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An Approach for Searching Unsupported Integer Solutions of the Bicriteria Network Flow Problem

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Abstract: *An approach for searching unsupported integer solutions of the bicriteria network flow problem is suggested. Some of the integer points in the triangles, defined by two neighbouring extreme nondominated points, are investigated by appropriately defined problems for a flow with one side constraint and for minimal cost flow.*

Keywords: *bicriteria network flow, Pareto optimal solutions, unsupported integer solution, one side constraint.*

1. Introduction

The constant interest towards the problems of flows in networks comes from their wide practical use and also from the “challenge” theoretical characteristics. The linear problem for minimal cost flow is decided with effective algorithms and has an integer solution, when the parameters of the problem are integer. But these characteristics could be lost with a slight generation in the constrains of the problems – defining coefficients different than ±1,0 in the flow conservation equations, or one linear sight constraint. The specialized linear algorithms, which decide general problems, use the network structure, embedded in the constraint matrix, but their complexity is not polynomial and the solutions are non-integer. The author is not familiar with existing specialized algorithms for searching integer solutions of these problems.

Although the problem for multicriteria flow optimization has simple flow constraints, the searching for decisions is complicated by the fact, that even in the continuous biobjective case they could be with exponential number. .

This paper concerns the problem of finding unsupported integer solutions of bicriteria network flow problem. In [1] three approaches are presented, in which appropriate flow problems are defined to generate these solutions. In the present paper an attempt is made, given an integer point in the triangle, determined by two neighbouring extreme nondominated points in the objective space, to determine, if there is such an integer flow in the network for which the values of the objective functions equals the coordinates of the integer point.

2. Formulation of the problem

Let the problem BCF of bicriteria network flow X on the network $G = \{N, U\}$ be defined. Here the set N consists of n nodes and a finite set U – of m directed arcs (i, j), $i, j \in N$, $X = (x_{ij}, (i, j) \in U)$. There are defined "cost" parameters c_{ij} and d_{ij} which are associated with each arc $(i, j) \in U$. The flow on the arc (i, j) is designed with $x(i, j) = x_{ij}$ The bicriteria flow problem on G is stated as follows:

$$\textbf{BCF:} \ \min^* (g_1(X), g_2(X))$$

subject to

$$(1) \qquad \sum_{j \in N} x_{ij} - \sum_{j \in N} x_{ji} = \begin{cases} v & \text{if } i = 1, \\ 0 & \text{if } i \neq s, t, \\ -v & \text{if } i = n; \end{cases}$$

$$(2) \qquad 0 \leq x_{ij} \leq u_{ij}, \ (i, j) \in U;$$

where the node 1 is the source node and the node n is the terminal node (the sink),

$$g_1(X) = \sum_{(i, j) \in U} c_{ij} x_{ij} ; \qquad g_2(X) = \sum_{(i, j) \in U} d_{ij} x_{ij},$$

$$v \leq v^*,$$

and v^* is the value of the maximal flow. The set of constraints (1)-(2) determines the set of feasible solutions $X(G, v, u)$. The functions g_i are called criteria or objectives. Every solution of the problem BCF is Pareto optimal and will be called PO solution.

When a condition for integrality of the flow is added to constraints (1) and (2), a problem of the integer bicriteria network flow (BCIF) is stated. It is true that any integer solution of BCF is a solution of BCIF, but not reverse.

The set $\{(g_1(X), g_2(X)): X \in X(G, v, u)\}$ is called objective space. The images of the PO solutions in this space are called nondominated solutions.

For problem BCF the set of nondominated solutions forms a piecewise linear and convex set, called efficient frontier (Ef). For BCIF when a nondominated integer solution lies on the Ef, it is called supported, otherwise – unsupported.

Let the flows X and Y be two neighbouring basic PO solutions in the set $X(G, v, u)$.

We denote $X_i = g_i(X)$ and $Y_i = g_i(Y)$ for $i = 1, 2$. The points $A_X = (X_1, X_2)$ and $A_Y = (Y_1, Y_2)$ are the images of the points X and Y in the criteria space (see Fig.1). They are breakpoints of Ef. The points A_X and A_Y are called *extreme nondominated points*.

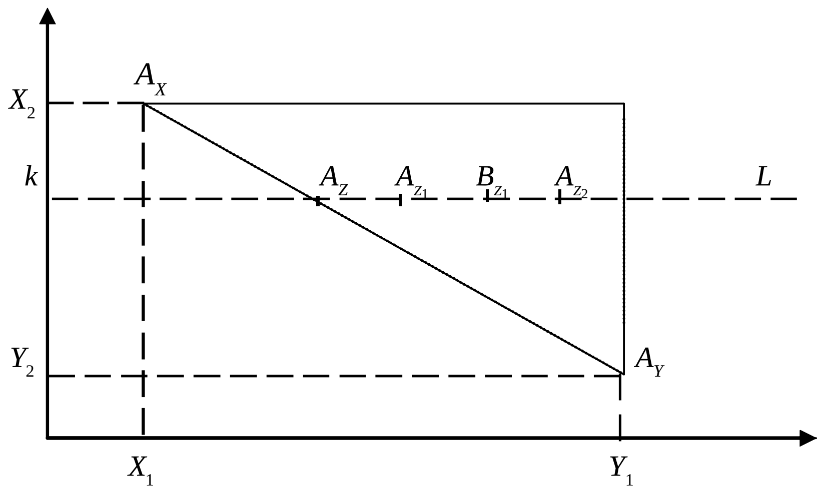

Fig. 1

3. Searching for unsupported nondominated solutions

Let the flows X and Y be two neighbouring basic PO solutions of the problem BCF. Let T_X and T_Y be the corresponding to X and Y basic trees. The tree T_Y is obtained by adding an appropriate arc (p, q) to the tree T_X, sending an integer flow of value ε along the cycle σ, formed by T_X and (p, q) and excluding an appropriate arc (a, b) from this cycle. If $g_1(\sigma)$ and $g_2(\sigma)$ denote the costs of σ, relative to the arc's costs in the objectives g_1 and g_2 respectively, it is true that $g_1(\sigma) > 0$ and $g_2(\sigma) < 0$.

Let $X_1 < Y_1$ and $X_2 > Y_2$ and $t = X_2 - Y_2 - 1$. Let A_i, $i=1,\ldots, t$, be points from the segment (A_X, A_Y), for which :

$A_i = (g_1(Z_i), g_2(Z_i))$, Z_i is a PO flow and $g_2(Z_i) = X_2 - i$, $i = i = 1,\ldots, t$, i. e., $g_2(Z_i)$ is integer valued.

Some of these points are integer, its corresponding flows are received from X by sending flows with values 1, …, ε–1 along the cycle σ, i.e., they are supported integer nondominated solutions [4]**.** The number of the remaining points is $t - \varepsilon + 1$. The corresponding flows of these points are obtained as convex linear combinations of the flows X and Y.

Let us fix our attention on some of these points, denoting it by $A_Z = (g_1(Z), g_2(Z))$. A structure T_{XY} corresponds to the flow Z. It is an union of the basic tree T_X and the arc (p, q), i.e.

$$T_{XY} = T_X \cup \{(p, q)\}.$$

For the flow Z is accomplished:

$$Z = X + X(\sigma).$$

$X(\sigma)$ is a flow along the cycle σ for which the value v_σ is not an integer number but the product $g_2(\sigma).v_\sigma$ is.

Let the line L be drawn across the point A_Z and parallel to the axis $g_1(X)$. For the points $(x, y) \in L$, the following is accomplished: $y = g_2(Z) = k$. There is no more than one unsupported integer nondominated solution of the problem BCF on the line L.

A flow problem OSC with one side constraint is defined:

$$\text{OSC: } \min g_1(X)$$

(3) $$\text{s.t. } X \in X(G, v, u)\},$$

(4) $$g_2(X) = k.$$

Following the statements in [2] and [3] we conclude, that Z is a basic optimal solution of the problem OSC. This basis consists of the tree T_X and the loop arc (p, q), i.e. $T_X = T_{XY}$.

Beginning from Z like an initial basic solution and minimizing the function $-g_1(X)$ over (3) and (4) by the primal algorithm, we move on the basis of the problem OSC which lies on L.

Let us determine two neighbouring basic solutions Z_1 and Z_2 for which at least one integer point (c, d) lies on the interval, determined by them. We determine the flow W, corresponding to (c, d) like a convex linear combination of the basic flows Z_1 and Z_2. (If some of Z_1, Z_2 or W is integer, the search on the line L is ended.) We define a minimal cost flow problem MF, for which W is an optimal solution using the reduced cost coefficients of the basis of Z_1 and changing the flow capacities of some nonbasic arcs and a loop arc. We find all optimal solutions of MF determining all circulations with zero cost in the incremental graph for W. We choose those of these circulations, which have zero cost relative to g_1 and g_2. If there is an integer solution, the search on the line L is ended. In the opposite case we continue this search investigating the integer point on the right of (c, d). If a point with integer corresponding flow on L is determined, it is included in a list for nondominance testing. The search continues on the line L, for which Z is not integer and $g_2(Z) = k - 1$.

The formal content of the suggested procedure applied to one triangle determined by two neighbouring extreme nondominated points, is as follows:

For every two neighbouring extreme nondominated point $A_X = (X_1, X_2)$ and $A_Y = (Y_1, Y_2)$, $X_1 < Y_1$, $X_2 > Y_2$ with corresponding basic trees T_X and T_Y, $T_Y = T_X \cup (p, q) \backslash (a, b)$ define:

$$L_i = 0, \; i = 1.$$

1. Let $A_Z = (Z_1, Y_2 - i))$ be a point from the segment (A_X, A_Y).

2. Determine λ: $\lambda X_2 + (1 - \lambda) Y_2 = X_2 - i$; denote $X_2 - i = k = Z_2$.
Determine Z_1: $Z_1 = \lambda X_1 + (1 - \lambda) Y_1$.
If Z_1 is not integer, go to 3. Otherwise $i = i + 1$.
If $i = X_2 - Y_2$, end; otherwise go to 1.

3. Determine Z: $Z = \lambda X + (1 - \lambda) Y$.

4. Let $\pi = c_B B^{-1}$, $v = d_B B^{-1}$ are node potentials corresponding to T_X and to objectives. B^{-1} is a basic matrix corresponding to T_X.

Calculate:

$$c_{ij} = c_{ij} + \pi_i - \pi_j,$$
$$d_{ij} = d_{ij} + \pi_i - \pi_j,$$
$$\delta = |k / d_{pq}|,$$
$$\alpha = -c_{pq}/d_{pq}.$$

5. Begin to solve the problem OSC with initial basic solution Z:

$$\text{OSC:} \quad \min(-g_1(X))$$
$$\text{s.t. } X \in X(G, v, u),$$
$$g_2(X) = k.$$

6. Let the flows Z_1 and Z_2 be two neighbouring basic solutions of OSC, for which there is an integer point $B_Z = (k_1, k)$ (the most left) in the segment $[A_{Z1}, A_{Z2}]$, $A_{Z1}=(g_1(Z_1), g_2(Z_1))$, $A_{Z2}=(g_1(Z_2), g_2(Z_2))$.

7. Let (p, q) denote the loop arc and (r, s) is the incoming arc.

The flow W, $g_2(W) = k$, for the point B_Z is determined like a convex linear combination of flows Z_1 and Z_2.

If W is integer, go to 10.

8. We define a new network $G_Z = \{N, U_Z\}$ in such a manner: for all nonbasic arcs (i, j) (with respect T_{Z1}), we determine:

a) if $w_{ij}=0$, and $c_{ij} < 0$, then a lower bound $l_{ij} = 0$ and an upper bound $u_{ij}=0$, i.e., we eliminate the arc (i, j);

b) if $w_{ij} = u_{ij}$, and $c_{ij} > 0$, then a lower bound $l_{ij} = u_{ij}$ i. e., we hold the flow on arc (i, j) equal to its upper boud;

c) if $c_{pq} < 0$, then $u_{pq}=w_{pq}$; if $c_{pq} > 0$, then $l_{pq}=w_{pq}$;

d) if $c_{rs} < 0$, then $u_{rs}=w_{rs}$; if $c_{rs} > 0$, then $l_{rs}=w_{rs}$.

The set of the constraints imposed on the flow in G_Z is $X(G_Z, v, u)$.

The flow W is an optimal solution of the problem MF:

$$\text{MF:} \quad \min \sum_{(i,\, j)\in U_Z} (c_{ij} + \alpha\, d_{ij}) x_{ij},$$

$$\text{s.t} \quad X \in X(G_Z, v, u)\}.$$

9. We find all optimal solutions of MF determining all circulations in the incremental graph $G_Z(W)$ with zero cost. If there is an integer optimal solution of MF received from W and from circulations which zero cost with respect to g_1 and g_2, go to 10.

The point B_Z receives the status “investigated”.

If there are other noninvestigated points in the segment $[A_{Z1}, A_{Z2}]$, the most left is denoted by B_Z, go to 7.

If $g_1(Z_2) < g_1(Y)$, go to 6.
$i = i + 1$, go to 1.

10. $L_i = L_i \text{U}((g_1 (W), g_2 (W))$, make a nondominance test.

3. Conclusion

The method proposed has more theoretical, than practical applications, describing the characteristics of the nonsupported integer solutions of the problem of the bicriteria network flow. The procedure is complicated by the fact that for each investigated integer point, all optimal solutions of the problem for minimal flow are searched for.

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