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Estimating the Minimum of a Function over the Efficient Set of a MOLP Problem – Some Experiments

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Abstract: *The MOLP problem is considered together with a linear function* φ *defined over the feasible set* $S \subseteq R^n$ *. A procedure is proposed for estimating the minimal value of* φ *over the efficient set* $E \subseteq S$ *using the reference point method. Some extensions of the procedure are proposed, too, and a short discussion is added. Three numerical examples are presented.*

Keywords: *Multiobjective linear programming, Optimization over the efficient set, Reference point method.*

1. Introduction

The multiobjective linear programming (MOLP) problem can be presented in the following way:

$$
\begin{array}{ll}
 & \max\ f_1(x) \\
 & \max\ f_2(x) \\
(1) & \quad \dots \\
 & \max\ f_m(x) \\
\text{s.t.} & \\
 & x \in S \subseteq R^n.
\end{array}
$$

Here $f_i(x)$, $i = 1, 2, \dots, m,$ are linear functions, they are the optimization criteria in MOLP problem (1). The vector $x \in S$ is called an argument vector and the vector $f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in R^m$ is called a criteria vector. The set S is called a decision space or feasible set in R^n. It is defined as follows

$$S = \{x \in R^n \,|\, c_i(x) \le 0,\ i = 1, 2, \dots, k\}.$$

In MOLP problems all $c_i(x)$ are linear functions. We will consider the list of constraints $c_i(x) \leq 0$, $i = 1, 2, \ldots, k$, as containing the inequalities $x_j \geq 0$ for all $j = 1, 2, \ldots, n$. The set S is nonempty and bounded.The set

$$Z = \{z \in R^m \mid z = f(x)\,,\ x \in S\}$$

is called an objective space or criteria space.

The point $z^1 = f(x^1) \in Z$, $x^1 \in S$, is called a nondominated (Pareto) point if there does not exist a point $x^2 \in S$, $x^2 \neq x^1$, such that the following two conditions are fulfilled simultaneously

$$f_i\,(x^2) \geq f_i\,(x^1) \text{ for all } i = 1, 2, \ldots, m;$$
$$f_j\,(x^2) > f_j\,(x^1) \text{ for one } j \text{ at least.}$$

If we have $z^1 = f\,(x^1)$, $z^1 \in Z$, $x^1 \in S$, and z^1 is nondominated, then the point x^1 is called an efficient point. The set $P \subseteq Z$ of all nondominated points in Z is called a nondominated (Pareto) set. The set $E \subseteq S$ of all efficient points in S is called an efficient set. For MOLP problems this set is closed.

Having in mind MOLP problem (1) and supposing that $\varphi\,(x)$ is a linear function on S, we will consider here the problem

$$\min_{x \in E} \varphi\,(x) = B \tag{2}$$

We will propose some ways to estimate the value of B. It is difficult to solve problem (2) directly because the set E is not convex.

2. A short review of the literature

Many papers describing methods for optimization over the set E can be found in the periodicals. Some of the first results are based on the idea to organize a movement in the set of efficient extreme points only. In the next years many attempts have been made to apply various optimization techniques for solving or analyzing problem (2). The survey of Yamamoto [19] contains a large amount of information (45 cited papers). Following the development of the ideas in the field, the author obtains as a result a classification of the existing algorithms for optimization over the efficient set. This classification contains seven classes:

- – adjacent vertex search algorithm;
- – nonadjacent vertex search algorithms;
- – face search algorithms;
- – branch and bound search algorithms;
- – lagrangean relaxation based algorithms;
- – dual approach;
- – bisection algorithms.

In Yamamoto’s paper each class is presented with one typical algorithm and these algorithms are compared with respect to the computational requirements.

Dauer [4] founds his work on the idea that the important case is when the efficient solutions are on the frontier of S .With the purpose to optimize over the set E

he uses the nondominated structure of the set $f(S)$ (corresponding to $E \subseteq S$). He proposes to solve a nonlinear programming problem (having a nonlinear constraint) and develops a method that uses only a portion of the function that forms the nonlinear constraint.

The utility function approach is used in the paper of Horst and Thoai [6]. They give a set of conditions that must be satisfied in order to use the utility function. The obtained solutions are ε-approximate.

D. J.White [18] gives several equivalent formulations of problem (2). For the case when $\varphi(x)$ is a linear function he describes an approach that uses a penalty function. Some computational aspects as well as ε-efficiency are discussed. Possible nonlinear extensions are pointed out.

Thoai [13] considers a special quasiconvex function of the criteria f_i and proposes a method based on maximization of this function. He outlines a class of problems where his method works satisfactorily.

A branch and bound type algorithm is proposed in the paper of Yamada, Tanino and Inuiguchi [21] for maximization of a concave function in a problem similar to problem (2).

We can mention here the papers concerning finding or estimating the nadir point in MOLP problems. Such procedures are of interest because to find the nadir point – this is a special case of optimizing over the efficient set. Some methods are cited in Miettinen [11] and Steuer [12]. The paper of Korhonen, Salo, Steuer [7] proposes to use the reference direction method for determining or estimating the nadir point.

The reference point method is chosen here for handling the problems connected with the nonconvexity of the set E. Several computational procedures are proposed that give upper bounds of the needed value B.

3. An auxiliary LP problem

Here we will not describe in details the reference point method proposed by Wierzbicki [16, 17]. Some information about this method may be found in Miettinen [11], Steuer [12], Vincke [14], too. With respect to problem (1) the reference point method recommends to solve the following LP problem

$$
\tag{3}
\begin{gathered}
\min\ D \\
\text{s.t.} \\
D \geq b_i\,(r_i - f_i(x)) \;-\; l \sum_{j=1}^{m} f_j(x), \quad i=1,\dots,m \\
x \in S.
\end{gathered}
$$

Here the set S and the functions $f_i(x)$ are defined as in problem (1), the coefficients b_i are positive real numbers for all i and l is a small positive number. The variable D can have positive or negative values. This LP problem has the following remarkable property: for an arbitrary reference point $r \in R^m$ the obtained solution of problem (3) determines an efficient point in the decision space of problem (1) (a nondominated point in the criteria space of the same problem).

4. An algorithm for estimating the minimal value of $\varphi(x)$ over the efficient set

Having in mind MOLP problem (1), we will use the notion of *a wall of the set* S. Remember that the set S in problem (1) is described by the constraints $c_i(x) \leq 0$, $i = 1, 2, \ldots, k$, and here the constraints $x_j \geq 0$ (for all $j = 1, 2, \ldots, n$) are included. Let the constraints $c_i(x) \leq 0$, $i = 1, 2, \ldots, p$, $p \leq k$, are not redundant and constitute the set S. Consider now the corresponding sets W_j where

(4) $$W_j = \{ x \in S \mid c_j = 0 \}, \quad j = 1, 2, \ldots, p.$$

Each one of these sets is *a wall of the set* S.

It must be noted that there is a more general notion of a *facet*. A definition of this notion can be found in Steuer [12]. So, each wall is a facet, but there can be a facet that is not a wall.

As we know in MOLP problems if an interior point of S is efficient, then all of S is efficient [12]. So we will suppose that each point $x^e \in E$ is a point from the frontier of S, i.e. for each point x^e we have $x^e \in W_j$ for some $j = 1, 2, \ldots, p$.

The main idea of the algorithm can be expressed as follows. For a fixed wall W_t we solve the problem $\min \{\varphi(x) \colon x \in W_t\}$. If the obtained solution x^t is an efficient point, it gives an estimate $\varphi(x^t)$ of B. This estimate is an *upper bound* of B. If x^t is not an efficient point, then the point $f(x^t)$ is used as a reference point in problem (3). The solution of the so formulated problem (3) is an efficient point that gives an estimate of B (an upper bound again). We repeat this procedure with all walls W_t. The minimal of the corresponding upper bounds is the obtained estimate of B.

Here below the algorithm is presented. The checking for efficiency of the solution of $\min \{\varphi(x) \colon x \in W_t\}$ is skipped because if $x^e \in E$ and $r_i = f_i(x^e)$, $\forall i$, then the solution x^s of problem (3) satisfies the equality $x^s = x^e$. So, it is sufficient to use problem (3) only.

The algorithm (version 1)

Let $W_0 = S$, W_i, $i = 1, 2, \ldots, p$, are the walls of S and u denotes the number of the current step.

```
1. Begin
   Set $u := 0$
2. Solve the problem $\min_{x \in W_u} \varphi$. The obtained solution is $x^{au}$.
3. Set $r_i = f_i(x^{au})$ and solve problem (3). The obtained solution is $x^{bu} \in E$.
4. Set $d_u = \varphi(x^{bu})$.
5. Check whether $u < p$.
   If $u < p$, then set $u := u + 1$, Go to 2.
   If $u = p$, then 6.
6. End of the algorithm.
```

With this algorithm we have the estimate

$$\min_{x \in E} \varphi \leq \min_{u} d_u, \quad u = 0, 1, 2, \ldots, p.$$

5. Numerical example

Example 1. For illustrative purposes we will consider example 8 from [12], p. 244. The additional data given by Steuer allow to estimate the work of the algorithm very easily. The example is defined as follows:

		x_1	x_2	x_3	x_4	x_5	
	f_1:	1	3	–2		1	max
Objs	f_2:	3	–1		3	1	max
	f_3:	1		2		3	max
s.t.	c_1:	2	4			3	$\leq$ 27
	c_2:			2	5	4	$\leq$ 35
	c_3:	5					$\leq$ 26
	c_4:				2		$\leq$ 24
	c_5:	5	5	2			$\leq$ 36

In addition: $x_i \geq 0$, $i = 1, 2, \ldots, 5$.

We will consider the following function φ:

$$\varphi(x) = 2f_2 + 4f_3 - f_1$$

Table 1 contains the list of the extreme nondominated points (in the criteria space) taken from [12]. The data here slightly differ from the original because we give more digits after the decimal point.

Table 1

	f_1	f_2	f_3	φ
z^1	20.25	14.25	0.00	8.25
z^2	19.80	17.40	0.90	18.60
z^3	9.31	8.675	26.25	113.04
z^4	14.06	30.583	13.816	102.37
z^5	9.12	12.00	26.25	119.88
z^6	10.7330	28.853	21.80	134.173
z^7	11.20	34.60	5.20	78.80
z^8	–1.2578	20.2648	34.04	177.9474
z^9	5.2	36.60	5.2	88.80
z^{10}	0.733	22.853	31.80	172.173
z^{11}	–34.80	0.60	35.20	176.8

The last column in Table 1 contains the corresponding values of φ. The table shows that z^1 is the best nondominated extreme point ($\varphi = 8.25$).

There is a list of walls W_i of S for the example.

$$\begin{aligned}
W_1 &= \{x \in S \mid c_1 = 27\},\\
W_2 &= \{x \in S \mid c_2 = 35\},\\
W_3 &= \{x \in S \mid c_3 = 26\},\\
W_4 &= \{x \in S \mid c_4 = 24\} \text{– this set is empty,}\\
W_5 &= \{x \in S \mid c_5 = 36\},
\end{aligned}$$

$$\begin{aligned}
W_6 &= \{x \in S \mid x_1 = 0\},\\
W_7 &= \{x \in S \mid x_2 = 0\},\\
W_8 &= \{x \in S \mid x_3 = 0\},\\
W_9 &= \{x \in S \mid x_4 = 0\},\\
W_{10} &= \{x \in S \mid x_5 = 0\}.
\end{aligned}$$

The algorithm works as follows. Solving the problem $\min\{\varphi : x \in S\}$ we obtain the point $x^{0s} \in S$ and $f(x^{0s}) = (20.25, -6.75, 0.00)$ in the criteria space. Using this point as a reference point in problem (3) we obtain the point $x^{0e} \in E$ and $f(x^{0e}) = z^1 = (20.25, 14.25, 0.00)$ in the criteria space, too, and the corresponding value $\varphi(x^{0e}) = 8.25$. The obtained points in the criteria space and the obtained value of φ are placed in the first row of Table 2.

Solving the problem $\min\{\varphi : x \in W_1\}$ we obtain the point $x^{1s} = x^{0s}$ and $f(x^{1s}) = (20.25, -6.75, 0.00)$. Solving problem (3) we obtain again the point z^1 in the criteria space (of course) and the value $\varphi = 8.25$. The obtained points in the criteria space and the corresponding value of φ are in the second row of Table 2.

Solving $\min\{\varphi : x \in W_2\}$ we obtain directly the point z^1. The corresponding results are in the third row of Table 2.

Proceeding in the same way and summarizing the results we obtain the whole Table 2.

Table 2

The checked set	$f(x^{is})$	$f(x^{ie})$	$\varphi(x^{ie})$
S	(20.25, –6.75, 0.0)	(20.25, 14.25, 0.0)	8.25
W_1	(20.25, –6.75, 0.0)	(20.25, 14.25, 0.0)	8.25
W_2	(20.25, 14.25, 0.0)	(20.25, 14.25, 0.0)	8.25
W_3	(11.2, 13.6, 5.2)	(15.96, 18.36, 9.96)	60.598
W_4	$W_4 = \varnothing$		
W_5	(19.8, –3.6, 0.9)	(19.85, 14.04 , 0.95)	12.046
W_6	(20.25, –6.75, 0.0)	(20.25, 14.25, 0.0)	8.25
W_7	(0.00, 0.00, 0.00)	(14.24, 14.24, 14.24)	71.227
W_8	(20.25, – 6.75, 0.0)	(20.25, 14.25, 0.0)	8.25
W_9	(20.25, – 6.75, 0.0)	(20.25, 14.25, 0.0)	8.25
W_{10}	(20.25, –6.75, 0.0)	(20.25, 14.25, 0.0)	8.25

In Table 2 the data for W_3, W_5 and W_7 are made round, but the changes are not significant. We see that the algorithm finds the needed value very surely. The result for W_5 is very good, too. The obtained results are relatively larger for two cases (W_3 and W_7) but Table 1 contains two points only that are better – points z^1 and z^2.

6. Some comments

Our computational experience shows very good behaviour of this algorithm. An application of this algorithm to the problem of estimating the nadir point in MOLP problems is described in [9]. The algorithm can be implemented without using any special optimization techniques. For MOLP problems it is sufficient to use standard LP programs.

We obtain feasible points $x \in S$ only at each step of the algorithm. This allows to use a Pareto test instead of the reference point method. If the point checked by the test is efficient, the solution of the test determines the same point. If the checked point is not efficient, the solution of the test determines an efficient point.

The proposed method gives an upper bound only for the needed minimal value and usually this bound is close to the minimal value.

7. Some advanced versions of the algorithm

It can be seen that in the proposed algorithm the walls are used for obtaining points with small value of φ. Then, in general, it is possible to use other ways for obtaining such points. For example, consider the linear function $\psi(x)$ and the numbers d_g, $g = 1, 2, \ldots, q$, such that

$$d_1 = \min_g d_g, \quad d_q = \max_g d_g$$

and

$$\min_{x \in S} \psi(x) < d_1 < d_2 < \ldots < d_q < \max_{x \in S} \psi(x).$$

Now we can consider the sets A_g

$$A_g = \{x \in S \mid \psi(x) = d_g\}, \ g = 1, 2, \ldots, q.$$

It is evident that $A_g \neq \varnothing$ for all g. In a version of the algorithm we replace the walls W_i with the sets A_g, $g = 1, \ldots, q$, and we solve the problem

$$\min \{\varphi : x \in A_g\} \text{ for all } g.$$

The obtained solutions are x^g and the corresponding points in the criteria space are $f(x^g)$. Then the points $f(x^g)$ are used as reference points in problem (3) and the obtained solutions are the efficient points x^{eg}. Now we have the estimate

$$\min_{x \in E} \varphi \le \min_g \varphi(x^{eg}), \ g = 1, 2, \ldots, q.$$

In such a version of the algorithm we are free to choose the numbers d_g as well as the function $\psi(x)$. The only condition is $A_g \neq 0$ for all g.

Another version of the algorithm can be obtained based on the following reasons. We use the sets A_g with the purpose to obtain the points $x^g \in S$ and the points $f(x^g) \in f(S)$. But the reference points can be everywhere in R^m. So, having the intention to determine a series of reference points we can use a set $S_1 \neq S$, $S_1 \subset R^n$. We suppose that S_1 is bounded and closed. (The function φ must be correspondingly defined, of course.) Now we define another series of sets C_g:

$$C_g = \{x \in S_1 \mid \psi(x) = d_g\}, \ g = 1, 2, \ldots, q.$$

Here $\psi(x)$ and d_g are defined like above and the only condition is $C_g \neq \varnothing$ for all g. Solving the problem $\min \{\varphi : x \in C_g\}$ for all g we obtain the series $x^g \in R^n$, $g = 1, 2, \ldots, q$, and the corresponding series of reference points $f(x^g) \in R^m$. Having this series we follow the rest part of the algorithm.

The algorithm (version 2)

```
1. Begin
Set $u := 1$.
2. Solve $\min_{x \in C_u} \varphi$. The obtained solution is $x^{1u}$. (It is possible that $x^{1u} \notin S$).
3. Solve problem (3) where $r_i = f_i(x^{1u})$, $i = 1, 2, \ldots, m$. The obtained solution is $x^{2u} \in E$.
4. Set $d_u = \varphi(x^{2u})$
5. Check whether $u < q$. If $u < q$, then set $u := u + 1$, Go to 2.
If $u = q$, then 6.
6. End of the algorithm.
```

So we obtain the estimate

$$\min_{x \in E} \varphi \le \min_{u} d_u, \quad u = 1, 2, \ldots, q.$$

Here some very natural questions arise: what are the good ways to choose the set S_1, the function $\psi(x)$ and the numbers d_g? Now we cannot give full answers to these questions. But if $\psi(x)$ is a linear function and the set S_1 satisfies the condition

$$f(S_1) \supset f(S),$$

then the reference points obtained by the last version of the algorithm are outer points for $f(S)$. In this case the obtained nondominated points $f(x^{2u})$ (Pareto points) have a minimal Tchebychev distance to the corresponding reference points $f(x^{1u})$.

Of course, increasing the number q (adding new points to the already inspected) we cannot make worse the obtained solution. For the second version of the algorithm we have the freedom to choose the set S_1, the function $\psi(x)$, and the numbers d_g under a very weak condition: $C_u \neq \varnothing$ for all u.

8. Some other examples

Example 2. Dauer [4] has considered the following MOLP problem:

$$\begin{array}{ll} & \max\ f_1(x) = 9x_1 + x_3, \\ & \max\ f_2(x) = 9x_2 + x_3, \\ \text{s.t.} & \\ & 9x_1 + 9x_2 + 2x_3 \le 81, \\ & 8x_1 + x_2 + 8x_3 \le 72, \\ & x_1 + 8x_2 + 8x_3 \le 72, \\ & 7x_1 + x_2 + x_3 \ge 9, \\ & x_1 + 7x_2 + x_3 \ge 9, \\ & x_1 + x_2 + 7x_3 \ge 9, \\ & x_1 \le 8,\ x_2 \le 8, \\ & x_i \ge 0,\ i = 1, 2, 3. \end{array}$$

We add to this example the function

$$\varphi(x) = 5x_1 + 3x_2 + x_3$$

and we wish to estimate the value

$$\min_{x \in E} \varphi(x).$$

Table 3 contains all the efficient extreme points for the example (given by Dauer) and the corresponding values of φ (in the last column)

Table 3

x_1	x_2	x_3	$\varphi(x)$
0.8	8	0.9	28.9
1	8	0.0	29
8	1	0.0	43
8	0.8	0.9	43.3
4	4	4.5	36.5
0.0	8	1	25
8	0.0	1	41

The minimal value of φ over S is equal to 9 and is obtained at point (1, 1, 1). This point is not efficient.

With this example we would like to illustrate the possibility to use a set S_1 containing S as a proper subset.. Minimizing the function φ on the walls of S_1 we obtain some points from the criterion space that do not belong to $f(S)$. Using these points as reference points in problem (3) we obtain Pareto points that are close to the reference points (in general).

We shall replace the set S, described in the example under consideration with the following set S_1:

$$\begin{gathered} 9x_1 + 9x_2 + 2x_3 \leq 96, \\ 8x_1 + x_2 + 8x_3 \leq 88, \\ x_1 + 8x_2 + 8x_3 \leq 88, \\ 7x_1 + x_2 + x_3 \geq 4, \\ x_1 + 7x_2 + x_3 \geq 4, \\ x_1 + x_2 + 7x_3 \geq 4, \\ x_1 \leq 12, \quad x_2 \leq 12, \\ x_i \geq 0, \ i = 1, 2, 3. \end{gathered}$$

We have to find min φ over each wall of this set.We shall not present here the full collection of computational results. Denoting

$$W_1 = \{x \in S_1 | 9x_1 + 9x_2 + 2x_3 = 96\}$$

and solving the problem

$$\min \{\varphi : x \in W_1\}$$

we obtain a solution, that gives

$$f_1 = 0, \qquad f_2 = 96.$$

These two numbers are coordinates of a point from the criterion space. Using this point as a reference point in problem (3) where $l = 0.01$, we obtain the following result:

$$x_1 = 0.0, \quad x_2 = 8.0, \quad x_3 = 1.0; \qquad f_1 = 1, \quad f_2 = 73 \ ; \ \varphi = 25.$$

Table 3 shows that this is the needed solution .

Example 3. We will consider again the example from Steuer's book [12, p.244]. The data for the MOLP problem are given above (in the text for example 1). But now we will consider the function

$$\mathrm{FIL} = (x_1 - 1)^2 + (x_2 - 2)^2 + (x_3 - 3)^2 + (x_4 - 2)^2 + (x_5 - 1)^2.$$

Our interest here is to estimate the minimal value of this function over the efficient set of the given MOLP problem.

Steuer has given the following 11 efficient extreme points. We give here these points once more for convenience:

$z^1 = (20.25, 14.25, 0.0)$,
$z^2 = (19.80, 17.40, 0.90)$,
$z^3 = (9.31, 8.675, 26.25)$,
$z^4 = (14.06, 30.583, 13.816)$,
$z^5 = (9.12, 12.0, 26.25)$,
$z^6 = (10.733, 28.853, 21.80)$,
$z^7 = (11.2, 34.6, 5.20)$,
$z^8 = (-1.2578, 20.2648, 34.04)$,
$z^9 = (5.2, 36.6, 5.20)$,
$z^{10} = (0.733, 22.853, 31.80)$,
$z^{11} = (-34.80, 0.60, 35.20)$.

The MOLP problem under consideration has 4 maximally efficient facets (MEF). The above given efficient extreme points constitute these MEFs as follows (Steuer):

MEF1 $\leftarrow$ z^1, z^2, z^3, z^4, z^5, z^6;
MEF2 $\leftarrow$ z^4, z^6, z^7, z^9;
MEF3 $\leftarrow$ z^5, z^6, z^8, z^{10};
MEF4 $\leftarrow$ z^8, z^{11}.

Here we must point out that in Steuer's book the above given extreme points as well as the maximally efficient facets belong to the criterion space and not to the argument space.

Using the constraints of the MOLP problem it is easy to see that each of points z^i has a corresponding point x^i that belongs to the intersection of some walls . The list of these intersections is as follows:

$z^1 \rightarrow x^1 \in W_1 \cap W_2 \cap W_6 \cap W_8 \cap W_{10}$,
$z^2 \rightarrow x^2 \in W_1 \cap W_2 \cap W_5 \cap W_8 \cap W_{10}$,
$z^3 \rightarrow x^3 \in W_1 \cap W_2 \cap W_8 \cap W_9$,
$z^4 \rightarrow x^4 \in W_1 \cap W_2 \cap W_3 \cap W_8$,
$z^5 \rightarrow x^5 \in W_1 \cap W_2 \cap W_7$,
$z^6 \rightarrow x^6 \in W_1 \cap W_2 \cap W_3 \cap W_7 \cap W_8$,
$z^7 \rightarrow x^7 \in W_2 \cap W_3 \cap W_5 \cap W_8 \cap W_{10}$,
$z^8 \rightarrow x^8 \in W_1 \cap W_2 \cap W_5 \cap W_7 \cap W_9$,
$z^9 \rightarrow x^9 \in W_2 \cap W_3 \cap W_7 \cap W_8$,
$z^{10} \rightarrow x^{10} \in W_1 \cap W_2 \cap W_3 \cap W_5 \cap W_7$,
$z^{11} \rightarrow x^{11} \in W_2 \cap W_5 \cap W_7 \cap W_9 \cap W_{10}$.

Comparing these intersections and using the description of maximally efficient facets given by Steuer we get the description of each MEF as a subset of S:

$\mathrm{MEF1} = \{x \in S \mid W_1 = 0,\ \ W_2 = 0,\ \ W_8 = 0\},$

$\mathrm{MEF2} = \{x \in S \mid W_2 = 0,\ \ W_3 = 0,\ \ W_8 = 0\},$

$\mathrm{MEF3} = \{x \in S \mid W_1 = 0,\ \ W_2 = 0,\ \ W_7 = 0\},$

$\mathrm{MEF4} = \{x \in S \mid W_2 = 0,\ \ W_5 = 0,\ \ W_7 = 0,\ \ W_9 = 0\}.$

The next step is to minimize the function FIL on each of these subsets of S. The minimal among obtained values is the needed minimum. Table 4 contains the results of these computations. The 1-st column contains the symbols of the used maximally efficient facets (subsets of S). The 2-nd column contains the corresponding minima of the function FIL, obtained on these subsets, the 3-rd column contains the corresponding nondominated vectors in criteria space.

Table 4

MEF1	**22.17599**	(14.926673, 17.543787, 12.453541)
MEF2	37.409894	(13.896882, 30.498379, 14.207482)
MEF3	42.061135	(5.17996, 23.55534, 27.70443)
MEF4	53.4344	(–3.29078, 19.07092, 34.11348)

Thus we see that the needed minimum is equal to 22.175997. *Now the question is: can we obtain this value or another one close to it using the proposed algorithm?*

We shall apply version 1 of the algorithm. The main computational results are collected in Table 5. In the **1-st column** we see the symbols of the active walls of S. We minimize the function FIL on these walls. The **2-nd column** contains the obtained corresponding minimal values of FIL . The **3-rd column** contains the corresponding points in the criterion space. All these points are dominated. But these points are used as reference points in problem (3) and the obtained Pareto points are written in the **4-th column**. The **5-th column** contains the corresponding values of FIL.

Table 5

1	2	3	4	5
W_1	6.758532	f_1=10.204541 f_2=10.422458 f_3=15.316346	12.7496 12.9676 17.8614	31.5781
W_2	4.999915	f_1=2.008572 f_2=14.33663 f_3=15.331489	10.1394 22.4744 23.469	51.0713
W_2	4.999915	f_1=2.008572 f_2=14.33663 f_3=15.331489	10.1394 22.4744 23.469	51.0713
W_3	18.851287	f_1=3.923564 f_2=21.638556 f_3=13.372526	11.211871 28.926863 20.660834	47.282117

Table 5 (continued)

1	2	3	4	5
W_4	$W_4 = \varnothing$			
W_5	4.166595	f_1=6.4444056 f_2=10.777268 f_3=12.500109	12.410140 16.743352 18.466192	30.855787
W_6	0.999902	f_1=1.000031 f_2=4.999498 f_3=8.998655	11.848236 15.847703 19.84686	35.605831
W_7	3.999945	f_1= – 4.003252 f_2=9.982783 f_3=9.985459	9.679067 23.665103 23.667779	48.672687
W_8	8.999911	f_1=8.000968 f_2=7.996032 f_3=3.994359	15.415049 15.410114 11.408440	**22.791814**
W_9	3.999908	f_1=1.992702 f_2=1.999540 f3=10.001240	11.907339 11.914177 19.915877	39.201898
W_{10}	0.999931	f_1=1.004126 f_2=6.980639 f_3=6.991470	12.4046 18.3811 18.392	30.400139

Thus this table gives the following estimate

$$\min_{x \in E} \text{FIL} \leq 22.791814$$

Having in mind the value 22.175997 we accept that the estimate 22.791814 is satisfactory.

9. Conclusion

We have shown (by examples) that it is possible to obtain an upper bound close to the minimal value of a linear function φ over the set E of a MOLP problem without using any special optimization techniques. The proposed versions of the algorithm use the well known reference point method for obtaining nondominated points. Therefore they differ from all algorithms cited in Yamamoto's paper. On the other hand all these versions substantively use the fact that the needed solutions belong to the frontier of S. And this is used in some other algorithms, too (in branch and bound algorithms, for example). The experiments show that the described algorithm (main version) can successfully work for estimating the minimum of a convex function on the set E. It must be pointed out that parallel computations can be used very easily. It is of interest now to have sufficiently good methods for obtaining lower bounds for the value of min φ over the set E.

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