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Invariant Spaces and Cosine Transforms

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Abstract: *The discrete cosine (and sine) transforms (DCT/DST) are analyzed in this paper on the basis of the linear representations of finite groups and geometrical approach. These transforms are extremely useful for multirate systems, adaptive filtering and compression of speech signals and images. It is shown that if the Discrete Fourier Transform (DFT) operator is referred to an appropriate basis it takes a block-diagonal form. These blocks coincide with DCT-1/DST-1 for even dimensions of signals space and with DCT-5/DST-5 for odd ones. The results enable the investigation of the full structure of DCT/DST.*

Keywords*: cosine transforms, orthogonality, signal processing, fast transforms, filter banks, characters of groups, theory of groups.*

1. Introduction

Each Discrete Cosine Transform (DCT) uses n real basis vectors $\{\vec{\boldsymbol{c}}_m\}$ with cosine coordinates. These basis vectors are orthogonal. For example, in the DCT-4, the k-component of $\vec{c}_m$ is $\frac{2}{\sqrt{n}}\cos(k+\frac{1}{2})(m+\frac{1}{2})\frac{\pi}{n}$. There are eight different types of DCT and this raises the question for their classification as a uniform structure. What connects these transforms, do they cover all the transforms of this structure?

Ahmed, Natarajan and Rao have found the first cosine transform in 1974. This is the so-called DCT-2 [1, 4]. There are four basic types – from DCT-1 to DCT-4. Two of them, DCT-2 and DCT-4, are applied actively in image processing, filter bancs and multirate systems [4]. It is important that they have got fast realizations.

This basic set was expanded in 1985 with four new transforms – from DCT-IO to DCT-IVO by W a n g and H u n t [2, 4].

All DCTs are orthogonal transforms and the usual proof is the direct calculation of the inner products of their basis vectors, applying trigonometric identities [4]. An useful source of such identities is this of H. D w i g h t [3].

The proof of orthogonality is obtained in Strang's paper [4] by a second indirect but neat way. The basis vectors are actually eigenvectors of symmetric second-difference matrices at different boundary conditions – of Neumann or Dirichlet. Orthogonality is proved automatically (matrices are symmetric) and all DCTs are connected in a fixed structure. Their multitude becomes largely comprehensible and explicable.

Does a more direct way exist to obtain these transforms, connecting them in a joint structure, proving orthogonality and giving fast realizations?

The objective of this paper is to give answers to these questions.

2. Invariant spaces and projectors on them

2.1. Convolution and the Dihedral group

The input and output signals of a linear time-invariant system are connected by the convolution operation [5, 6]:

(1) $$\boldsymbol{y} = \boldsymbol{x} * \boldsymbol{h}.$$

Here $\boldsymbol{h}$ is the impulse response of the system. The sets of real numbers $\boldsymbol{R}$, integer numbers $\boldsymbol{Z}$ and the integer numbers – multiple of some integer number n (i.e. $n\boldsymbol{Z}$), with addition as a binary operation, are groups [7, 8]. Of course we have $\boldsymbol{R} \supset \boldsymbol{Z} \supset n\boldsymbol{Z}$. Periodical functions – continuous and discrete, could be defined on the factor-groups $\boldsymbol{R}/\boldsymbol{Z}$ and $\boldsymbol{Z}/n\boldsymbol{Z}$. The signals are functions usually defined on the $\boldsymbol{R}$, $\boldsymbol{Z}$, the torus group $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$, the residue system (mod n) $\boldsymbol{Z}/n\boldsymbol{Z}$, or their Cartesian products. As such they are elements of some functional space, most often a Hilbert space $\boldsymbol{H}$, which is supplied with the form $(x|\, y)$ that takes values in the field of the complex numbers $\boldsymbol{C}$. This form, called an inner or scalar product, is Hermitian and positive definite [8, 9].

If $\boldsymbol{L}^2(a, b)$ denotes the space of square summable functions on the interval (a, b), and $\boldsymbol{L}^2(\boldsymbol{Z}/n\boldsymbol{Z})$ denotes n-dimensional complex vector space of functions (vectors), we have Hilbert spaces with inner products [6, 8, 9, 10]:

(2) $$(x|\, y) = \int_a^b x^*(t)\, y(t)\, dt, \qquad (\vec{x}\,|\vec{y}) = \sum_{k \in Z/nZ} x_k^* y_k .$$

The convolution (1) could be written as an inner product if the right shift operator ρ and the sign operator σ are used: $\rho\colon x(t) \to x(t-1)$, $\sigma\colon x(t) \to x(-t)$. If $x, h \in \boldsymbol{L}^2(-\infty, \infty)$, then $(x\,|\,\rho^t \sigma\, h) = \int_{-\infty}^{\infty} x(t)\; h(t-\tau) d\tau = x * h.$

In the canonical basis of $\boldsymbol{L}^2(\boldsymbol{Z}/n\boldsymbol{Z})$, formed by vectors, $\{\vec{\boldsymbol{e}}_k\}$, $\vec{\boldsymbol{e}}_k = \left[\boldsymbol{\delta}_{l,k}\right]$, $l, k = 0, 1, \ldots, (n-1)$ (mod n), ($\delta_{l,k}$ is the Kroneker symbol), the two endomorphisms have the following (orthogonal) matrices:

$$\rho_n = [\delta_{k-1,l}], \sigma_n = [\delta_{k,n-l}],$$
$$k, l = 0,1,\ldots,(n-1)\pmod{n}$$

(3) $$\rho_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

In this case the k-th coordinate of the output vector $\vec{\boldsymbol{y}}$ in (1) has the form

(4) $$\boldsymbol{y}_k = (\vec{\boldsymbol{x}} \mid \rho^k \sigma \vec{\boldsymbol{h}}),$$

and the convolution is cyclic. In (4) ρ and σ define a linear representation of the dihedral group $\boldsymbol{D}_n$ [7, 8, 9] (see the definitions at the end of the paper):

(5a) $$\boldsymbol{D}_n = \langle \sigma, \rho \mid \sigma^2 = \rho^n = (\sigma\rho)^2 = 1 \rangle.$$

One verifies from (3) that the matrices of two endomorphisms satisfy the defining relations of $\boldsymbol{D}_n$. The group $\boldsymbol{D}_n$ has a second presentation, which is isomorphic to (5a) ($\sigma \to \sigma, \tilde{\sigma} \to \sigma\rho$):

(5b) $$\boldsymbol{D}_n = \langle \sigma, \tilde{\sigma} \mid \sigma^2 = \tilde{\sigma}^2 = (\sigma\tilde{\sigma})^n = 1 \rangle.$$

In this case $\tilde{\sigma}_n = \sigma_n \rho_n$ and if n=4,

$$\tilde{\sigma}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The defining relations for $\boldsymbol{D}_\infty$ have the form (ρ could have arbitrary power ρ^t, $t \in \boldsymbol{R}$ (or Z)):

(5c) $$\boldsymbol{D}_\infty = \langle \sigma, \rho \mid \sigma^2 = (\sigma\rho)^2 = 1 \rangle.$$

It follows from (5) that not only the obvious symmetry σ (which reflects functions in ordinate axis) is involution, i.e. $\sigma^2 = 1$. Involutions are the elements of dihedral group $\rho^k\sigma$ (the “inner” polygon of the Cayley’s colored graph of $\boldsymbol{D}_n$ [7]) in (4), that reflects in the vertical lines $t = k/2$. These “parallel lines” coincide in the discrete case with diameters of the unit circle. If we apply the Strang’s terminology [4], for even k this symmetry realizes “meshpoint (or whole-sample) symmetry”, and for odd k – “midpoint (or half-sample) symmetry”. The most simple representatives of these two classes are σ (reflects in the ordinate) and $\sigma\rho \sim \tilde{\sigma}$ (reflects in the vertical line $t = -1/2$), i.e. the two generators of the dihedral group into the second presentation of this group (5b).

2.2. Properties of the Fourier operator

2.2.1. Some definitions

The continuous Fourier operator $\boldsymbol{F}$ is defined by the relation $\boldsymbol{F}x(t)=\int_{-\infty}^{\infty}e^{-j2\pi\nu t}x(t)dt=X(\nu)$ and the discrete Fourier operator $\boldsymbol{F}_n$ – by the square matrix ($\boldsymbol{F}_n$ = $(1/\sqrt{n})[w^{kl}]=\boldsymbol{C}_n - j\boldsymbol{S}_n$, $w=e^{-j2\pi/n}; k,l=0,1,...,n-1$). These two operators are unitary [6, 8, 11, 12, 13], i.e. the inverse one coincides with the Hermitian conjugate: $\boldsymbol{F}^{-1}=\boldsymbol{F}^*$. In the abstract harmonic analysis continuous and discrete transforms are considered to be an expansion by characters of group of the real numbers $\boldsymbol{R}$ and the cyclic group $\boldsymbol{Z}/n\boldsymbol{Z}$ respectively [14]. A fundamental property of the Fourier operator is that it transforms convolution into algebraic multiplication [6], i.e. $\boldsymbol{F}(x*h)=\boldsymbol{F}x.\boldsymbol{F}h$.

2.2.2. Eigenvalues and eigenspaces of the Fourier operator

The question about the eigenstructure of the Fourier operator has a long history. In the early 70's it was stated as the "multiplicity problem" for $\boldsymbol{F}$ [10]. The involution σ introduced in (3) connects $\boldsymbol{F}=\boldsymbol{C}-j\,\boldsymbol{S}$ and $\boldsymbol{F}^{-1}=\boldsymbol{F}^*$:

$$(6)\qquad \sigma\boldsymbol{C}=\boldsymbol{C}\sigma=\boldsymbol{C};\quad \sigma\,\boldsymbol{S}=\boldsymbol{S}\,\sigma=-\boldsymbol{S};\quad \boldsymbol{F}^{-1}=\sigma\,\boldsymbol{F}=\boldsymbol{F}\,\sigma\ \Rightarrow\ \boldsymbol{F}=\sigma\,\boldsymbol{F}\,\sigma.$$

It follows from (6) that $\boldsymbol{F}^4=\sigma^2\boldsymbol{F}^4=(\sigma\,\boldsymbol{F}.\boldsymbol{F})(\sigma\,\boldsymbol{F}.\boldsymbol{F})=1$. This well-known result (hereby quite evident) shows that $\{1, \boldsymbol{F}, \boldsymbol{F}^2, \boldsymbol{F}^3\}$ is a linear representation of the cyclic group $\boldsymbol{C}_4$. If, then $\boldsymbol{F}\vec{\boldsymbol{x}}=\lambda\vec{\boldsymbol{x}}$ then $\lambda^4=1$ and the eigenvalues of the Fourier operator are $\{1, j, -1, -j\}$. Determining the multiplicity of the eigenvalues for $\boldsymbol{L}^2(\boldsymbol{Z}/n\boldsymbol{Z})$ is completed by determining the trace $\mathrm{tr}(\boldsymbol{F}_n)$ [15].

If E is n-dimensional vector space over the field of the complex numbers $\boldsymbol{C}$ then general linear group GL(E) is the group of automorphisms of E [9]. It is identified with the group of the invertible matrices. The linear representation of G in E is an arbitrary morphism φ of G in GL(E), i.e. $\varphi(st)=\varphi(s)\varphi(t)$, $\varphi(1)=1$, $\varphi(s^{-1})=(\varphi(s))^{-1}$, $s, t\in G$. The vector space $F\subset E$ is invariant relative to the group G in the representation φ if for each $\vec{\boldsymbol{x}}\in F$ all $\varphi_s(\vec{\boldsymbol{x}})$, $s\in G$, belong to F too. A representation φ is irreducible if E is not zero and has no G-invariant spaces except 0 and E. Each representation is a direct sum of irreducible representations [9]. This means that a basis exists in which the matrices of the representation have a block-diagonal form. The function on G, $\chi_\varphi(s)=\mathrm{tr}(\varphi_s)$ for $\forall s\in G$ is a character of the representation. We are interested in the canonical decomposition of the linear representation [9]. Let $\chi_0,\chi_1,...,\chi_{h-1}$, be the characters of all irreducible representations $F_{\tilde{0}}$, ..., F_{h-1} of the group G and $n_{\tilde{0}}$,..., n_{h-1} are their orders. Let $E=U_0\oplus...\oplus U_{m-1}$ be a decomposition of the representation φ into a direct sum [9] of irreducible representations. Let us denote by E_i (for i = 0, 1,..., h–1) the direct sum of those U_0,...,U_{m-1}, which are isomorphic of F_i. Thus we shall get the canonical decomposition $E=E_0\oplus...\oplus E_{h-1}$. The projector p_i on E_i is given by the following important formula [9]:

(7) $$p_i = \frac{n_i}{g} \sum_{t \in G} \chi_i^*(t)\, \varphi_t; \qquad g = \text{card } G\ .$$

The projectors for the operator $\boldsymbol{F}$, which is a linear representation of the cyclic group $\boldsymbol{C}_4$ have – according to (7), the form:

(8) $$q_{\text{ep}} = \frac{1}{4}(1 + \boldsymbol{F} + \boldsymbol{F}^2 + \boldsymbol{F}^3), \quad q_{\text{en}} = \frac{1}{4}(1 - j\boldsymbol{F} - \boldsymbol{F}^2 + j\boldsymbol{F}^3),$$
$$q_{\text{op}} = \frac{1}{4}(1 - \boldsymbol{F} + \boldsymbol{F}^2 - \boldsymbol{F}^3), \quad q_{\text{on}} = \frac{1}{4}(1 + j\boldsymbol{F} - \boldsymbol{F}^2 - j\boldsymbol{F}^3).$$

We could obtain these projectors from the resolvent [11] of the operators $\boldsymbol{F}$, $R(\xi\, \boldsymbol{F})$ ($\xi \in \boldsymbol{C}$ too). One verifies that [15]:

(9) $$R(\boldsymbol{F}, \zeta) = (\boldsymbol{F} - \zeta)^{-1} = \{\zeta^3 + \zeta^2\ \boldsymbol{F} + \zeta\ \boldsymbol{F}^2 + \boldsymbol{F}^3\}/(1 - \zeta^4).$$

If we introduce the Hartley operator [16, 17] (which is an involution: $\tau^2 = \boldsymbol{C}^2 + \boldsymbol{S}^2 = 1$, $\boldsymbol{F} = \boldsymbol{C} - j\,\boldsymbol{S}$), we could obtain the following projectors:

(10) $$q_{\text{e}} = \frac{1+\sigma}{2};\ q_{\text{o}} = \frac{1-\sigma}{2};\ q_{\text{p}} = \frac{1+\tau}{2};\ q_{\text{n}} = \frac{1-\tau}{2}.$$

The projectors in (10) make it possible for the mutually orthogonal projectors on the eigenspaces $\{\Re_{\mathbf{ep}}, \Re_{\mathbf{en}}, \Re_{\mathbf{op}}, \Re_{\mathbf{on}}\}$of $\boldsymbol{F}$ from (8) to be composed:

(11) $$q_{\text{ep}} = q_{\text{e}} q_{\text{p}},\ q_{\text{en}} = q_{\text{e}} q_{\text{n}},\ q_{\text{op}} = q_{\text{o}} q_{\text{p}},\ q_{\text{on}} = q_{\text{o}} q_{\text{n}}.$$

The solution of the problem for the multiplicity of the eigenvalues and dimensions of the eigenspaces of $\boldsymbol{F}$ now turns out to be trivial – it coincides with the trace of the corresponding projector [15].

Table 1. Dimensions of the invariant spaces

n	$\Re_{\text{ep}}$	$\Re_{\text{en}}$	$\Re_{\text{op}}$	$\Re_{\text{on}}$
$4k$	k+1	k	k	k–1
$4k$+1	k+1	k	k	k
$4k$+2	k+1	k+1	k	k
$4k$+3	k+1	k+1	k+1	k

These dimensions are shown in a table for a different n (mod 4) in [10] (see for alternative approaches).

2.2.3. Invariant spaces of the cyclic and Dihedral groups

The dihedral group $\boldsymbol{D}_n$ in (5) has a cyclic subgroup $\boldsymbol{C}_n = \{1, \rho, \rho^2, \ldots, \rho^{n-1}\}$, or $\boldsymbol{C}_n = \langle \rho \mid \rho^n = 1 \rangle$. The canonical decomposition of this representation allows the convolution (4) to be simplified. The cyclic group has n irreducible representations, which coincide with its characters and have the form:

(12)
$$\chi_h(\rho^k) = e^{j\frac{2\pi}{n}hk}.$$

According to (7) the projectors on the one-dimensional invariant spaces are in the form

(13)
$$p_h = \frac{1}{n}\sum_{k=0}^{n-1} \chi_h^*(\rho^k)\,\rho^k = \vec{\boldsymbol{f}}_h.\vec{\boldsymbol{f}}_h^* = H_h(\rho),$$

where $\vec{\boldsymbol{f}}_h^*$ is the *h*-th column of $\boldsymbol{F}_n$, and * denotes Hermitian conjugation. The set $\{H_h(\rho)\}$ is called uniform-DFT analysis filter bank [18, 19]. In this basis ρ is represented by the diagonal "modulation" matrix μ_n:

(14)
$$?_n = \operatorname{diag}(1,\ e^{j\frac{2\pi}{n}},\dots,e^{j\frac{2\pi}{n}k},\dots,e^{j\frac{2\pi}{n}(n-1)}),\ \rho^k\boldsymbol{F} = \boldsymbol{F}\,?^k,\quad \boldsymbol{F}\rho^k = ?^{-k}\boldsymbol{F}.$$

μ and ρ define a linear representation of the Heisenberg-Weyl nilpotent group $\boldsymbol{N}_n$:

(15)
$$N_n = \langle ?,\rho \,|\, ?^n = \rho^n = [?,\rho]^n = 1;\ \rho.[\mu,\rho] = [?,\rho].\rho,\ ?.[?,\rho] = [?,\rho].\mu\rangle,$$
$$[?,\rho] = \mu.\rho.\mu^{-1}.\rho^{-1}.$$

Obviously

$$H_k(\rho) = ?^{-k}H_0(\rho)?^k = H_0(w^k\rho), w = e^{-j2\pi/n}, H_0(\rho) = \frac{1}{n}\vec{1}\vec{1}^{\mathrm{T}} = \frac{1}{n}(1+\rho+\dots+\rho^{n-1});\ \vec{1}$$

is the vector of all 1's. The bandpass filters $H_k(\rho)$, $k>0$, are realised through modulation and demo-dulation of the prototype lowpass filter $H_0(\rho)$. Each filter $H_k(\rho)$ offers about 13 dB of the minimum stopband attenuation with respect of the zero-frequency gain [18, p.116]. The filters are not very good because $H_0(\rho)$ executes simple averaging.

In the new basis the convolution will be in the form (if $\vec{\boldsymbol{X}} = \boldsymbol{F}\vec{\boldsymbol{x}}$, $\vec{\boldsymbol{H}} = \boldsymbol{F}\vec{\boldsymbol{h}}$):

(16)
$$y_k = (\vec{\boldsymbol{x}}\,|\,\rho^k\sigma\vec{\boldsymbol{h}}) = (\boldsymbol{F}^*\vec{\boldsymbol{x}}\,|\,?^k\,\boldsymbol{F}\,\vec{\boldsymbol{h}}) = \sum_{l=0}^{n-1} X_l\,H_l\,e^{j\frac{2\pi}{n}lk}.$$

It must be pointed out that the involutions $\rho^k\sigma$ and not ρ^k participate in the convolution. Because of this we are interested in the canonical decomposition of D_n. If n is even, $\boldsymbol{D}_n$ has four representations of first order, which map σ and ρ into $\{\pm1\}$ in all possible ways. The rest of irreducible representations are of a second order [9]:

(17)
$$r^m(\rho^k) = \begin{bmatrix} \cos\frac{2\pi}{n}mk & -\sin\frac{2\pi}{n}mk \\ \sin\frac{2\pi}{n}mk & \cos\frac{2\pi}{n}mk \end{bmatrix},\ r^m(\rho^k\sigma) = \begin{bmatrix} \cos\frac{2\pi}{n}mk & \sin\frac{2\pi}{n}mk \\ \sin\frac{2\pi}{n}mk & -\cos\frac{2\pi}{n}mk \end{bmatrix},\ 0 < m < n/2.$$

From (7) the following projectors on E_m can be obtained [15]:

$$p_m = \frac{\varepsilon_m}{n}\sum_{k=0}^{n-1}\cos\left(\frac{2\pi}{n}m\,k\right).\rho^k = \varepsilon_m \operatorname{Re}(H_m(\rho)) = ?_{c,m} H_0(\rho)?_{c,m} + ?_{s,m} H_0(\rho)?_{s,m},$$

(18)
$$0 \le m \le \frac{n}{2}, \qquad \varepsilon_m = \begin{cases} 1, \text{ if } m = 0, \frac{n}{2}, \\ 2, \text{ if } \forall\ m \ne 0, \frac{n}{2}. \end{cases}$$

Here $\sqrt{\varepsilon_m}\,?^m = ?_{c,m} + j\,?_{s,m}, \{H_m(\rho) = \vec{f}_m.\vec{f}_m^{\,*}\}$ is the uniform-DFT analysis filter bank, so that the two vectors

(19)
$$\vec{c}_m = \sqrt{\frac{\varepsilon_m}{n}}\left[\cos\frac{2\pi}{n}lm\right] \text{ and } \vec{s}_m = \sqrt{\frac{\varepsilon_m}{n}}\left[\sin\frac{2\pi}{n}lm\right]$$

form an orthogonal basis of E_m and $p_m = \vec{c}_m\vec{c}_m^{\mathrm{T}} + \vec{s}_m\vec{s}_m^{\mathrm{T}}$. This is a sine-cosine modulated filter bank with the same prototype $H_0(\rho)$. In order to refer E to the basis so constructed the matrix of conversion P_n should be used [15],

(20)
$$P_n = \left[\vec{c}_0 \mid \vec{c}_1 \mid \vec{s}_1 \mid \dots \mid \vec{c}_{n/2-1} \mid \vec{s}_{n/2-1} \mid \vec{c}_{n/2}\right],$$

$$P_n^{\mathrm{T}}.\rho^k\sigma.P_n = \operatorname{diag}(1, r(\rho^k\sigma),\dots, r^m(\rho^k\sigma),\dots, r^{n/2-1}(\rho^k\sigma),(-1)^k),$$

$r^m(\rho^k\sigma)$ is from (17).

If $H_m = (F\,\vec{h})_m = H_{r,m} + j\,H_{i,m}, X_m = (F\,\vec{x})_m = X_{r,m} + j\,X_{i,m}$,

(21)
$$y_k = (\vec{x} \mid \rho^k\sigma\,\vec{h}) = \operatorname{Re}\{\sum_{m=0}^{n/2}\varepsilon_m X_m H_m e^{j\frac{2\pi}{n}mk}\}.$$

Referring E to the orthogonal bases of $\{E_m\}$ rationalises the convolution.

3. Discrete cosine/sine transform

3.1. Invariant spaces of the DFT and DCT/DST

The four invariant subspaces of the Fourier operator $\{\Re_{\mathrm{ep}}, \Re_{\mathrm{en}}, \Re_{\mathrm{op}}, \Re_{\mathrm{on}}\}$ could be joined in more rough decomposition of the space: $\Re_{\mathrm{ep}} + \Re_{\mathrm{en}} = \Re_{\mathrm{e}}, \Re_{\mathrm{ep}} + \Re_{\mathrm{en}} = \Re_{\mathrm{o}}$ – the subspaces of the even and odd vectors (functions on $\boldsymbol{Z}/n\boldsymbol{Z}$). Projectors into these subspaces are $q_{\mathrm{e}} = \frac{1+\sigma}{2}$, $q_{\mathrm{o}} = \frac{1-\sigma}{2}$ from (10).

It is easy to be shown that σ and the identity 1 have two presentations:

(22)
$$\sigma = \sum_{0\le k<n}\rho^k\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^k = \sum_{0\le k<n}\rho^{-k}\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{-k},$$
$$1 = \sum_{0\le k<n}\rho^k\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{-k} = \sum_{0\le k<n}\rho^{-k}\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{k}.$$

Here $\vec{\delta} = [1, 0, 0, ..., 0]^{\mathrm{T}}$ is the *n*-dimensional vector of Dirac. It is important that the two projectors q_e, q_0 have obviously orthogonal columns:

$$(23) \qquad q_e = \frac{1}{2}\sum_{0\le k<n} (\rho^k + \rho^{-k})\,\vec{\delta}\,\vec{\delta}^{\mathrm{T}}\rho^{-k}\ , \quad q_o = \frac{1}{2}\sum_{0\le k<n} (\rho^k - \rho^{-k})\,\vec{\delta}\,\vec{\delta}^{\mathrm{T}}\rho^{-k}\ .$$

If linear-independent columns of these projectors are chosen as the basis of the space, the Fourier operator will take a block-diagonal form. The dimensions of these invariant spaces coincide with the trace of the projectors i.e.

$$(24) \qquad \dim \mathfrak{R}_e = \begin{cases} n/2+1, \text{if } n \text{ even} \\ (n+1)/2, \text{ if } n \text{ odd} \end{cases}; \quad \dim \mathfrak{R}_o = \begin{cases} n/2-1, \text{if } n \text{ even} \\ (n-1)/2, \text{if } n \text{ odd} \end{cases}.$$

These dimensions could be defined from Tabl. 1 too. Of course they coincide with the number of the projectors' linearly independent columns. Two cases could be considered– even and odd dimension of the space.

3.2. Even n – DCT-1/DST-1

We take the first $n/2 + 1$ (orthonormal) columns of the first projector and the first $n/2 - 1$ (orthonormal) columns of the second projector to construct the endomorphism α. This will be orthogonal operator, i.e. the inverse one coincides with the transpose:

$$(25) \quad \alpha = \vec{\delta}\vec{\delta}^{\mathrm{T}} + \rho^{\frac{n}{2}}\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{-\frac{n}{2}} + \frac{1}{\sqrt{2}}\sum_{0<k<n/2}((\rho^k+\rho^{-k})\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{-k} + (\rho^k-\rho^{-k})\vec{\delta}\vec{\delta}^{\mathrm{T}}\rho^{-k-n/2})\ .$$

If $n = 8$, α takes this "stealth aircraft" configuration :

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & -1/\sqrt{2} & 0 & 0 \end{pmatrix}.$$

Fourier operator $\boldsymbol{F}$ will be presented in this new basis from the block-diagonal matrix $\alpha^{\mathrm{T}}\boldsymbol{F}\,\alpha$:

$$\begin{pmatrix} 1/2\sqrt{2} & 1/2 & 1/2 & 1/2 & 1/2\sqrt{2} & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & -1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & 0 & -1/\sqrt{2} & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 1/2 & -1/2 & 0 & 0 & 0 \\ 1/2\sqrt{2} & -1/2 & 1/2 & -1/2 & 1/2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i/2 & -i/\sqrt{2} & -i/2 \\ 0 & 0 & 0 & 0 & 0 & -i/\sqrt{2} & 0 & i/\sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & -i/2 & i/\sqrt{2} & -i/2 \end{pmatrix}.$$

From (25) and the properties of the Fourier operator (14):

(26) $$\boldsymbol{F}\ \rho = ?^{-1}\ \boldsymbol{F}; \quad \boldsymbol{F}\vec{\delta} = \frac{1}{\sqrt{n}}\vec{\mathbf{1}}; \quad \vec{\mathbf{1}} = [1, 1, 1, ..., 1]^{\mathrm{T}},$$

it follows:

(27a) $\boldsymbol{F}\alpha = [\ \vec{\boldsymbol{c}}_0 \mid \ \vec{\boldsymbol{c}}_1 \mid \ \vec{\boldsymbol{c}}_2 \mid ... \ \mid \vec{\boldsymbol{c}}_{n/2} \mid \ -j\ \vec{\boldsymbol{s}}_1 \mid \ -j\ \vec{\boldsymbol{s}}_2 \mid ... \ \mid -j\ \vec{\boldsymbol{s}}_{n/2-1}]$.

Here $\{\vec{\boldsymbol{c}}_k, \vec{\boldsymbol{s}}_k\}$ are the basis orthonormal vectors of the invariant spaces E_k of the dihedral group (19). Finally

(27b) $$\begin{aligned} \alpha^{\mathrm{T}}\vec{\boldsymbol{c}}_k &= \sqrt{\varepsilon_k / n}(\vec{\delta} + \cos(\pi k).\ \rho^{n/2}\vec{\delta}) + \sqrt{2\varepsilon_k / n}) \sum_{0<l<n/2} \cos(2\pi kl/n)\ \rho^l \vec{\delta}, \\ \alpha^{\mathrm{T}}\vec{\boldsymbol{s}}_k &= 2/\sqrt{n} \sum_{0<l<n/2} \sin(2\pi kl/n)\ .\rho^{l+n/2}\ \vec{\delta}. \end{aligned}.$$

This is obviously the block-diagonal matrix:

(27c) $$\alpha^{\mathrm{T}}\boldsymbol{F}\alpha = \begin{bmatrix} \boldsymbol{C1} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{j}\ \boldsymbol{S1} \end{bmatrix}.$$

Here $\boldsymbol{C1}$ and $\boldsymbol{S1}$ are respectively Discrete Cosine Transform-1 (DCT-1) and Discrete Sine Transform-1 (DST-1):

(28) $$\begin{aligned} \boldsymbol{C1} &= \frac{2}{\sqrt{n}}\boldsymbol{D1}[\cos(\frac{2\pi}{n}kl)]\ \boldsymbol{D1};\ \ 0 \le k, l \le n/2. \\ \boldsymbol{S1} &= \frac{2}{\sqrt{n}}\sin(\frac{2\pi}{n}kl);\quad 0 < k, l < n/2. \\ \boldsymbol{D1} &= \mathrm{diag}\ (\frac{1}{\sqrt{2}}, 1, 1, ... 1, 1, \frac{1}{\sqrt{2}}). \end{aligned}$$

3.3. Odd n – DCT-5/DST-5

In this case α is constructed from the first $(n+1)/2$ normalized columns of the first projector and the first $(n-1)/2$ normalized columns of the second projector:

(29) $$\alpha = \vec{\delta}\vec{\delta}^{\mathrm{T}} + \frac{1}{\sqrt{2}} \sum_{0<k<(n+1)/2} ((\rho^{k}+\rho^{-k})\vec{\delta}\,\vec{\delta}^{\mathrm{T}}\rho^{-k} + (\rho^{k}-\rho^{-k})\vec{\delta}\,\vec{\delta}^{\mathrm{T}}\rho^{-k-n/2})\,.$$

If n = 5, α will be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{pmatrix}.$$

In the new basis the Fourier-operator $\boldsymbol{F}$ will take this block-diagonal form $\alpha^{\mathrm{T}}\boldsymbol{F}\,\alpha$:

$$\begin{pmatrix} 0.447214 & 0.632456 & 0.632456 & 0 & 0 \\ 0.632456 & 0.276393 & -0.723607 & 0 & 0 \\ 0.632456 & -0.723607 & 0.276393 & 0 & 0 \\ 0 & 0 & 0 & -0.850651i & -0.525731i \\ 0 & 0 & 0 & -0.525731i & 0.850651i \end{pmatrix}.$$

It can be obtained as in previous part that this matrix has the form

(30) $$\alpha^{\mathrm{T}}\boldsymbol{F}\alpha = \begin{bmatrix} \boldsymbol{C5} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{j}\,\boldsymbol{S5} \end{bmatrix}.$$

In this formula

(31) $$\begin{aligned} &\boldsymbol{C5} = \frac{2}{\sqrt{n}}\boldsymbol{D5}[\cos(\frac{2\pi}{n}kl)]\boldsymbol{D5}; && 0 \le k,l < (n+1)/2; \\ &\boldsymbol{S5} = \frac{2}{\sqrt{n}}[\sin(\frac{2\pi}{n}kl)]; && 0 < k,l < (n+1)/2; \\ &\boldsymbol{D5} = \mathrm{diag}\,(\frac{1}{\sqrt{2}}, 1, 1, ..., 1, 1, 1). \end{aligned}$$

Here $\boldsymbol{C5}$ and $\boldsymbol{S5}$ are the well known DCT-5 and DST-5.

In (27c)–(30) one of the blocks is real and the other – imaginary. It is easy to be proved, that ($\boldsymbol{F} = \boldsymbol{C} - j\,\boldsymbol{S}$):

(32) $$\boldsymbol{C}^2 = \frac{1+\sigma}{2}; \qquad \boldsymbol{S}^2 = \frac{1-\sigma}{2}.$$

For every linear mapping, presented by the $m \times n$ matrix $\boldsymbol{H}$ in a fixed basis for the space of columns (i.e. the range of the values) we have $\Re(\boldsymbol{H}) = \Re(\boldsymbol{H}\boldsymbol{H}^{\mathrm{T}})$ [13, 20]. Hence for this case

(33) $$\Re(\boldsymbol{C}) = \Re(\boldsymbol{C}^2) = \Re(\frac{1+\sigma}{2}) = \Re_e; \quad \Re(\boldsymbol{S}) = \Re(\boldsymbol{S}^2) = \Re(\frac{1-\sigma}{2}) = \Re_o .$$

Referring the space to the orthogonal basis of $\Re_\mathbf{e}$ and $\Re_\mathbf{o}$ transforms the matrices ***C*** and ***S*** to a block-diagonal form in which upper-left/down-right corners are respectively DCT/DST and all the other components are zeros. Wickerhauser proposed presentation like (27c) and (30) for DCT-4 and DST-4 ("the easiest case") and it is obtained by factorization of $2n \times 2n$ matrix [4,19]. Nearly analogical factorization is proposed for DCT-2 and DST-2 in [19].

Geometrical approach was proposed in this paper that allows not only all DCT/DST transforms to be received, but it could be generalized for unitary matrix with definite properties. This generalization will be realized in another paper. Results received in this paper solve the task for the fast realizations of these transforms; all they are derivatives of the DFT, and sparse matrices implement the connections.

4. Conclusions

The block-diagonal form of the Discrete Fourier Transform (DFT) was obtained on the basis of groups theory representations and geometrical approach. Diagonal blocks of this form consist of DCT-1/DST-1 for even dimensions of the signals space and of DCT-5/DST-5 for odd ones. These results allow the full structure of DCT/DST to be constructed. This is important for multirate systems, adaptive filtering, compression of speech signals and images. Along with this it solves the problem of fast realizations of the transforms, which are generically connected with DFT.

5. Some definitions

Some definitions have been used in the paper. The group G is a set G with binary operation $G \times G \to G$, noted as $(a,b) \to ab$ and such, that: 1) It is associative; 2) Identity element $u \in G$ exists, $ua = a = au$, for every $a \in G$; 3) For u and every element $a \in G$ an inverse element $a' \in G$ exists, and $a\,a' = u = a'\,a$.

If G and H are groups, the morphism $\varphi: G \to H$ of these groups is a function from G to H, which is morphism of their binary operations, i.e. $\varphi(ab) = \varphi(a)\,\varphi(b)$ for all $a,b \in G$ [7, 8].

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Инвариантни пространства и косинусови трансформации

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(Р е з ю м е)

В статията се анализират дискретни косинусови и синусови трансформации (DCT/DST) на базата на линейни представяния на крайни групи и на геометричен подход. Тези трансформации са изключително полезни за многоскоростните системи, адаптивното филтриране и компресия на говорни сигнали и образи. Показано е, че ако операторът за дискретна трансформация на Фурие (DFT) се отнесе към подходящ базис, той приема блочно-диагонална форма. Тези блокове съвпадат с DCT-1/DST-1 за четни размерности на пространството на сигналите и с DCT-5/DST-5 – за нечетни. Резултатите позволяват да се изследва пълната структура на DCT/DST.