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An Application of Caratheodory's Theorem to the Spectral Set Problem for Convex Matrix Sets*

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Abstract: *Let K be a compact and convex set of n×n real matrices. The paper presents an application of the well known Caratheodory's theorem to the problem of characterizing the spectral set of K. In particular, using this theorem, it is shown that the entire spectral set can be obtained from the spectra of convex polytopes in K having dimension no greater than 2n. In general, this result enables us to study the spectral properties of K by examining lower-dimensional convex subsets of K.*

Keywords: *Caratheodory's theorem, convex matrix sets, convex polytopes.*

1. Introduction

R^n and $M_n(R)$ will denote the vector spaces of n-dimensional vectors and $n\times n$ matrices with real elements, respectively. For any matrix set $S \subset M_n(R)$ the spectral set (spectrum) of S is defined as

$$\sigma(S) = \{\lambda \in C: \det(\lambda I - A) = 0,\ A \in S\}.$$

The notations $\sigma_R(S)$ and $\sigma_{C\backslash R}(S)$ are respectively given by $\sigma_R(S) = \sigma(S) \cap R$ and $\sigma_{C\backslash R}(S) = \sigma(S) \cap C\backslash R$ where $C\backslash R = \{\lambda \in C: \lambda \neq R\}$. We shall say that a set of matrices has some property if each matrix in the set has this property.

Let K denote a closed and bounded (compact) convex set in $M_n(R)$. This type of matrix sets arises in a variety of applications, e.g. in modelling and robustness analysis

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of control systems with uncertain physical parameters [5], in linear complementarity problems [8], in the analysis and determination of the solution set of systems of linear interval equations [7]. In most cases, it is of essential importance to establish criteria which guarantee that each element in K has some property such as Hurwitz or Schur stability, nonsingularity etc. Since these properties are basically determined by $\sigma(K)$ and its location in the complex plane, this leads to the problem of studying and characterizing the properties of $\sigma(K)$ itself.

The purpose of this paper is to present an application of a standard result from the finite dimensional convex analysis to the spectral set problem for convex matrix sets. In particular, using the well known Caratheodory's theorem, it is shown that $\sigma_R(K)$ and $\sigma_{C\backslash R}(K)$ can be determined from the spectra of convex polytopes in K having dimension no greater than n and $2n$, respectively. This result is formulated as Theorem 2.1 and in general, it enables us to study the spectral properties of K by examining lower-dimensional convex subsets of K. In the literature, similar results are available in some special cases of convex matrix sets. For example, the problem of determining the spectral set of an interval matrix is considered in [4] and stability characterizations for a polytope of matrices in terms of the stability of its exposed sets are obtained in [1] and [2]. Theorems 2.2 and 2.3 in this paper deal with the spectral set problem for two families of convex matrix sets introduced in [3]. The obtained spectral characterizations employ the specific structure of these matrix families and enable us to improve the corresponding stability and nonsingularity criteria formulated in this reference.

The following notions from the convex analysis will be used. Let V denote a finite dimensional vector space. The dimension of a convex set C in V (dim C) is defined as the dimension of the affine hull of C. For any set $S \subset V$, the convex hull of S is denoted by conv S. If S consists of a finite number of elements, i.e. $S = \{x_0, ..., x_p\}$ then conv$S$ is a (convex) polytope in V with vertices $x_0, ..., x_p$. If, in addition, $x_0, ..., x_p$ are affine independent, conv$\{x_0, ..., x_p\}$ is a p-dimensional simplex. It is well known that any closed and bounded convex set $C \subset V$ can be represented as $C = \text{conv}\, S$ where $S \subset C$ is the set of extreme points of C. In this case, the Caratheodory's theorem [6, Theorem 17.1] simply states that C is a union of all d-dimensional simplexes with vertices in S, i.e.

$$C = \bigcup_{x_0,\dots,x_d \in S} \text{conv}\{x_0, ..., x_d\}, \tag{1}$$

where $d = \dim C$. We shall use this result in both cases $V = R^n$ and $V = M_n(R)$.

2. Results

The following theorem provides a spectral set characterization for any compact and convex set in $M_n(R)$.

Theorem 2.1. Let $K \in M_n(R)$ be a compact convex set and $\mathcal{E}$ be the set of its extreme points. Let also $d = \dim K$, $\nu = \min\{n, d\}$ and $\mu = \min\{2n, d\}$. Then

(2) $$\sigma_R(K) = \bigcup_{A_0,\dots,A_\nu \in \mathcal{E}} \sigma_R(\mathrm{conv}\{A_0, \dots, A_\nu\})$$

(3) $$\sigma_{C\setminus R}(K) = \bigcup_{A_0,\dots,A_\mu \in \mathcal{E}} \sigma_{C\setminus R}(\mathrm{conv}\{A_0, \dots, A_\mu\}).$$

P r o o f. If $d \leq n$ then $\nu = \mu = \delta$ and both (2) and (3) follow from (1) with $C = K$. Also, (3) is a consequence of (1) in the case $d \leq 2n$. Thus, we have to prove (2) for $\nu = n < d$ and (3) for $\mu = 2n < d$. (Note that $2n < d$ is possible only if $n > 2$.)

Let $\nu = n$ and $\lambda \in \sigma_R(K)$, i.e. $Ax = \lambda x$ for some $A \in K$ and $x \in R^n$, $x \neq 0$. By the Caratheodory's theorem A can be written as $A = \alpha_0 A_0 + \dots + \alpha_d A_d$ and hence

(4) $$\alpha_0 A_0 x + \dots + \alpha_d A_d x = Ax,$$

where $A_i \in \mathcal{E}$, $\alpha_i \geq 0$, $i = 0, \dots, d$, and $\sum_{i=0}^{d} \alpha_i = 1$. Applying this theorem again with $C = \mathrm{conv}\{A_0 x, \dots, A_d x\} \subset R^n$, it follows that Ax can be represented as a convex combination of no more than $\nu + 1$ affine independent vectors among $A_0 x, \dots, A_d x$. Up to reindexing these vectors, we can then write

(5) $$(\alpha_0' A_0 + \dots + \alpha_\nu' A_\nu)x = Ax = \lambda x,$$

where $\alpha_i' \geq 0$, $i = 0, \dots, \nu$, and $\sum_{i=0}^{\nu} \alpha_i' = 1$. This implies that $\lambda \in \sigma_R(\mathrm{conv}\{A_0, \dots, A_\nu\})$. Conversely, if λ is an element of the union of sets in the right hand side of (2) then the inclusion $\mathrm{conv}\{A_0, \dots, A_\nu\} \subset K$ for any $A_0, \dots, A_\nu$ implies that $\lambda \in \sigma_R(K)$. This proves equality (2).

In order to prove (3) let $\mu = 2n$ and $\lambda = \alpha + i\beta \in \sigma_{C\setminus R}(K)$ where $\alpha, \beta \in R$, $\beta \neq 0$. Following the same approach as in the real case, we note that this is equivalent to

(6) $$\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} z = \begin{bmatrix} \alpha I & -\beta I \\ \beta I & \alpha I \end{bmatrix} z,$$

where $A \in K$ and $z \in R^{2n}$, $z \neq 0$. Using the same arguments in the vector space R^{2n}, it is now obtained that $\lambda \in \sigma_{C\setminus R}(\mathrm{conv}\{A_0, \dots, A_\mu\})$ for some $A_0, \dots, A_\mu \in \mathcal{E}$. Also, if λ is an element of the the right hand side of (3) then the inclusion $\mathrm{conv}\{A_0, \dots, A_\mu\} \subset K$ implies that $\lambda \in \sigma_{C\setminus R}(K)$. This completes the proof.

Both spectral characterizations (2) and (3) are immediate consequences of (1) and the only difference between them is the dimension of the convex substes of K in the right hand side of (2) and (3), respectively. This motivates the separate treatment of the real and imaginary (nonreal) part $\sigma(K)$. On the other hand, it is easily seen from (2) and (3) that

(7) $$\sigma(K) = \bigcup_{A_0,\dots,A_\mu \in \mathcal{E}} \sigma(\mathrm{conv}\{A_0, \dots, A_\mu\}).$$

Thus, the entire spectral set of K can be obtained from the spectra of convex polytopes with vertices in $\mathcal{E}$ and dimension no greater than $2n$.

In what follows, we consider the problem of determining the spectra of two families of convex matrix sets studied in [3]. Applying a similar approach, we utilize the specific structure of these matrix families in order to obtain the corresponding spectral characterizations. The following notations are used.

Let $N = \{1, ..., n\}$ and $N(k)$ be a partition of N into k, $1 \le k \le n$, pair-wise disjoint nonvoid subsets N_j of cardinality n_j, $j = 1, ..., k$. We shall denote by $n_{\max}$ the maximum of n_j, i.e. $n_{\max} = \max\{n_j\colon j = 1, ..., k\}$. For a given $N(k)$, $D_n^{(k)}$ is defined to be the set of all diagonal matrices $D \in M_n(R)$ such that $D[N_j] = d_j I$, $d_j \ge 0$, $j = 1, ..., k$, where $D[N_j]$ is the principal submatrix of D with row and column indices in N_j.

For any $A_0, ..., A_m \in M_n(R)$ and a partition $N(k)$, let $K_r^{(k)}(A_0, ..., A_m)$ and $K_c^{(k)}(A_0, ..., A_m)$ be defined, respectively, by

$$(8) \qquad K_r^{(k)}(A_0, ..., A_m) = \{A \in M_n(R)\colon A = \sum_{i=0}^{m} D_i A_i,\ D_i \in D_n^{(k)},\ i = 0, ..., m,\ \sum_{i=0}^{m} D_i = I\},$$

$$(9) \qquad K_c^{(k)}(A_0, ..., A_m) = \{A \in M_n(R)\colon A = \sum_{i=0}^{m} A_i D_i,\ D_i \in D_n^{(k)},\ i = 0, ..., m,\ \sum_{i=0}^{m} D_i = I\}.$$

Obviously, for a fixed $N(k)$, (8) and (9) describe particular compact and convex matrix sets. We note the two special partitions of N corresponding to the cases $k = 1$, i.e. $N_1 = N = \{1, ..., n\}$ and $k = n$, i.e. $N_j = \{j\}$, $j = 1, ..., n$. It is easily seen that in the former case $K_c^{(1)}(A_0, ..., A_m) = K_r^{(1)}(A_0, ..., A_m)$ is the convex hull of $A_0, ..., A_m$ whereas in the latter case $K_c^{(n)}(A_0, ..., A_m)$ and $K_r^{(n)}(A_0, ..., A_m)$ represent the sets of all matrices whose columns (resp. rows) are independent convex combinations of the columns (resp. rows) of matrices $A_0, ..., A_m$. Given the sets (8) and (9), let $\mathcal{E}_r^{(k)} \subset K_r^{(k)}(A_0, ..., A_m)$ and $\mathcal{E}_c^{(k)} \subset K_c^{(k)}(A_0, ..., A_m)$ be defined, respectively, by

$$(10) \qquad \mathcal{E}_r^{(k)} = \{E = \sum_{i=0}^{m} D_i A_i\colon D_i \in D_n^{(k)},\ D_i[N_j] \in \{0, I\},\ i = 0, ..., m,\ j = 1, ..., k,\ \sum_{i=0}^{m} D_i = I\},$$

$$(11) \qquad \mathcal{E}_c^{(k)} = \{E = \sum_{i=0}^{m} D_i A_i\colon D_i \in D_n^{(k)},\ D_i[N_j] \in \{0, I\},\ i = 0, ..., m,\ j = 1, ..., k,\ \sum_{i=0}^{m} D_i = I\},$$

It is easily seen that each of the sets $\mathcal{E}_r^{(k)}$ and $\mathcal{E}_c^{(k)}$ includes $A_0, ..., A_m$ and that each of these sets contains at most $(m + 1)^k$ different matrices. Also, it can be shown that every matrix in $K_r^{(k)}(A_0, ..., A_m)$ (respectively, $K_c^{(k)}(A_0, ..., A_m)$) can be represented as a convex combination of matrices belonging to $\mathcal{E}_r^{(k)}$ (respectively, $\mathcal{E}_c^{(k)}$). Thus, Theorem 2.1 can be applied to (8) and (9) with $K_r^{(k)}(A_0, ..., A_m) = \operatorname{conv}\mathcal{E}_r^{(k)}$ and $K_c^{(k)}(A_0, ..., A_m) = \operatorname{conv}\mathcal{E}_c^{(k)}$. However, the following lemma allows to obtain spectral set characterizations preserving the particular form of (8) and (9).

Lemma 2.1. Let $A_i \in M_n(R)$, $i = 0, ..., m$, $N(k)$ be a partition of N and $\nu = \min\{n_{\max}, m\}$. For any $x \in R^n$ and $A \in K_r^{(k)}(A_0, ..., A_m)$

$$(12) \qquad Ax = (D_0 E_0 + ... + D_\nu E_\nu)\, x$$

for some $E_i \in \mathcal{E}_r^{(k)}$ and $D_i \in D_n^{(k)}$, $i = 0, ..., \nu$, $\sum_{i=0}^{\nu} D_i = I$.

P r o o f . Since $A \in K_r^{(k)}(A_0, ..., A_m)$, we have

(13) $$Ax = (\bar{D}_0 A_0 + \dots + \bar{D}_m A_m)x$$

where $\bar{D}_i \in D_n^{(k)}$, $i = 0, \dots, m$, $\sum_{i=0}^{m} \bar{D}_i = I$. If $m \le n_{\max}$ then $\nu = m$ and the assertion of the lemma is obvious since $A_0, \dots, A_m \in \mathcal{E}_r^{(k)}$. Assume that $\nu = n_{\max} < m$. According to the partition $N_1 \text{И} \dots \text{И} N_k = N$, (13) can be written as a system of equalities

(14) $$A(N_j)x = \bar{D}_0[N_j]A_0(N_j)x + \dots + \bar{D}_m[N_j]A_m(N_j)x, \ j = 1, \dots, k,$$

where $A(N_j)$ (respectively, $A_i(N_j)$, $i = 0, \dots, m$) is the $n_j \times n$ submatrix of A (respectively, A_i, $i = 0, \dots, m$) with row indeces in N_j, $j = 1, \dots, k$. Since $\bar{D}_0[N_j], \dots, \bar{D}_m[N_j]$ are scalar matrices satisfying $\sum_{i=0}^{m} \bar{D}_i[N_j] = I$, $j = 1, \dots, k$, $A(N_j)x$ is a convex combination of $A_0(N_j)x, \dots, A_m(N_j)x$. By the Caratheodory's theorem, each $A(N_j)x$ can be represented as a convex combination of no more than $n_j + 1$ vectors among $A_0(N_j)x, \dots, A_m(N_j)x$. With $\nu = n_{\max}$ this implies that $A(N_1)x, \dots, A(N_k)x$ can be written as

(15) $$\begin{aligned} A(N_1)x &= \alpha_{10}A_{10}(N_1)x + \dots + \alpha_{1\nu}A_{1\nu}(N_1)x \\ \dots & \quad \dots \qquad \dots \\ A(N_k)x &= \alpha_{k0}A_{k0}(N_1)x + \dots + \alpha_{k\nu}A_{k\nu}(N_k)x \end{aligned}$$

where $\alpha_{ji} \ge 0$, $\sum_{i=0}^{\nu} \alpha_{ji} = 1$ and $A_{ji} \in \{A_0, \dots, A_m\}$, $j = 1, \dots, k$, $i = 0, \dots, \nu$.

Obviously, in the cases $n_j < n_{\max}$, $j = 1, \dots, k$ the corresponding equalities in (15) contain $n_{\max} - n_j$ coefficients $\alpha_{ji} = 0$, respectively. Let matrices $E_0, \dots, E_\nu$ be given by $E_i = \tilde{D}_1 A_{1i} + \dots + \tilde{D}_k A_{ki}$, $i = 0, \dots, \nu$ where $\tilde{D}_1, \dots, \tilde{D}_k \in D_n^{(k)}$ are diagonal matrices such that $\tilde{D}_i[N_i] = I$, $i = 1, \dots, k$ and $\tilde{D}_i[N_i] = 0$, $i, j = 1, \dots, k$, $i \ne j$. Thus, $E_0, \dots, E_\nu \in \mathcal{E}_r^{(k)}$. Let also $D_0, \dots, D_\nu \in D_n^{(k)}$ be determined by $D_i[N_i] = \alpha_{ji} I$, $j = 1, \dots, k$, $i = 0, \dots, \nu$. Since $\sum_{i=0}^{\nu} \alpha_{ji} = 1$, $j = 1, \dots, k$ we have $\sum_{i=0}^{\nu} D_i = I$. With this notation, it is easily seen that (15) can be written in the form (12) which completes the proof.

Now, we can state the following results.

Theorem 2.2. Let $A_i \in M_n(R)$, $i = 0, \dots, m$, N_k be a partition of N, $\nu = \min\{n_{\max}, m\}$, and $\mu = \min\{2n_{\max}, m\}$. Then

(16) $$\sigma_R(K_r^{(k)}(A_0, \dots, A_m)) = \underset{E_0,\dots,E_\nu \in \mathcal{E}_r^{(k)}}{\text{И}} \sigma_R(K_r^{(k)}(E_0, \dots, E_\nu)),$$

(17) $$\sigma_{C\backslash R}(K_r^{(k)}(A_0, \dots, A_m)) = \underset{E_0,\dots,E_\mu \in \mathcal{E}_r^{(k)}}{\text{И}} \sigma_{C\backslash R}(K_r^{(k)}(E_0, \dots, E_\mu)).$$

The proof of this theorem is essentially based on equality (12) and follows the same steps as in the proof of Theorem 2.1. By applying Theorem 2.2 to $K_r^{(k)}(A_0^{\mathrm{T}}, \dots, A_m^{\mathrm{T}})$ and noting that $\sigma(K_r^{(k)}(A_0^T, \dots, A_m^T)) = \sigma(K_c^{(k)}((A_0, \dots, A_m))$, we obtain an analogous result for the spectrum of (9).

Theorem 2.3. Let $A_i \in M_n(R)$, $i = 0, \dots, m$, N_k be a partition of N, $\nu = \min\{n_{\max}, m\}$, and $\mu = \min\{2n_{\max}, m\}$. Then

(18) $$\sigma_R(K_c^{(k)}(A_0, \dots, A_m)) = \underset{E_0,\dots,E_\nu \in \mathcal{E}_c^{(k)}}{\text{И}} \sigma_R(K_c^{(k)}(E_0, \dots, E_\nu)),$$

(19) $$\sigma_{C\backslash R}(K_c^{(k)}(A_0, \dots, A_m)) = \underset{E_0,\dots,E_\mu \in \mathcal{E}_c^{(k)}}{\text{И}} \sigma_{C\backslash R}(K_c^{(k)}(E_0, \dots, E_\mu)).$$

The above theorems provide spectral set characterizations of (8) and (9) in terms of the spectra of convex sets which are generated by the extreme matrices and are of the same form as $K_r^{(k)}(A_0, ..., A_m)$ and $K_c^{(k)}(A_0, ..., A_m)$, respectively. Depending on the partition $N(k)$ and matrices $A_0, ..., A_m$, these results enable us to reduce the "dimensionality" of the spectral set problem under consideration. Equalities (16) – (19) show that such a reduction is possible with respect to the entire spectral set if $2n_{\max} < m$ and with respect to the real part of the spectrum if $n_{\max} < m$. We note that in the two special cases of partitions with $k = 1$ and $k = n$, the corresponding values of $n_{\max}$ are $n_{\max} = n$ and $n_{\max} = 1$.

Theorems 2.2 and 2.3 are particularly relevant to the work presented in [3] where various properties of (8) and (9) are studied by means of P-matrices and block P-matrices. For example, it is shown in this reference that the nonsingularity of (8) (respectively, (9)) is equivalent to the block P-property of a specially constructed test matrix with dimension $mn \times mn$. In view of the above results, however, one can apply this nonsingularity criterion to the matrix sets in the right hand side of (16) (respectively, (18)) in which case the dimension of the resulting test matrices is reduced to $\nu n \times \nu n$. In a similar way, Theorems 2.2 and 2.3 can be used to improve the criteria for Hurwitz and Schur stability of (8) and (9) obtained in [3].

3. Conclusion

The results in this paper illustrate an application of the classical Caratheodory's theorem to the problem of determining the spectral set of a general compact and convex set in $M_n(R)$. In the cited literature, a special case of this problem is considered in [4] where it is shown that the spectral set of an interval matrix can be obtained as a union of the spectra of its n-dimensional exposed sets. This result, however, is no longer valid for more general convex sets in $M_n(R)$ which can be seen from the stability characterizations for a polytope of matrices obtained in [1] and [2]. In this context, Theorem 2.1 gives a spectral set characterization which is applicable to any compact and convex matrix set. Theorems 2.2 and 2.3 represent more specialized results related to the spectral properties of matrix sets described in the form (8) and (9). These theorems are primarily motivated by the work in [3] and can be used to improve some of the results in this reference.

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Приложение на теоремата на Каратеодори в задачата за определяне на спектралното множество на изпъкнали матрични множества

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(Р е з ю м е)

Нека K представлява компактно и изпъкнало множество от реални матрици с размерност $n \times n$. Обсъжда се приложението на добре познатата теорема на Каратеодори в задачата за определяне на спектралното множество на K. Показано е, че използването на тази теорема дава възможност да се получи цялото спектрално множество от спектрите на изпъкналите многостени в K, които имат размерност, не по-голяма от $2n$. Този резултат позволява изследването на спектралните свойства на K, изучавайки изпъкналите подмножества в K с по-малка размерност.