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A Classification Based Approach for Finding Pareto Optimal Solutions of the Multicriteria Network Flow

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Abstract: *The paper discusses the difficulties in finding integer Pareto optimal solutions of the problem of multicriteria network flow, using efficient flow algorithms only. The application of simplified rules of the classification approach is suggested, with the help of which a Pareto optimal solution is found and the value of one criterion, pointed by the DM, is improved.*

Keywords: *network flow, multicriteria problems, bicriteria linear problems.*

1. Introduction

Let the set N be a set of n nodes and the set U – of m directed arcs (i,j), $i,j\in N$, for a given oriented connected network $G = \{N,U\}$. There are k "cost" parameters a_{ij}^r, $r\in K$ defined, where K is the set of natural numbers from 1 up to k, associated with each arc (i,j). The flow on the arc (i,j) is denoted by $x(i,j) = x_{ij}$. The multicriteria flow problem (**MCF**) may be stated as follows:

$$\textbf{MCF:}\qquad \min{}^*(g_1(x), g_2(x),\dots ,g_k(x))$$

subject to

$$(1)\qquad \sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = \begin{cases} v & \text{if } i = s; \\ 0 & \text{if } i \neq s,t; \\ -v & \text{if } i = t; \end{cases}$$

$$(2)\qquad 0 \le x_{ij} \le c_{ij},\ (i,j)\in U;$$

where s is the source node and t – the terminal node (the sink);

$$g_r(x) = \sum_{(i,j)\in U} a_{ij}^r x_{ij} ,$$

$$v \le v^*,$$

and v^* is the value of the maximal flow. The set of constraints (1)–(2) determines the set of feasible solutions X. The functions g_r are called criteria or objectives and a_{ij}^r is the value for criterion r.

The requirement $v \le v^*$ adds in fact one more criterion to **MCF** problem, that is why it can be avoided introducing the arc (t,s), the first and the last of the constraints (1) being replaced correspondingly:

$$\sum_{j\in N} x_{sj} - x_{ts} = 0 ;$$

$$\sum_{i\in N} x_{it} - x_{ts} = 0.$$

At such a configuration of the network and a non-zero flow in it, the basis of the problem corresponds to a spanning tree T, which contains always the arc (t,s).

A flow $x^1 = \{x_{ij}^1, (i,j)\in U\}$, $x^1\in X$, is a Pareto-optimal feasible solution or flow (P.o.s.), if from the inequality $g_i(x) \le g_i(x^1)$, $i\in K$, for some $x\in X$, it follows that $x = x^1$.

When $k = 1$, the problem is reduced to the single criterion problem for min-cost flow (MF). This is a linear programming problem in general and some polynomial algorithms exist for its solving. The efficiency, the efficient data support of these algorithms is due to the unimodularity of the constraints matrix. That is why the flow problems are distinguished in the class of linear programming problems.

The situation is not similar in multicriteria flow problems. Given that the number of P.o.s. in multicriteria problems is usually very large, in order to find a satisfactory solution, the user, called decision maker (DM) is also engaged in the solution process. He evaluates the current P.o.s. and sets his/her requirements towards the values of some criteria. These preferences are mathematically achieved formulating single criterion problems, called scalarizing problems (SP). In them the optimization of a flow linear function on the set of constraints is required, this set consisting of the initial problem constraints and some added constraints also. In flow problems however, the addition of constraints in the general case destroys the specifics of the constraints matrix and hence no efficient flow algorithms can be applied in the solution of such a SP.

The algorithms solving MCF are of two types usually. The first one, the lexicographic, store the flow structure of the constraints and find integer P.o.s. [1, 4]. In this case the DM can point only the criteria priorities and he/she cannot make choice in a way other than re-ranking of the criteria with respect to new priorities at the beginning of the algorithm. The other type of algorithms allows the DM specify desired values or intervals, in which the values of some criteria are altered and in this case the scalarizing problems are solved by adaptations of the simplex-method, which consider the presence of a flow structure in the set of constraints, the P.o.s. obtained being not obligatory integer [6, 7].

When k=2, methods for solving bicriteria flow problem (BCF) are developed in [2, 3].

The issue of the present paper is the study of the possibility to apply a classification based interactive algorithm [8] in the solution of the problem set.

2. A classification method for solving a continuous multicriteria flow problem

The following classification oriented interactive algorithm is developed in [8], solving linear integer problems:

The set of the criteria $K = \{1, 2, \ldots, k\}$ of the problem formulated is divided by the DM in seven classes as follows:

$K^{<}$ is the set of indices of the criteria that the DM wishes to improve; $K^{\leq}$ is the set of indices of the criteria, which the DM wants to improve by a desired (aspiration) value Δ_k;

$K^{>}$ is the set of indices of the criteria that the DM agrees to be deteriorated;

$K^{\geq}$ is the set of indices of the criteria which the DM agrees to be deteriorated by a certain value δ_k;

$K^{><}$ is the set of indices of the criteria when the DM requires their values to lie in intervals, defined by the values of the corresponding criterion: $g_k - t_k^{-} \leq g_k(x) \leq g_k + t_k^{+}$.

$K^{=}$ is the set of indices of the criteria, the current values of which the DM wants to preserve;

K^{0} is the set of indices of the criteria, in the values of which the DM allows arbitrary changes.

In order to obtain a more satisfactory for the DM P.o.s., the following scalarizing problem P1 is set:

$$\min F(x) = \max \left[\max_{k\in K^{\leq}} (g_k(x) - \bar{g}_k)/|g'_k|, \max_{k\in K^{>}\cup K^{\geq}} (g_k(x) - g_k)/|g'_k|\right] + r \max_{k\in K^{<}} (g_k(x) - g_k)/|g'_k|$$

subject to constraints

$$g_k(x) \geq g_k, \quad k \in K^{<} \cup K^{=};$$

$$g_k(x) \geq g_k - \delta_k, \quad k \in K^{=};$$

$$g_k - t_k^{-} \leq g_k(x) \leq g_k + t_k^{+}, \quad k \in K^{><};$$

$$x \in X,$$

where g_k is the value of the criterion k in the current P.o.s; $\bar{g}_k = g_k - \delta_k$ – the desired value (aspiration level) for the criterion; $\bar{g}_k$ – a scaling coefficient; $r \geq 1$.

In order to find a solution of the optimization problem set, the linear problem P2 is solved:

$$\min(\alpha + r\beta)$$

under constraints:

$$\alpha \geq (g_k(x) - \bar{g}_k)/|g_k'|, \quad k \in K^{\leq};$$

$$\alpha \geq (g_k(x) - g_k)/|g_k'|, \quad k \in K^{>} \cup K^{\geq};$$

$$\beta \geq (g_k(x) - g_k)/|g_k'|, \quad k \in K^{<};$$

$$x \in X_1;$$

α, β – arbitrary numbers;

X_1 – the set of feasible solutions of problem P1.

Problem P2 is a linear programming problem. The set X is however included in its set of constraints, which enables the presence of a flow structure in the set of the linear problem constraints. From another point of view, this is a problem for an optimal flow in a network, which has linear constraints of general type (SLC) added to its constraints set. As mentioned in the introduction, this problem is solved by specialized simplex algorithms, which use network representation of the basic solution and simplified pivoting rules. The basis of such a problem comprises a spanning tree of the network, in which the arc $\{t,s\}$ is always included and p in number additional arcs from the network, where p is the number of the linearly independent additional constraints. In SLC, unlike in flow problems, the solution obtained is not apriori integer, since the matrix of the constraints system is not unimodular anymore.

In order to find a P.o.s. of MCF, an optimization problem for a flow on a network, is formulated, to the constraints of which some linear constraints are added and the condition for its solution integrity is required. No theoretic investigations and algorithms solving this special type of problems are known, except the general approximate or exact algorithms of integer linear programming.

3. Classification oriented approach to the solution of integer MCF

Having in mind the difficulties in MCF problems solving above discussed, the use of simplified rules for classification is offered, in which the DM is allowed to improve the value of one criterion only at each iteration, solving problems for an optimal flow on a network, obtained after the removal of certain arcs.

3.1. Improvement of the value of a criterion, chosen by the DM

Iteration 0: Finding an initial P.o.s.

Problem **FP1** is solved:

$$\textbf{FP1:}\ \text{Min } G_0(x) = \sum_{i\in K} \lambda_i\, g_i(x),$$
$$x\in X\ ,$$

where $\lambda_i > 0,\ \sum_{i\in K} \lambda_i = 1.$

The solution is realized by a network simplex method for finding a flow with a minimal value. Let T^* be the spanning tree corresponding to the optimal solution x^* and l=1.

Iteration *l*: For a given iteration, in which the values of the criteria for the current P.o.s. denoted by x^*, are $g_i(x)$, the set of indices of the criteria are represented as a sum of the following subsets:

$$K = \{k_0\} \cup K^{>} \cup K^{=} \cup K^{0},$$

where k_0 is the index of one criterion only, the value of which is selected for improvement (decrease) by the DM, allowed by the algorithm. If $\{k_0\} = \varnothing$, end of the algorithm.

For each $q\in K^{=}$, the values of the potentials π of the network nodes are computed:

$$\pi^q(s) = 0,$$
$$\pi^q(j) = \pi^q(i) + a_{ij}^q \text{ for each arc } (i,j)\in U,\ (i,j)\in T^*.$$

A network $G = \{N, U^*\}$ is constructed, in which the set of arcs U^* is obtained from the set of arcs of the original network G after eliminating those out-of-basis (not belonging to T^*) arcs, the reduced values of which for criterion q are non zero, i.e.

$$U^* = U \setminus \{(i,j): (i,j) \in T^*,\ a_{ij}^q + \pi^q(i) - \pi^q(j) \neq 0\}.$$

This means elimination of these cycles in the network, each one of which has a positive or a negative value, because along them the flow would be increased or decreased, something, that would break the requirement for the criteria with indices from the set $K^=$.

The set X^* is defined:

$$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = \begin{cases} v & \text{if } i=s, \\ 0 & \text{if } i\neq s,t, \\ -v & \text{if } i=t; \end{cases} \tag{3}$$

$$0 \le x_{ij} \le c_{ij},\ (i,j) \in U^*.$$

The problem

$$\textbf{MCF1}:\ \min{}^*(g(x): i\in K\setminus K^=),$$
$$x \in X^*,$$

is denoted by **MCF1**.

The following problem for minimal flow in a network is solved:

$$\textbf{FP2}:\ \min = \sum_{i\in K\setminus K^=} P_i\, g_i(x),$$
$$x \in X^*,$$

where $0<P_{k0} \equiv 1$, $\sum_{i\in K\setminus K^=} P_i = 1$.

In case the problem has no solution, end of the algorithm or the DM has to alter his/her requirements.

Otherwise, the solution of the problem is denoted by x^*. Then $l:=l+1$ is set and iteration l is executed.

The algorithm proposed finds a P.o.s. of the problem set. Really, x^* is a P.o.s. of MCF1 problem. But if one solution is P.o.s. for part of the criteria of the multiobjective problem put, then it is a P.o.s. for the problem itself.

3.2. Improvement with a given value of the criterion, selected by the DM

In this case the classes of indices of the criteria are defined as follows:

$K^{\le}$ is $\{k_0\}$ index of a criterion, which the DM wishes to improve (decrease) by a value not greater than a given value Δ_{k0};

$K^{>}$ is the set of indices of the criteria that the DM agrees to be deteriorated;

$K^=$ is the set of indices of the criteria, for which the DM's wish is to preserve their current values,

K^0 is the set of indices of the criteria, about which the DM agrees to get arbitrary values;

$$K=\{k_0\} \cup K^{>} \cup K^{=} \cup K^{0}.$$

Iteration 0. Finding an initial P.o.s.

Problem FP1 is solved, $l=1$.

Iteration *l*. Network G^* is constructed.

The bicriteria problem for a flow in a network MCF3 is solved:

$$\textbf{MCF3}: \min\left(g = \sum_{i \in K^{=}} \lambda_i g_i(x)\ g_{k0}(x)\right), \sum \lambda_i = 1.$$

$$x \in X^*.$$

In order to solve the problem, an algorithm, described in [3] is used, which finds adjacent basic P.o.s. of the problem set.

It is known that the P.o.s. basic solutions of the bicriteria linear problem can be ranked according to the increasing value of g, so that:

$$g(x^1) < g(x^2) < g(x^3) < \ldots < g(x^p) < \ldots,$$

$$g_{k0}(x^1) > g_{k0}(x^2) > g_{k0}(x^3) > \ldots > g_{k0}(x^p) > \ldots$$

The solution x^p is adjacent basic solution to x^{p-1} and x^{p+1}.

The adjacent basic P.o.s. can be determined from a basic P.o.s. investigating the reduced cost matrix CR associated with this solution. A column vector CR(i,j) of dimension 2 in CR, associated with a nonbasic arc (i,j) is called effective if CR$(i,j)\neq 0$ and if there exists a vector of weights $\mu = (\mu_1, \mu_2)$, such that

$$\mu\lambda \mathrm{CR} \geq 0 \text{ and } \mu\lambda \mathrm{CR}\,(i,j)=0.$$

Then the basis of the solution (the spanning tree) of the basic P.o.s. obtained is altered by pivoting an effective arc according to certain rules and exclusion of the arc from the tree.

The search for basic P.o.s. continues till the first P.o.s. x^p is found, for which the following is satisfied:

$$g_{k0}(x^p) < g_{k0}(x^*) - \Delta_{k0}.$$

Then x^{p-1} is the solution, with which the algorithm continues its operation.

If the problem has no solution, end of the algorithm or the DM has to change his/her requirements.

The next confirmation [3] proves that x^{p-1} is a P.o.s. of the problem studied: let for a multicriteria problem with k objectives $g_i(\mathrm{x})$, $i \in K$, the numbers λ_i, $i \in K$ be arbitrary nonnegative numbers, the sum of which is 1. Then a P.o.s. of the bicriteria network flow problem with two objectives $\sum_{i \in K-1} \lambda_i g_i$ and g_k is a P.o.s. of the given k-criteria problem; $x^* = x^{p-1}$, $l := l+1$ is set and iteration l is executed.

4. Conclusion

The paper discusses a classification approach applied to the search for Pareto optimal solutions of the problem for multicriteria network flow, satisfying DM's requirements. Depending on the current solution and DM's preferences for improvement of the value of one criterion only and for preserving the values of a set of others, the network is modified in an appropriate way. After that a problem for single criterion flow in a network is solved in case the DM wishes improvement of the criterion value, not defining the bounds of this improvement. If the DM points the maximal value, by which the criterion value is to be decreased, which cannot be greater than the difference between the value of the criterion for the current solution and its minimum on the modified network, a problem of bicriteria flow is solved and this P.o.s. is selected, which satisfies the conditions placed.

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Подход, основан на класификация за намиране на Парето оптимални решения за многокритериален мрежов поток

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(Р е з ю м е)

В статията се дискутират затрудненията при намиране на целочислени Парето оптимални решения на задачата за многокритериален мрежов поток, като се използват единствено ефективните потокови алгоритми. Предложено е прилагането на опростени правила на метод на класификацията, при което се намира Парето оптимално решение, подобряващо стойността на един от критериите, посочени от лицето, вземащо решение.