БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ . BULGARIAN ACADEMY OF SCIENCES

КИБЕРНЕТИКА И ИНФОРМАЦИОННИ ТЕХНОЛОГИИ · Том **4**, № 2
CYBERNETICS AND INFORMATION TECHNOLOGIES · Volume **4**, No 2

София . 2004 . Sofia

Approximation of Portfolio Moment States

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Abstract: *A portfolio optimization problem with assets is discussed in the paper. The portfolio motion equations are deduced in the phase space based on the theorem for existence and identity of Cauchy's problem for ordinary differential equations (ODE).*

These equations are used for approximation of the portfolio-phase trajectory.

Keywords: *Finance Management, portfolio, assets.*

Introduction

In the financial world there exist two main directions of solving the problem of portfolio optimization. Namely, these are the discrete time optimization and the continuous time optimization. The first direction was initially developed by H. M a r k o w i t z [4, 5], and besides J. Tobin's work (see T o b i n [10, 11]), etc. M e r t o n [8] and W i e s e - m a n n [12] have also some contributions.

The philosophy of the first approach according to Markowitz's point of view can be summarized:in the following two formulae.

• For a given upper bound σ^2 of the portfolio-return variance, choose a feasible portfolio π such that the expectation $\mu(\pi)$ is maximal under all feasible portfolios π, with $\sigma^2(\pi) \le \sigma^2$. This corresponds to the optimization problem:

(1)
$$\left| \begin{array}{l} \max\limits_{\pi \in R^n} \mu(\pi) \\ \sigma^2(\pi) \le \sigma^2 \\ \sum\limits_{i=1}^{n} \pi_i = 1, \pi_i \ge 0, i = 1, 2, ..., n \end{array} \right. .$$

• For a given lower bound μ for the mean portfolio return, choose an admissible portfolio π such that $\sigma^2(\pi)$ is minimal under all admissible portfolios π, with $\mu \leq \mu(\pi)$ Analogically, this corresponding to the optimization problem:

$$(2)\qquad \begin{vmatrix} \min\limits_{\pi \in R^n} \sigma^2(\pi) \\ \mu \leq \mu(\pi) \\ \sum\limits_{i=1}^{n} \pi_i = 1,\ \pi_i \geq 0,\ i = 1, 2, \ldots, n. \end{vmatrix}$$

Finally we can deduce the following relation between the two problems (1) and (2), which is widespread formulation of the Markowitz's idea, thus:

$$(3)\qquad \begin{vmatrix} \max\limits_{\pi \in R^n} \left(\mu(\pi) - \alpha.\sigma^2(\pi)\right) \\ \sum\limits_{i=1}^{n} \pi_i = 1,\ \pi_i \geq 0,\ i = 1, 2, \ldots, n. \end{vmatrix}$$

Merton uses the following formulation:

$$(4)\qquad \begin{vmatrix} \max\limits_{\pi \in R^n} E\left(U(R)\right) \\ \sum\limits_{i=1}^{n} \pi_i = 1,\ \pi_i \geq 0,\ i = 1, 2, \ldots, n, \end{vmatrix}$$

with a utility function $U(.)$, where R is the portfolio return.

Besides all aforesaid, we remark that the future price of the securities are modeled via returns, at the end of the fixed trading period, i.e. at moment T.

The second direction (the continuous time portfolio problem) consists of maximizing the total expected utility of consumption over the trading interval $[0, T]$ and/or the terminal wealth. Again the dynamics of the owned stocks price are driven by independent Wiener processes.

There are two main approaches for solving the continuous time portfolio problem:

• the stochastic control approach developed by M e r t o n [6, 7];

• the martingale approach, presented in P l i s k a [9], K a r a t z a s, L e h o c z k y, S h r e v e [3], K a r a t z a s [2].

The first approach is based on the standard results of stochastic control theory. The optimal solution is computed by solving the so-called Hamilton-Jacobi-Bellman Equation (HJBE) in two steps. The first step consists of searching for the optimal portfolio strategy as a function of the (unknown) optimal expected utility. When inserting this portfolio and consumption strategy into the HJBE, the result is a non-linear partial differential equation and this is the second step. In the special case of the Black-Scholes model and HARA (Hyperbolic Absolute Risk-Averse) utility function, Merton was able to find closed form solutions for this problem. In general, it is very hard to get explicit solution to the HJBE.

The second approach is based on the martingale theory and the stochastic integration. Besides, he depends crucially on the completeness of the market model.

The model that we suggest is based on popular result of dynamic optimization theory. Namely, this is the principle of maximum of Pontryagin. We construct an approach for flow (moving) states of the portfolio via simple observations over the nature of the portfolio management. The result is a system of Ordinary Differential Equation (ODE). So, the portfolio problem receives a typified dynamic optimization problem formulation when maximizing the market price of a portfolio at each moment. Finally, we hope that the optimal solution of the aforesaid problem may be computed after we use the Pontryagin's principle of maximum.

Equations for describing portfolio motion

Here we will introduce a system of equations describing the portfolio motion. Thus it will be possible to approximate the phase trajectory of a portfolio (according to the chosen control with its characteristics).

Now, we have a set of n assets, an initial time moment t_0 and an interval $[t_0, +\infty)$.

We will use the following five assumptions.

1) Let $c_{kj}(t)$ is the price at moment t, $t \geq t_0$, for one unit of k-th asset expressed in the units of j-th asset, $k \neq j$. For all these functions c_{kj}, $k \neq j$, $k, j = 1, 2, ..., n$ (a total number of $n(n-1)$), it can be said that

(1.1) $$\begin{cases} c_{kj}(t) > 0 \\ k \neq j; k, j = 1, 2, ..., n \end{cases},$$

on the base that every asset has a positive price at each moment t ; $t \geq t_0$. Further, a number of $n(n-1)/2$ functions are sufficient for calculations. These functions we will call a *price system.*

They satisfy the following relations:

(1.2) $$\begin{cases} c_{kj}(t)c_{jk}(t) = 1 \\ k \neq j; k, j = 1, 2, ..., n, t \geq t_0 \end{cases}.$$

based on the cross price property.

Let us assume that these functions are:

• differentiable within the interval $[t_0, t_1]$ except a finite number (possibly equal to zero) points from (t_0, t_1) (different points for different functions in general), i.e. the right derivative exists;

• the right derivatives are right partial continuously at $[t_0, t_1]$; here t_1 is a time moment in the future, it will be discussed in details later on.

2) The portfolio state at each moment will be defined by a number of "n" real parameters (values), denoted by $x_1, x_2, ..., x_n$ and called phase coordinates. These parameters will represent corresponding parts (positions) from fixed "n" number assets, which we will hold at our portfolio. In other words, x_j represents the position from j-th asset, $j = 1, 2, ..., n$, in current moment. For more convenience we will consider the values $x_1, x_2, ..., x_n$ as the coordinates of the point $x = (x_1, x_2, ..., x_n)$ from R^n (n-dimensional Euclidean space). This point is called the phase state of a portfolio.

3) Feasible control:

(1.3) $$u_{kj} : t \to u_{kj}(t) \in R; t \in [t_0, t_1], j \neq k; k, j = 1, 2, ..., n.$$

These values $u_{kj}(t) = u_{kj}\left(c(t), \dot{c}^{+}(t)\right)$ at moment t are equal to the managerial decisions for defining the positions that have to be sold not later than a given future moment $t + \delta t$. In other words, the value $u_{kj}(t)$ will represent the position from k-th asset, which we must sell through j-th asset, $j \neq k$, at the moment $t + {}_j^k\delta t$. We will remember that the time period ${}_j^k\delta t$ is the period when we must perform transaction from k-th asset to j-th asset.

This means that the value $u_{kj}(t)$ represents the position from j-th asset from the interval $[t, t + {}_j^k\delta t)$. According to that the feasible control must satisfy the following inequalities:

$$(1.4)\qquad \begin{cases} \sum\limits_{j\neq k,\, j=1}^{n} u_{kj}(t) \leq 1,\, t_0 \leq t \leq t_1, \\ u_{kj}(t) \geq 0,\, j \neq k, \\ j, k = 1, 2, ..., n. \end{cases}$$

These inequalities express the consideration that a time moment for control does not exist for which we could sell a position bigger than the kept one at the moment. The so defined functions (1.2) are called by us control parameters. Their number is $n(n-1)$ and their characteristics are the parameters ${}_j^k\delta t,\, j \neq k,\, j, k = 1, 2, ..., n$. The vector function $u: t \to u(t) \in R^{n(n-1)}, t_0 \leq t \leq t_1$, with a value of $u(t) = \left(u_{12}(t), u_{13}(t), ..., u_{1n}(t), u_{21}(t), u_{23}(t), ..., u_{2n}(t), ..., u_{n1}(t), u_{n2}(t), ..., u_{nn-1}(t)\right)$ is called by us "feasible control" with characteristics for realization $p = (\delta t, {}_2^1\delta t, {}_3^1\delta t, ..., {}_n^1\delta t, {}_1^2\delta t, {}_3^2\delta t, ..., {}_n^2\delta t, ..., {}_1^n\delta t, {}_2^n\delta t, ..., {}_{n-1}^{\;n}\delta t)$ (of managerial decisions. Here $\delta t = \max\{ {}_j^k\delta t,\, j \neq k,\, j = 1, 2, ..., n\}_{k=1}^{n}$ represents a period between two neighbour feasible controls. It follows from the above said that a set U exists, $U \subset R^{n(n-1)}$. The control parameters have such values that the point $u(t)$ belongs to U for each time moment t of control $[t_0, t_1]$. We will call the set U a set of feasible control. According to (1.3) the set U is defined as

$$(1.5)\qquad U: \begin{cases} \sum\limits_{j\neq k,\, j=1}^{n} u_{kj} \leq 1 \\ 0 \leq u_{kj},\, j \neq k,\ k, j = 1, 2, \ldots, n. \end{cases}$$

4) We will postulate that the characteristics of control u (the times for realization ${}_j^k\delta t$) must be constant in the interval $[t_0, t_1]$. On this base the inertia time could be regarded as constant for the time interval $[t_0, t_1]$ with a parameter ${}_j^k\delta t$ when relaxing a position of k-th asset, $k = 1, 2, ..., n$ through j-th asset, $j \neq k, j = 1, 2, ..., n$. This value is non-negative and close to zero; we will assume also the time for decision making to be to be sufficiently small and it will not be taken into account (non-inertia decision making). This means the times $\{ {}_j^k\delta t,\, j \neq k,\, j = 1, 2, ..., n\}_{k=1}^{n}$ will be used for realization of corresponding controls. Also the next decision making will be made after time, $\delta t = \max\{ {}_j^k\delta t,\, j \neq k,\, j = 1, 2, ..., n\}_{k=1}^{n}$ i.e., after realizing all the predecessor decisions.

5) The condition for "non-inertia" of control decisions means that they can be switched instantly when it is necessary from one condition to another feasible condition. Mathematically we accept that the control parameters have the possibility to change their values at a leap. In other words, feasible control parameters are not only

continuous functions, but partial continuous ones, too. For more convenience (and definiteness) in our presentation we will assume that the controls in each break-point from I type are right-continuous. Or their values could be defined by the equality $u(\tau) = u(\tau+0)$ at any break-point τ for any component of control $u = u(t), t_0 \le t \le t_1$. We will assume also that all break points are internal for the interval $[t_0, t_1]$. And now the conclusion is that a feasible control is every partially continuous function from the right vector function $u : t \to u(t) \in R^{n(n-1)}, t_0 \le t \le t_1$, with values in the set U of feasible decisions and it is also continuous at the ends of the interval $[t_0, t_1]$.

The motion-equations' system will describe the approximate change of the moment portfolio condition for the time of control. Let as assume that t is an arbitrary and a fixed moment from the interval of control $[t_0, t_1]$. The corresponding managerial decisions for the portfolio are accepted in this moment. As we postulated, the realization of these decisions (the vector function $u(t)$) needs time δt. Let us denote:

- $a_k(t+\delta t)$ is the position to be released from k-th asset, $k = 1, 2, \ldots, n$ by the moment $t+\delta t$ (this is realized by using another asset on the basis of a decision made at the moment t). Then we have the following expression:

(1.6)
$$\begin{cases} a_k(t+\delta t) = x_k(t) \sum\limits_{\substack{j=1 \\ j\ne k}}^{n} u_{kj}(t) \\ k = 1, 2, \ldots, n, \ t_0 \le t \le t_1, \ \delta t > 0; \end{cases}$$

- $b_j(t+\delta t)$ is the amount that could be adopted by j-th asset from the other ones up to the moment $t+\delta t$ then we have

(1.7)
$$\begin{cases} b_j(t+\delta t) = \sum\limits_{\substack{k=1 \\ k\ne j}}^{n} u_{kj}(t) x_k(t) c_{kj}(t + {}_j^k\delta t) \\ j = 1, 2, \ldots, n, \quad t_0 \le t \le t_1. \end{cases}$$

It is easy to see that at the moment $t+\delta t$ of realization of manager decisions (it was accepted at the moment t in the form of value $u(t)$) the change of the holding position amount over j-th asset is given by

(1.8) $$x_j(t+\delta t) - x_j(t) = b_j(t+\delta t) - a_j(t+\delta t)\ .$$

Let us assume that:

- The function $x_j : t \to x_j(t) \in R, t_0 \le t \le t_1$ describing a position in the portfolio of the j-th asset, is differentiable at the control time interval $[t_0, t_1]$.

It is known that we can substitute (with some approximation) the increase $x_j(t+\delta t) - x_j(t)$ of function x_j with its differential at the moment, i.e. with $\delta t \dot{x}_j(t)$.

NB! It is known that when δt is small (but $\delta t > 0$ for us, intuitively it is clear for successful management), the increment of the function x_j differs very little from its differential. In the linear case the difference is zero.

In other words the following relations are true

(1.9) $$x_j(t+\delta t) = x_j(t) + \delta t \dot{x}_j(t)\ .$$

• They realize the assumption that future position amounts of the j-th asset, which we will hold after a fixed moment t, depend on the value of the future time δt and by the position amount at the current moment t and by its behavior $\dot{x}_j$.

Analogically and according to the definition of ideal price processes we could accept with some approximation the following relations to be true:

$$(1.10)\qquad \begin{cases} c_{kj}(t+{}_j^k\delta t)=c_{kj}(t)+{}_j^k\delta t\dot{c}_{kj}^{+}(t) \\ k<j,\quad k,j=1,2,...,n;\quad t_0\le t\le t_1. \end{cases}$$

• They express the assumption that the future asset price depends on both the current level $c_{kj}(t)$ and also the “future” tendency. The last is expressed in terms of the right derivative $\dot{c}_{kj}^{+}(t)$ and the duration ${}_j^k\delta t$: ${}_j^k\delta t\dot{c}_{kj}^{+}(t)$.

And now according to (1.6) and (1.7) the equality (1.8) has the form

$$x_j(t+\delta t)-x_j(t)=-x_j(t)\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}(t)+\sum_{\substack{k=1\\k\neq j}}^{n}x_k(t)u_{kj}(t)c_{kj}(t+{}_j^k\delta t)$$

and after substituting with (1.8) and (1.9) the equality below follows

$$(1.11)\qquad \dot{x}_j(t)=-\frac{x_j(t)}{\delta t}\sum_{\substack{k=1\\k\neq j}}^{n}u_{jk}(t)+\frac{1}{\delta t}\sum_{\substack{k=1\\k\neq j}}^{n}x_k(t)u_{kj}(t)\left(c_{kj}(t)+{}_j^k\delta t\dot{c}_{kj}^{+}(t)\right).$$

After differentiating the equalities (1.2) (according to the right differential) the following equalities are true:

$$\begin{cases} \dot{c}_{kj}^{+}(t)=\dfrac{\dot{c}_{jk}^{+}(t)}{c_{jk}^{2}(t)} \\ k\neq j, k,j=1,2,...,n,\quad t_0\le t\le t_1, \end{cases}$$

and using the introduced price system $c = c(t)$, the equation (1.11) has the form:

$$(1.12)\qquad \begin{aligned} \dot{x}_j(t)=&-\frac{x_j(t)}{\delta t}\sum_{k\neq j,k=1}^{n}u_{jk}(t)+\frac{1}{\delta t}\sum_{k<j,k=1}^{n}x_k(t)u_{kj}(t)\left(c_{kj}(t)+{}_j^k\delta t\dot{c}_{kj}^{+}(t)\right)+\\ &+\frac{1}{\delta t}\sum_{k>j,k=1}^{n}x_k(t)u_{kj}(t)\left(\frac{1}{c_{jk}(t)}-{}_j^k\delta t\frac{\dot{c}_{jk}^{+}(t)}{c_{jk}^{2}(t)}\right). \end{aligned}$$

Let us mention that the above equation was derived without taking into account the time moment, the manager decisions and the finance asset j Then, equality (1.12) is true for each time moment, each feasible manager’s decision (with corresponding characteristics) and for each financial asset (a total number of n) forming the portfolio. The last means that the following system of differential equations is true:

$$
(1.13)\quad \begin{cases}
\dot{x}_j = -\dfrac{x_j}{\delta t}\sum\limits_{k\neq j,k=1}^{n} u_{jk} + \dfrac{1}{\delta t}\sum\limits_{k<j,k=1}^{n} x_k u_{kj}\left(c_{kj}(t) + {}_j^k\delta t\,\dot{c}_{kj}^{+}(t)\right) + \\
+\dfrac{1}{\delta t}\sum\limits_{k>j,k=1}^{n} x_k u_{kj}\left(\dfrac{1}{c_{jk}(t)} - {}_j^k\delta t\,\dfrac{\dot{c}_{jk}^{+}(t)}{c_{jk}^{2}(t)}\right), \\
t_0 \le t \le t_1, \\
j = 1, 2, \ldots, n
\end{cases}
$$

(according to the phase coordinates $x_1,\ x_2, \ldots, x_n$). Here the unknown functions are the phase coordinates and the control functions $\{u_{kj}, j\neq k, j=1, 2, \ldots, n\}_{k=1}^{n}$ and their corresponding characteristics $\{{}_j^k\delta t, j\neq k, j=1, 2, \ldots, n\}_{k=1}^{n}$. The price system and resp. their right derivatives have a sense of input data (given and defined functions in the interval $[t_0, t_1]$). Note that for a given type of control, i.e.:

- chosen way of reaction or suitable defining of control parameters $\{u_{kj}, j\neq k, j=1, 2, \ldots, n\}_{k=1}^{n}$ as functions of time;
- chosen speed of reaction, i.e. fixing of parameters $\{\delta t, {}_j^k\delta t, j\neq k, j=1, 2, \ldots, n\}_{k=1}^{n}$ in the interval $[t_0, t_1]$.

The system (1.13) will have the type of non-autonomous system in a normal form:

$$
\begin{vmatrix}
\dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n \\
\dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n \\
\cdots \\
\dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n
\end{vmatrix},
$$

$t_0 \le t \le t_1, (x_1, x_2, \ldots, x_n) \in X$ or in a vector form $\dot{x} = A(t)x$, where $x = (x_1,\ x_2, \ldots, x_n)$, and $A(t) = \left(a_{ij}(t)\right)_{n\times n}$ is the matrix of order "n" with elements

$$
a_{ij}(t) = \begin{cases}
-\dfrac{1}{\delta t}\sum\limits_{k\neq i,k=1}^{n} u_{ik}(t),\ j = i, \\
\dfrac{u_{ji}(t)}{\delta t}\left[c_{ji}(t) + {}_i^j\delta t\,\dot{c}_{ji}^{+}(t)\right] j < i, \\
\dfrac{u_{ji}(t)}{\delta t}\left[\dfrac{1}{c_{ij}(t)} - {}_i^j\delta t\,\dfrac{\dot{c}_{ij}^{+}(t)}{c_{ij}^{2}(t)}\right],\ j > i,
\end{cases}
$$

and according to the theory of differential equations.

NB! We rely on the mathematical result known as the Theorem for existence and uniqueness of Cauchy's problem for non-autonomous homogenous system of differentiall equations in a normal form. According to this theorem we deduce the solution of (1.13). It is differentiable anywhere in the interval $[t_0, t_1]$ with the exception of a finite number of points from the considered interval.

For a given initial condition (at the moment $t = t_0$) of a portfolio ($x(t_0) = x^0$), we could synonymously derive from the above system one approximation of its (true) trajectory, resp. one approximation of the law of portfolio motion according to some conditions that we will discuss in the next paragraph. Therefore the system (1.13) we will call also the law of motion (in differential form) of a portfolio.

During the above discussion a need for initial condition arises. Such initial condition could easily be constructed. Indeed, let us imagine an existing portfolio for management after a given moment t_0. This moment t_0 could be considered as a starting point for the examined management period. Also, the financial assets' positions of the portfolio at the moment t_0 could be considered as coordinates of a point from the phase space. This point we will use as the initial phase state of a portfolio in the phase space.

In case we still do not have a portfolio, then we will construct it after some initial time moment t_0 with the suitable controls, and again this moment could be assumed to be a starting point for the management period. The resources (different types of financial assets) given for the portfolio construction could be considered as an initial state of the portfolio at the moment t_0, their amounts respectively.

Finally, when we change the management in (1.13) with another one (but keeping the initial state x^0), we will get another trajectory with the point x^0 as its left end; by using new change we will get another trajectory with the point x^0 as its left end, and so on. In this way different possible managements could lead one portfolio at one moment to have different states of positions.

Theorem for existence and uniqueness

Again, we solve the problem for portfolio management starting from a given initial moment and a given initial state. By using feasible controls $u = u(t), t_0 \le t \le t_1$, we want to have the portfolio value to be maximal at each moment from a given time interval (horizon) $[t_0, t_1]$. Additionally, we could (or not) have final values/goals for some of the assets. We will use for that purpose the Theorem for existence and uniqueness for Caushy's problem. It is applied for a non-autonomous system of first order linear homogenous ordinary differential equations in a normal form:

$$(2.1) \qquad \left| \begin{array}{l} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n \\ \qquad \cdots \\ \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n . \end{array} \right.$$

The system of functions

$$(2.2) \qquad \{x_k : t \to x_k(t) \in R, k = 1, 2, .., n\},$$

defined in some open interval $(\theta_0,\ \theta_1)$ is called a solution of the above system (2.1), when:

- these functions are differentiable everywhere in the interval $(\theta_0,\ \theta_1)$;
- they satisfy the above system after the substitution.

The space line of variables $x_1, x_2, \dots, x_n, t$, defined by parametric equations:

(2.3)
$$\left| \begin{array}{l} x_k = x_k(\theta), \theta_0 < \theta < \theta_1 \\ k = 1, 2, \dots, n \\ t = \theta, \end{array} \right.$$

where $\{x_k : t \to x_k(\theta), \theta_0 < \theta < \theta_1, k = 1, 2, \dots, n\}$ are the solutions of the system (2.1), is called an integral curve of the system. It is known also that this curve will go through the point $\gamma^0 = (x_1^0, x_2^0, \dots, x_n^0, t_0)$ from the space $\{x_1, x_2, \dots, x_n, t\}$ if and only if the corresponding solution $\{x_k : t \to x_k(\theta), \theta_0 < \theta < \theta_1, k = 1, 2, \dots, n\}$ of system (2.1) satisfies the relations

(2.4)
$$x_1(t_0) = x_1^0, x_2(t_0) = x_2^0, \dots, x_n(t_0) = x_n^0$$

(it is necessary that the number t_0 belongs to the interval $(\theta_0,\ \theta_1)$; otherwise, the values $x_1^0, x_2^0, \dots, x_n^0$ cannot be defined). The relations (2.4) are called initial conditions and the above problem is just Caushy's problem.

Theorem for existence and uniqueness

Let the functions $a_{ij} : t \to a_{ij}(t) \in R, \quad i, j = 1, 2, \dots, n,$ are defined and continuous in the interval (a, b). Then there exists an unique solution for Caushy's problem (2.1),(2.4) defined in the interval (a, b) if we take a number t_0 from the interval (a, b), and $x_1^0, x_2^0, \dots, x_n^0$ are real numbers.

The proof of this theorem could be found in the courses of ordinary differential equations.

We will use the following formulation for our purposes.

Corrolary. Let us consider Caushy's problem (2.1),(2.4) and that the functions $a_{ij} : t \to a_{ij}(t) \in R, i, j = 1, 2, \dots, n,$ are defined and continuous in the interval (a, b) and also that they have left and right bounds at points "b" and "a" respectively and t_0 belongs to the interval (a, b). Then a number of "*n*" functions defined and continuous in the closed interval $[a, b]$ exist and they are an unique solution of Caushy's problem in the open interval (a, b).

P r o o f. To prove the above corrolary it is sufficient to extend the functions $a_{ij} : t \to a_{ij}(t) \in R, i, j = 1, 2, \dots, n,$ for each values of t outside the interval (a, b):

$$a_{ij}(t) = a_{ij}(a+0) = \lim_{\substack{t \to a \\ t > a}} a_{ij}(t), t \le a,$$
$$a_{ij}(t) = a_{ij}(b-0) = \lim_{\substack{t \to b \\ t < b}} a_{ij}(t), t \ge b,\ i, j = 1, 2, \dots, n.$$

In this way the functions $a_{ij} : t \to a_{ij}(t) \in R, \quad i, j = 1, 2, \dots, n,$ are defined and continuous over the real line $-\infty < t < +\infty$ and it contains t_0. According to the Theorem of Caushy for the extended problem an unique solution exists and it is defined on the real line. In other words, a number of "*n*" functions $x_k : t \to x_k(t) \in R, k = 1, 2, ..., n$, exist and they are defined and continuous in the interval. They are an unique solution of Caushy's problem in the open interval (a, b).

An application of the theorem of existence and identity

Let us see the equations of motion (1.11) to finish the discussion from the above paragraph. Namely, to illustrate the application of the theorem for existence and identity (corollary) let us choose the phase trajectory (the change of moment conditions) of a portfolio as a result of an applied/used definite feasible control $u : t \to u(t), t_0 \le t \le t_1$ with characteristics for realization $p = (\delta t, {}_2^1\delta t, {}_3^1\delta t, ..., {}_n^1\delta t, ..., {}_1^n\delta t, {}_2^n\delta t, ..., {}_{n-1}^n\delta t)$. For that reason let us take the components of the chosen control $u: t \to u(t)$ and "*p*" in the equation (1.11). Then we can represent these expressions in the following system:

(3.1)
$$\begin{vmatrix} \dot{x}_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + ... + a_{1n}(t)x_n \\ \dot{x}_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + ... + a_{2n}(t)x_n \\ \cdots \\ \dot{x}_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + ... + a_{nn}(t)x_n \end{vmatrix},$$

where $t \in [t_0, t_1]$,

(3.2)
$$a_{ij}(t) = \begin{cases} -\dfrac{1}{\delta t} \sum\limits_{k \ne i, k=1}^{n} u_{ik}(t),\ j = i, \\ \dfrac{u_{ji}(t)}{\delta t} \Big[c_{ji}(t) + {}_i^j\delta t \dot{c}_{ji}^+(t) \Big] j < i, \\ \dfrac{u_{ji}(t)}{\delta t} \left[\dfrac{1}{c_{ij}(t)} - {}_i^j\delta t \dfrac{\dot{c}_{ij}^+(t)}{c_{ij}^2(t)} \right],\ j > i. \end{cases}$$

Let us denote with $\tau_1, \tau_2, ..., \tau_k$ all the time moments (in ascending order) from the time interval $[t_0, t_1]$, in which some of the control components $u = u(t)$ have a breakpoint, i.e. $t_0 < \tau_1 < \tau_2, ..., \tau_n < t_1$. Also we consider the system (3.1) for all values of t from the interval $[t_0, \tau_1]$. It is easy to see that the functions $a_{ij} : t \to a_{ij}(t) \in R, i, j = 1, 2, ..., n$, forming the system (3.1), whose values are calculated from (3.2), are continuous in the open interval (t_0, τ_1) if and only if the "input data" – the functions: $c : t \to c(t)$ and $\dot{c}^+ : t \to \dot{c}^+(t)$ are continuous at each point from the interval.

• Let us assume that the above is true, i.e. vector-functions $c : t \to c(t)$ and $\dot{c}^+ : t \to \dot{c}^+(t)$ are continuous at each point from the interval (t_0, τ_1).

Then for some of the components of u: $t \to u(t)$, c: $t \to c(t)$, and $\dot{c}^+ : t \to \dot{c}^+(t)$ the ends of the interval t_0 and τ_1 will be the break-points from the first kind (according to the introductory remarks from point 1) and for the rest of the functions they will be simply points of continuity. This will guarantee that the functions $a_{ij} : t \to a_{ij}(t) \in R, i, j = 1, 2, ..., n,$ will have a finite right limit when t leans to t_0 from the right and to τ_1 from the left. At the end taking the initial condition $x^0 = (x_1^0, x_2^0, ..., x_n^0)$ it follows that for Caushy's problem:

$$\text{(3.3)} \qquad \begin{cases} \dot{x} = A(t)x \\ x(t_0) = x^0, \quad t_0 \le t \le \tau_1, \end{cases}$$

we can apply the corollary from the above paragraph. The answer is that the functions, $x_k : t \to x_k(t) \in R, k = 1, 2, ..., n,$ exist, that they are defined and continuous in the whole $[t_0, \tau_1]$ closed interval. These functions are unique solution of (3.3) in the open interval (t_0, τ_1). Analogically we can consider the system (3.1) for values of t from the closed interval $[\tau_1, \tau_2]$ and by the help of the point $x(\tau_1 - 0)$we define the following Cauchy's problem:

$$\text{(3.4)} \qquad \begin{cases} \dot{x} = A(t)x \\ x(\tau_1) = x(\tau_1 - 0), \quad \tau_1 \le t \le \tau_2. \end{cases}$$

• For this problem according to the assumptions: vector functions c: $t \to c(t)$ and $\dot{c}^+ : t \to \dot{c}^+(t)$ are continuous at each point in the open interval (τ_1, τ_2).

The corollary from the above paragraph would be applicable again and according to it a continuous vector function defined in the interval $[\tau_1, \tau_2]$ and with values R^n exists as unique solution in the interval (τ_1, τ_2) of (3.4). Let us denote this function again by $x : t \to x(t) \in R^n$. In this way the change of moment conditions of a portfolio from the time point t_0 to the time point τ_2 will be described by the function. It is defined in the interval $[t_0, \tau_2]$ and it is continuous at each point (rf. to Fig. 1).

Further, by using the point $x(\tau_2 - 0)$ as an initial condition for the system (3.1) we could enhance the function $x : t \to x(t) \in R^n$, $t_0 \le t \le \tau_2$, on the closed interval $[\tau_2, \tau_3]$ and so on. At the end we will receive a function defined on the whole interval $[t_0, t_1]$, see Fig. 2. This function is:

– continuous at each point in the interval $[t_0, t_1]$;

– partially differentiable (namely at points $(\tau_1, \tau_2, ..., \tau_k)$;

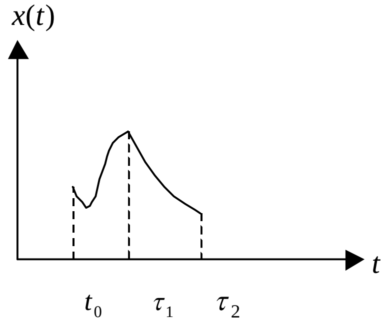

Fig. 1

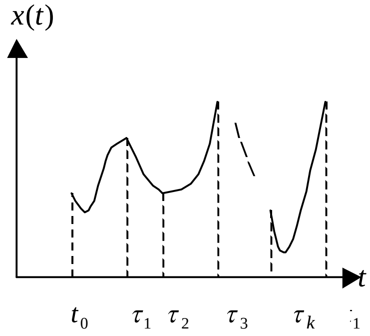

Fig. 2

– satisfying the initial condition $x(t_0) = x^0$;

– describing uniquely the change of time conditions of a portfolio as a result of the given control $u : t \to u(t), t_0 \leq t \leq t_1$, with times for realizing managerial decisions (transactions) "p".

Let us repeat that the above discussions were based on the assumption: the functions $c = c(t)$ and $\dot{c}^+ = \dot{c}^+(t)$ are continuous in every interval (t_0, τ_1), (τ_1, τ_2), (τ_2, τ_3),..., (τ_k, t_1). This guarantees that all time points of prices' jumping are between points $\tau_1, \tau_2, ..., \tau_k$ (for the selected price system). And this means that the so defined control decisions will react immediately for each jump of the price motion.

Acknowledgments: The authors wish to thank Prof. Racho Denchev from Sofia University and Prof. Svetlozar Rachev from University of Santa Barbara – USA, for their encouragement. The authors are very grateful to res. associate Stefan Koynov from IIT – BAS for improving the English style.

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Апроксимация на моментните състояния на портфейл

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(Р е з ю м е)

В настоящата статия се разглежда задачата за оптимално управление на портфейл от финансови активи.

Изведени са уравнения за движението във фазовото пространство на базата на теоремата за съществуване и единственост на задачата на Коши за обикновени диференциални уравнения.

Чрез изведените уравнения се апроксимира фазовата траектория на портфейла.