated by a classical, partially linear distribution function, using a criterion under strict uncertainty *S*, with knots

(30) $$\{\ (x_l^{(i)}; \hat{F}_l^{S,(i)}) \mid l=1, 2,\ldots, z_i\}, \text{ where}$$
$$0 = F_1^{S,(i)} \le F_2^{S,(i)} \le \ldots \le F_{z_i}^{S,(i)} = 1,$$
$$\hat{F}_{d,l}^{(i)} \le \hat{F}_l^{S,(i)} \le \hat{F}_{u,l}^{(i)}, \text{ for } l = 2, 3,\ldots, z_i-1.$$

Then

(31) $$F_i^{pR}(x)=\begin{cases} 0 & \text{for } x < x_1^{(i)} \\ \hat{F}_l^{S,(i)} & \text{for } x_l^{(i)} = x < x_{l+1}^{(i)},\ l=1,2,...,z_i-1 \\ \hat{F}_l^{S,(i)} + \dfrac{(x - x_l^{(i)})(\hat{F}_{l+1}^{S,(i)} - \hat{F}_l^{S,(i)})}{x_{l+1}^{(i)} - x_l^{(i)}} & \text{for } x_l^{(i)} < x < x_{l+1}^{(i)},\ l=1,2,...,z_i-1 \\ 1 & \text{for } x_{z_i}^{(i)} \le x \end{cases}.$$

The resulting alternatives are approximated by a 1D classical-risky generalized lottery of I type, which may be called *pS-generalized*:

(32) $$g_i^{pS} = \langle F_i^{pS}(x); x\rangle, \text{ for } i=1, 2,\ldots, q.$$

2) Alternatives are ranked in descending order of the expected utility of the *pS*-generalized lotteries. The resulting ranking criterion may be called *pS-expected utility* of the fuzzy-rational generalized lotteries:

(33) $$E_i^{pS}(u/F_i^{pR}) = \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)dF_i^{pS}(x) =$$
$$= \sum_{\substack{l=1 \\ x_{l+1}^{(i)} > x_l^{(i)}}}^{z_i-1} \frac{\hat{F}_{l+1}^{S,(i)} - \hat{F}_l^{S,(i)}}{x_{l+1}^{(i)} - x_l^{(i)}} \int_{x_l^{(i)}}^{x_{l+1}^{(i)}} u(x)dx + \sum_{\substack{l=1 \\ x_{l+1}^{(i)} = x_l^{(i)}}}^{z_i-1} (\hat{F}_{l+1}^{S,(i)} - \hat{F}_l^{S,(i)})u(x_l^{(i)}).$$

5. Ranking 1D fuzzy-rational generalized lotteries of I type with Laplace approximation

5.1. Laplace approximation of *x*-ribbon distributions

The values of the required quantiles $\hat{x}_l^{L,(i)}$, for l=2, 3,…, z_i–1, do not depend on the utility function. According to the Laplace principle of insufficient reason, if no information is available for the quantiles (i.e. $\hat{x}_{d,l}^{(i)} = \hat{x}_{d,1}^{(i)} = \hat{x}_{u,1}^{(i)}$ and $\hat{x}_{u,l}^{(i)} = \hat{x}_{d,z_i}^{(i)} = \hat{x}_{u,z_i}^{(i)}$ for l=2, 3,…, z_i–1), then the distribution must be uniform in the interval $[\hat{x}_{d,1}^{(i)}; \hat{x}_{d,z_i}^{(i)}]$. Let the quantile with the $F_l^{(i)}$ index of this uniform distribution be called *quantile of the complete ignorance* and be denoted as $x_l^{aL,(i)}$:

(34) $$x_l^{aL,(i)} = \hat{x}_{d,1}^{(i)} + (\hat{x}_{d,z_i}^{(i)} - \hat{x}_{d,1}^{(i)})F_l^{(i)} \quad \text{for } l = 2, 3, \ldots, z_i - 1.$$

Let $h_l^{x,(i)}$ be the affine transformation of the maximal uncertainty interval under strict uncertainty of the l-th quantile $[\hat{x}_{d,1}^{(i)}; \hat{x}_{d,z_i}^{(i)}]$ into the actual uncertainty interval $[\hat{x}_{d,l}^{(i)}; \hat{x}_{u,l}^{(i)}]$ of that same quantile. Then the required quantile $\hat{x}_l^{L,(i)}$ will be the image of the quantile of complete ignorance $x_l^{aL,(i)}$ at the affine transformation $h_l^{x,(i)}$:

(35) $$\hat{x}_l^{L,(i)} = \hat{x}_{d,l}^{(i)} + (\hat{x}_{u,l}^{(i)} - \hat{x}_{d,l}^{(i)})\frac{x_l^{aL,(i)} - \hat{x}_{d,1}^{(i)}}{\hat{x}_{d,z_i}^{(i)} - \hat{x}_{d,1}^{(i)}} = \hat{x}_{d,l}^{(i)} + (\hat{x}_{u,l}^{(i)} - \hat{x}_{d,l}^{(i)})F_l^{(i)} \quad \text{for } l = 2, 3, \ldots, z_i - 1.$$

5.2. Laplace approximation of p-ribbon distributions

The values of the required quantile indices $\hat{F}_l^{L,(i)}$ for l = 2, 3, ..., z_i–1 do not depend on the utility function. According to Laplace principle of insufficient reason, if no information is available for the quantile indices (i. e. $\hat{F}_{d,l}^{(i)} = 0$ and $\hat{F}_{u,l}^{(i)} = 1$ for l = 2, 3,..., z_i–1), then the distribution must be uniform in the interval $[x_1^{(i)}; x_{z_i}^{(i)}]$. Let the quantile index of the quantile $x_l^{(i)}$ of that distribution be called *quantile index of complete ignorance* and be denoted $F_l^{aL,(i)}$:

(36) $$F_l^{aL,(i)} = \frac{x_l^{(i)} - x_1^{(i)}}{x_{z_i}^{(i)} - x_1^{(i)}} \quad \text{for } l = 2, 3, \ldots, z_i - 1.$$

Let $h_l^{p,(i)}$ be the affine transformation of the maximal uncertainty interval under strict uncertainty of the l-th quantile index [0; 1] into the actual uncertainty interval $[\hat{F}_{d,l}^{(i)}; \hat{F}_{u,l}^{(i)}]$ of the same quantile index. Then the required quantile index $\hat{F}_l^{L,(i)}$ will be the image of the quantile index of complete ignorance $F_l^{aL,(i)}$ at the affine transformation $h_l^{p,(i)}$:

(37) $$\hat{F}_l^{L,(i)} = \hat{F}_{d,l}^{(i)} + (\hat{F}_{u,l}^{(i)} - \hat{F}_{d,l}^{(i)})F_l^{aL,(i)} = \hat{F}_{d,l}^{(i)} + (\hat{F}_{u,l}^{(i)} - \hat{F}_{d,l}^{(i)})\frac{x_l^{(i)} - x_1^{(i)}}{x_{z_i}^{(i)} - x_1^{(i)}} \quad \text{for } l = 2, 3, \ldots, z_i - 1.$$

5.3. Laplace expected utility criterion for 1D fuzzy-rational generalized lotteries of I type

The dependency (35) generates a set of Laplace approximated knots for the *x*-ribbon distribution functions, with which to build classical, partially linear distribution functions. On that basis it is possible to calculate the Laplace expected utility for the *x*-fuzzy-rational generalized lotteries of I type, by substituting (35) into (28):

$$(38) \qquad E_i^{xL}(u/F_i^{xR}) = \int_{\hat{x}_1^{L,(i)}}^{\hat{x}_{z_i}^{L,(i)}} u(x)dF_i^{xL}(x) =$$

$$= \sum_{\substack{l=1 \\ \hat{x}_{l+1}^{L,(i)} > \hat{x}_l^{L,(i)}}}^{z_i-1} \frac{F_{l+1}^{(i)} - F_l^{(i)}}{\hat{x}_{l+1}^{L,(i)} - \hat{x}_l^{L,(i)}} \int_{\hat{x}_l^{L,(i)}}^{\hat{x}_{l+1}^{L,(i)}} u(x)dx + \sum_{\substack{l=1 \\ \hat{x}_{l+1}^{L,(i)} = \hat{x}_l^{L,(i)}}}^{z_i-1} (F_{l+1}^{(i)} - F_l^{(i)})u(\hat{x}_l^{L,(i)}).$$

The Laplace expected utility criterion in that case has the following three properties:

1) The estimated knots using (35) obey the conditions in (25);

2) If the uncertainty intervals of the quantiles are of zero length (i.e. $\hat{x}_{d,l}^{(i)} = \hat{x}_{u,l}^{(i)}$), then the Laplace expected utility criterion (38) transforms into the classical expected utility criterion under risk;

3) Under strict uncertainty nothing is known, and the uncertainty interval of all inner quantiles are of maximal length (i.e. $\hat{x}_{d,l}^{(i)} = \hat{x}_1^{(i)}$ and $\hat{x}_{u,l}^{(i)} = \hat{x}_{z_i}^{(i)}$). Then according to (35) $\hat{x}_l^{L,(i)} = x_l^{\alpha L,(i)}$, and the distribution is uniform in the interval $[\hat{x}_{d,1}^{(i)}; \hat{x}_{d,z_i}^{(i)}]$ with density $f_i(x) = \frac{1}{\hat{x}_{z_i}^{L,(i)} - \hat{x}_1^{L,(i)}}$. In that case the Laplace expected utility criterion (38) transforms into the classical Laplace criterion under strict uncertainty:

$$(39) \qquad E_i^{xL}(u/F_i^{xR}) = \int_{\hat{x}_1^{L,(i)}}^{\hat{x}_{z_i}^{L,(i)}} u(x)dF_i^{xL}(x) = \int_{\hat{x}_1^{L,(i)}}^{\hat{x}_z^{L,(i)}} u(x)f_i(x)dx =$$

$$= \int_{\hat{x}_1^{L,(i)}}^{\hat{x}_{z_i}^{L,(i)}} u(x)\frac{1}{\hat{x}_{z_i}^{L,(i)} - \hat{x}_1^{L,(i)}}dx = \frac{1}{\hat{x}_{z_i}^{L,(i)} - \hat{x}_1^{L,(i)}} \int_{\hat{x}_1^{L,(i)}}^{\hat{x}_{z_i}^{L,(i)}} .u(x)dx .$$

The dependence (35) generates a set of Laplace approximated knots for the *p*-ribbon distribution functions, with which to build classical, partially linear distribution functions. On that basis it is possible to calculate the Laplace expected utility for the *p*-fuzzy-rational generalized lotteries of I type, by transforming (33) into (40):

(40) $$E_i^{pL}(u/F_i^{pR}) = \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)dF_i^{pL}(x) =$$

$$= \sum_{\substack{l=1 \\ x_{l+1}^{(i)} > x_l^{(i)}}}^{z_i-1} \frac{\hat{F}_{l+1}^{L,(i)} - \hat{F}_l^{L,(i)}}{x_{l+1}^{(i)} - x_l^{(i)}} \int_{x_l^{(i)}}^{x_{l+1}^{(i)}} u(x)dx + \sum_{\substack{l=1 \\ x_{l+1}^{(i)} = x_l^{(i)}}}^{z_i-1} (\hat{F}_{l+1}^{S,(i)} - \hat{F}_l^{S,(i)}) u(x_l^{(i)})$$

The Laplace expected utility criterion in that case also has the following three properties:

1) the estimated knots using (37) obey the conditions in (30);

2) if the uncertainty intervals of the quantile indices are of zero length (i.e. $\hat{F}_{d,l}^{(i)} = \hat{F}_{u,l}^{(i)}$), then the Laplace expected utility criterion (40) transforms into the classical expected utility criterion under risk;

3) under strict uncertainty nothing is known, and the uncertainty intervals of all inner quantile indices are of maximal length (i.e. $\hat{F}_{d,l}^{(i)}=0$, $\hat{F}_{u,l}^{(i)}=1$). Then according to (37) $\hat{F}_l^{L,(i)} = F_l^{\alpha L,(i)}$, and the distribution is uniform in the interval $[x_1^{(i)}; x_{z_i}^{(i)}]$ with density $f_i(x) = \frac{1}{x_{z_i}^{(i)} - x_1^{(i)}}$. In that case the Laplace expected utility criterion (40) transforms into the classical Laplace criterion under strict uncertainty:

(41) $$E_i^{pL}(u/F_i^{pR}) = \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)dF_i^{pL}(x) = \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)f_i(x)dx =$$

$$= \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)\frac{1}{x_{z_i}^{(i)} - x_1^{(i)}}dx = \frac{1}{x_{z_i}^{(i)} - x_1^{(i)}} \int_{x_1^{(i)}}^{x_{z_i}^{(i)}} u(x)dx .$$

6. Example problem with fuzzy-rational generalized lotteries of I type

In order to facilitate a businessman's decision, her/his utility function was constructed in the interval [–$2000; $28 000] using the lottery equivalence method (M c C o r d, D e N e u f v i l l e [7]). Five uniformly distributed prize values, given in the second column of Table 1 are chosen in the interval. The corresponding uncertainty intervals of the elicited utilities are given in columns 3 and 4 of Table 1, whereas column 5 contains their point estimates, calculated as the mean values of the uncertainty intervals. The utility function, depicted on Fig. 1, is constructed using linear interpolation on the seven knots.

Let's assume that the businessman has to choose between three investment projects, for which the NPV in US dollars are calculated in the form of probability

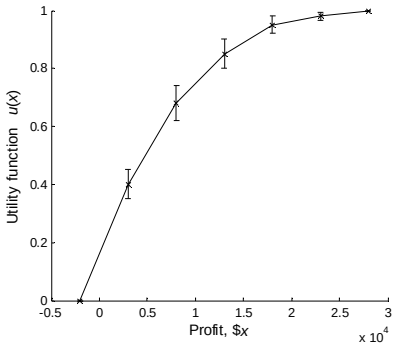

Fig. 1. Utility function over prizes in the interval [–$2 000;$28 000]

Table 1. Utility knots and uncertainty intervals for prizes in the interval [–$2 000;$28 000]

i	x_i	u_i	$u_{i,d}$	$u_{i,u}$
1	-2000	0	0	0
2	3000	0.4	0.35	0.45
3	8000	0.68	0.62	0.74
4	13000	0.85	0.8	0.9
5	18000	0.95	0.92	0.98
6	23000	0.98	0.97	0.99
7	28000	1	1	1

distributions. The first project is expected to yield profits in the interval [$0; $18 000], the second project – in the interval [$500; $18 000], and the third project – in the interval [$0; $17 500]. Then the set of prizes X consists of all profits in the continuous interval [$0; $18 000]. Since there are only three alternatives and a countless number of consequences (profits), the problem may be modeled with generalized lotteries of I type.

The subjective distribution functions of the businessman over the prize set in each of the alternatives are constructed by interpolation on elicited knots. For the first two alternatives, three inner quantile uncertainty intervals corresponding to three initially defined quantile indices were elicited. The acquired knots are given in columns 2 to 4 of rows 2 to 6, and 8 to 12 of Table 2. For the third alternative, the quantile index uncertainty intervals of four inner quantiles were elicited. The acquired knots are given in columns 2 to 4 of rows 14 to 19 of Table 2. Ranking the alternatives shall be performed using classical (based on point estimates of probabilities) and fuzzy-rational generalized lotteries of I type. The Laplace expected utility criterion shall be used to approximate and rank the fuzzy-rational lotteries.

Table 2. Elicited subjective data for the three projects

l	$F_l^{x,(1)}$	$x_{d,l}^{(1)}$	$x_{u,l}^{(1)}$	$x_l^{(1)}$	$x_l^{aL,(1)}$	$x_l^{L,(1)}$
1	0	0	0	0	–	0
2	0.25	4000	12000	8000	4500	6000
3	0.5	5000	13000	9000	9000	9000
4	0.75	6000	14000	10000	13500	12000
5	1	18000	18000	18000	–	18000
l	$F_l^{x,(2)}$	$x_{d,l}^{(2)}$	$x_{u,l}^{(2)}$	$x_l^{(2)}$	$x_l^{aL,(2)}$	$x_l^{L,(2)}$
1	0	500	500	500	–	500
2	0.25	1500	9500	5500	5000	3500
3	0.5	5500	13500	9500	10000	9500
4	0.75	9500	17500	13500	14000	15500
5	1	18500	18500	18500	–	18500
l	$x_l^{(3)}$	$F_{d,l}^{p,(3)}$	$F_{u,l}^{p,(3)}$	$F_l^{p,(3)}$	$F_l^{aL,(3)}$	$F_l^{L,(3)}$
1	0	0	0	0	–	0
2	1000	0.01	0.12	0.065	0.057	0.016
3	5000	0.2	0.4	0.3	0.286	0.211
4	9000	0.45	0.75	0.6	0.514	0.467
5	14000	0.65	0.92	0.785	0.824	0.667
6	17500	1	1	1	–	1

6.1. Problem solving via classical risky generalized lotteries of I type

Classical risky generalized lotteries of I type require a classical distribution function. This could be provided by interpolation on the point estimates of the uncertainty intervals of the distribution function knots, calculated as mean values of those intervals. The former are given in column 5 of Table 2. The graphics of the distribution functions and the densities, constructed using the point estimates, are depicted on Fig. 2.

Then the expected utilities of the alternatives may be calculated using (20). Another possibility is to use the approximate algorithm to calculate the integral of expected utility of classical risky generalized lotteries of I type, proposed in (T e n e k e d j i e v [16]. The algorithm does not generate error in the integration of linearly interpolated utility function and density function (where the latter is acquired via differentiation of a linearly interpolated distribution function). The calculated expected utilities with this algorithm are: $E_1(u/F_1)$=0.683, $E_2(u/F_2)$=0.685, and $E_3(u/F_3)$=0.597. Since $E_2(u/F_2)>E_1(u/F_1)>E_3(u/F_3)$, then the preferences of the DM should be: at first place – Project 2, followed by Projects 1 and 3.

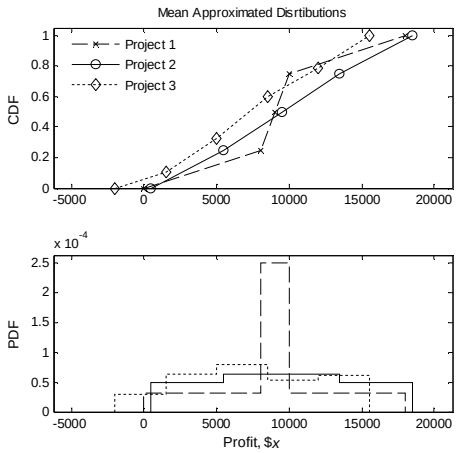

Fig. 2. Distribution function (CDF) and density (PDF) of prizes for the three alternatives, constructed on point estimates

6.2. Problem solving via fuzzy-rational generalized lotteries of I type

The data in columns 2 to 4 of Table 2 allow constructing two *x*-ribbon and one *p*-ribbon distribution functions over the prizes. In that case, the investment projects could be modeled by two *x*-fuzzy-rational (first and second alternative) and one *p*-fuzzy-rational (third alternative) generalized lotteries of I type. The procedures from Sections 4.2.2 and 5.1, and Sections 4.2.3 and 5.2 can be applied to rank lotteries from both types.

6.2.1. *Approximation of the x-ribbon distributions*

A first step in the approximation is to find the quantiles of complete ignorance for the probability distributions of the first two alternatives, using (34). For example, the first quantile of complete ignorance for the distribution function of the first alternative is $x_2^{aL,(1)} = \hat{x}_{d,1}^{(1)} + (\hat{x}_{d,5}^{(1)} - \hat{x}_{d,1}^{(1)})F_2^{(1)} = 0+(18000-0)0.25=4500$.

All other quantiles are calculated in the same fashion, and are given in the sixth column of Table 2 in rows 2 to 6 and 8 to 12. It is now possible to calculate the required Laplace quantiles, using (35). For example, the first quantile for the distribution function of the first alternative is $\hat{x}_2^{L,(1)} = \hat{x}_{d,2}^{(1)} + (\hat{x}_{u,2}^{(1)} - \hat{x}_{d,2}^{(1)})F_2^{(1)} =$ =4000+(12000–4000)0.25=6000. All other quantiles are calculated in the same fashion, and are given in the column 7 of Table 2 in rows 2 to 6 and 8 to 12. Then both *x*-ribbon distributions may be approximated on the acquired knots. The graphics of the Laplace approximated distribution functions for both alternatives, along with their densities, are given on Figs. 3 and 4.

6.2.2. *Approximation of the p-ribbon distribution*

The quantile indices of complete ignorance for the probability distributions of the third alternative could be calculated using (36). For example, the first quantile index of complete ignorance is $F_2^{aL,(3)} = \dfrac{x_2^{(3)} - x_1^{(3)}}{x_6^{(3)} - x_1^{(3)}} = \dfrac{1000-0}{17500\text{-}0} = 0.057$. All other quantile

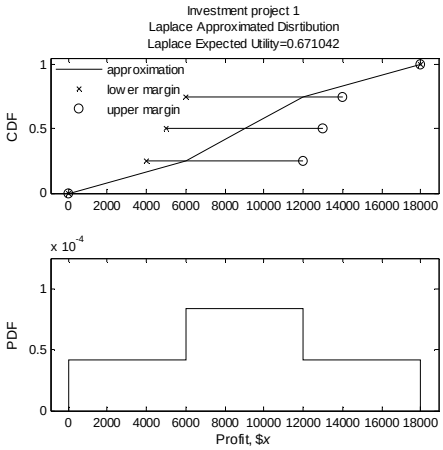

Fig. 3. Distribution function (CDF) and density (PDF) of prizes for the first alternative

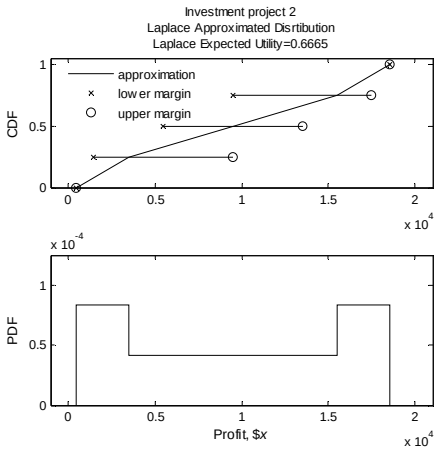

Fig. 4. Distribution function (CDF) and density (PDF)of prizes for the second alternative

indices are calculated in the same fashion, and are given in column 6 of Table 2 in rows 14 to 19. It is now possible to calculate the required Laplace quantile indices, using (37). For example, the first quantile index shall be

$$\hat{F}_2^{pL,(3)} = \hat{F}_{d,2}^{(3)} + (\hat{F}_{u,2}^{(3)} - \hat{F}_{d,2}^{(3)}) \frac{x_2^{(3)} - x_1^{(3)}}{x_6^{(3)} - x_1^{(3)}} = 0.01 + (0.12 - 0.01)[(1000 - 0)/(17500 - 0)] = 0.016.$$

All other quantile indices are calculated in the same fashion, and are given in column 7 of Table 2 in rows 14 to 19. Then the *p*-ribbon distribution may be approximated on the acquired knots. The graphics of the Laplace approximated distribution function for the third alternative, along with its density, is given on Fig. 5.

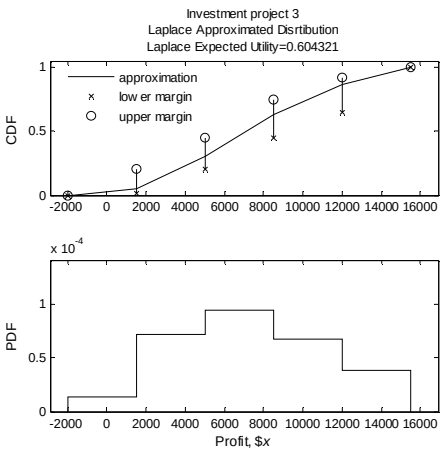

Fig. 5. Distribution function (CDF) and density (PDF)of prizes for the third alternative

6.2.3. *Calculation of the Laplace expected utility*

The *xL*-expected utility of the first two alternatives are calculated using (28), and the data in Table 1 and 2, and are: $E_1^{xL}(u/F_1^{xR}) = 0.671$, $E_2^{xL}(u/F_2^{xR}) = 0.667$. The *pL*-expected utility of the third alternative is calculated using (33) and the data in Tables 1 and 2, and is $E_3^{pL}(u/F_3^{pR}) = 0.604$. Since $E_1^{xL}(u/F_1^{xR}) > E_2^{xL}(u/F_2^{xR}) > E_3^{pL}(u/F_3^{pR})$, then the preferences of the DM should be: at first place – Project 1, followed by Projects 2 and 3.

The example case demonstrated that the final ranking is dependent on the proper modeling of unquantified uncertainty. In our example the ranking of alternatives is

different when the unquantified uncertainty is approximated according to Laplace expected utility criterion from the case when unquantified uncertainty is neglected by the expected utility criterion applied on middle points of uncertainty intervals.

7. Conclusions

The paper dealt with the fuzzy rationality of real DMs and its influence on the construction of probability distributions. Two methods were outlined to construct the latter using linear interpolation on several elicited knots. Those envisaged the elicitation of either quantiles or quantile indices. Due to the reality of DMs both methods resulted in uncertainty intervals of the required estimates. Because of that ribbon distribution functions were introduced, which depending on the type of uncertainty intervals (on prizes or on probabilities) could be *x*-ribbon and *p*-ribbon ones.

1D generalized lotteries of I type envisaged probability distributions of the prize. Due to the use of ribbon functions, the classical risky generalized lotteries of I types transformed into fuzzy-rational ones of either *x*-fuzzy-rational or *p*-fuzzy-rational type. A two-step procedure to rank those was formalized in both cases. It envisaged approximation of the ribbon functions by classical, partially linear ones in order to apply expected utility. The Laplace criterion under strict uncertainty was proposed for that purpose, and its use was formalized both for *x*-ribbon and *p*-ribbon functions. Quantiles and quantile indices of complete ignorance were introduced to facilitate the approximation of the ribbon distributions with classical distribution functions. The Laplace expected utility criterion was introduced, and formalized both for the case of *x*-fuzzy-rational and *p*-fuzzy-rational generalized lotteries of I type. An example was developed which demonstrated how generalized lotteries could be ranked in the classical case (with point estimates of quantiles/quantile indices, which are the mean values of the uncertainty intervals) and in the fuzzy-rational case (i.e. using Laplace approximation and Laplace expected utility). All calculations of Laplace expected utility in the example were performed with the help of a specially constructed MATLAB program function, which is available free upon request from the authors. The example results showed that problem modeling and more detailed analysis of the information available are crucial for the final decision.

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