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An Approach for Finding Pareto Optimal Solutions of the Multicriteria Network Flow Problem

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Abstract: *A survey is done of Pareto optimal solutions of multicriteria problems of flows in a network. An algorithm is proposed that finds solutions when the decision maker (DM) sets a requirement about the criteria upper limit. Analysis of the approach suggested and the results obtained is represented.*

Keywords: *multicriteria network flow problem, Pareto optimal solutions, DM.*

1. Introduction

Let the network $G=\{N,U\}$ consists of a set N of n nodes and a finite set U of m directed arcs (i, j), $i, j \in N$. There are defined k "cost" parameters $a_{ij}^{\,r}$, $r \in I_k$, where I_k is the set of natural numbers from 1 to k, which are associated with each arc (i, j). The flow on the arc (i, j) is designed with $x(i, j)=x_{ij}$. The multicriteria flow problem (MCF) may be stated as follows:

MCF: $$\min{}^* \left(g_1(x), g_2(x), \dots, g_k(x)\right)$$

subject to

(1) $$\sum_{j\in N} x_{ij} - \sum_{j\in N} x_{ji} = \begin{cases} v & \text{if } i=s, \\ 0 & \text{if } i\neq s, t \\ -v & \text{if } i=t; \end{cases}$$

(2) $$0 \le x_{ij} \le c_{ij}, \quad (i, j) \in U,$$

where s is the source node and t is the terminal node (the sink),

$$g_r(X) = \sum_{(i,j)\in U} a_{ij}^{\,r}\, x_{ij},$$

$$v \le v^*,$$

and v^* is the value of the maximal flow. The set of constraints (1) – (2) determines the set of feasible solutions X. The functions g_i are called criteria or objectives.

The requirement $v \leq v^*$ adds in fact one more criterion to MCF problem, that is why it can be avoided introducing the arc (t, s), the first and the last of the constraints (1) being replaced correspondingly:

$$\sum_{j \in N} x_{sj} - x_{ts} = 0,$$

$$\sum_{i \in N} x_{it} - x_{ts} = 0.$$

At such a configuration of the network and a non-zero flow in it, the basis of the problem corresponds to spanning tree T, which always contains the arc (t, s). Since T is a network structure, which does not include cycles, the set of its remaining arcs defines the cutting set $(X, \bar{X})$ in G. The pivoting of a new arc in the basis causes either a change in the cut, or a change in the elements of any of the sets X or $\bar{X}$. A flow $x^1 = \{x_{ij}^1, (i,j) \in U)\}$, $x^1 \in X$, is a Pareto-optimal feasible solution or flow (P.o.), if from the inequality $g_i(x) \leq g_i(x^1)$, $i \in I_k$, for some $x \in X$, it follows that $x = x^1$.

When k=1 the problem is reduced to the single criterion problem for min-cost flow (MF). This is a linear programming problem in general and some polynomial algorithms exist for its solving. The efficiency, the efficient data support of these algorithms are due to the unimodularity of the constraints matrix. That is why the flow problems are distinguished in the class of the linear programming problems.

When k=2, methods for solving bicritera flow problem (BCF) are developed in [2, 3]. The idea of the algorithm in [3] is described in the next section.

The problem MCF is a multicriteria linear programming problem. The methods for solving this class of problems can be applied to it. In [1] a network specializations of the primal simplex multicriteria algorithms is developed.

The advantages of the properties of pure network models are used in solving MCF with preemptive priorities, assigned to the objectives. In this case the criteria are previously ordered by the decision maker according to their importance. The first criterion is the most important, the second – less than the first, the third – less than the second and so on. It is proved in this case that there exists a scalar $M^0 > 0$ such that for any $M > M^0$, a solution x^* is a preemptive optimal solution if and only if it solves the problem

$$\min \sum_{i \in I_k} M^{k-1} g_i$$

i.e., it is an P.o. solution. In [4] this problem is worked out solving a sequence of single criterion flow problems, optimizing the criterion of highest priority over an appropriately modified network, then minimizing the next in order criterion over a network modified again and etc.

In many cases the finding of ef.s. is performed by solving single criterion problems where the set of the constraints of the MCF is enlarged by additional constraints, which are linear constraints in the case investigated. These methods destroy the unimodular structure of the original matrix of the constraints. Adaptations of the simplex methods, which use the properties of the embedded network structure are designed in [5] and [6].

2. Setting the problem

In a lot of the cases of multicriteria problems solving, the decision maker sets the condition to obtain a solution, in which the values of the criteria have real practical sense – for example path length, value, time, etc., and are within definite limits, i.e.:

$$g_i(x) \leq b_i,\ i \in I_k,\ \text{s.t. } x \in X.$$

These constraints are defined as equalities in the following way: the set of the network arcs is enlarged by the arcs $(t, s)_i$, $i \in I_k$, i.e. – κ in number arcs from the sink node to the source node:

$$g_i(x) + x\,(s, t)_i = b_i,\ i \in I_k,\ \text{s.t. } x \in X.$$

Statement 1

In [3] an algorithm finding basic P.o. solutions of BCF problem is described. The fact is used that the P.o. basic solutions x^i can be ranked according to the increasing value of $g_1(x^i)$, so that:

$$\begin{aligned} &g_1(x^1) < g_1(x^2) < g_1(x^3) < \ldots, \\ (3)\qquad &g_2(x^1) > g_2(x^2) > g_2(x^3) > \ldots. \end{aligned}$$

The solution x^p is adjacent basic solution to x^{p-1} and x^{p+1}.

The adjacent basic P.o. solution can be determined from a basic P.o. solution investigating the reduced cost matrix CR associated with this solution. A column vector $CR(i, j)$ of dimension 2 in CR, associated with a nonbasic arc (i, j) is called effective if $CR(i, j) \neq 0$ and if there exists a vector of weights $\lambda = (\lambda_1, \lambda_2)$, such that

(4) $$\lambda CR \geq 0 \text{ and } \lambda CR(i, j) = 0.$$

Let $u^1(i)$ and $u^2(i)$ be dual variables (potentials) associated with a node i for the first and the second objective functions respectively. The potentials of node j, the ending node of the arc (i, j) in the spanning tree corresponding to the basic P.o. solutions, are determined by the equations

$$u^1(j) = u^1(i) + a_{ij}^{\,1},$$
$$u^2(j) = u^2(i) + a_{ij}^{\,2}.$$

For each arc $(i, j) \in U$, the vector $CR(i, j)$ is determined as follows:

$$CR^1(i, j) = u^1(i) - u^1(j) + a_{ij}^{\,1},$$
$$CR^2(i, j) = u^2(i) - u^2(j) + a_{ij}^{\,2}.$$

Moving from x^p to x^{p+1}, in order to obtain the basic tree associated with x^{p+1}, we must remove an arc from the tree T_p corresponding to x^p and pivot another arc. The appropriate arc to enter the basis is that arc which results in minimum increase of the first objective for a unit decrease in the second objective. To satisfy (3) the potentials of this arc must satisfy the inequalities

$$CR^1(i, j) > 0,$$
$$CR^2(i, j) < 0.$$

We determine the function $d(i,j)$ on the set of nonbasic arcs as follows:

$$d(i, j) = \begin{cases} |CR^1(i, j)\ /\ CR^2(i, j)| & \text{if } CR^1(i, j) > 0 \text{ and } CR^2(i, j) < 0, \\ \propto & \text{otherwise.} \end{cases}$$

The arc (p, l) to be pivoted into the basis corresponds to the $d(p, l)$ value for which

(5) $$d(p, l) = \min \{\, d(i,j)\ /\ (i,j) \in U \text{ and is a nonbasic arc} \}.$$

The components of vector λ corresponding to the basic ef.s. are determined as follows

$$\lambda_1 = d(p, l) / (1+d(p, l)),$$

(6)

$$\lambda_2 = 1/(1+d(p, l).$$

In order to obtain the new basic solution, on the cycle $\sigma_p(p, l)$, formed by the arc (p, l) and T_p, oriented according to the arc (p, l), if $x_{ij}=0$ and in the opposite direction – if $c_{ij}=x_{ij}$, the values $\varepsilon_1 = \min\ (c_{ij} - x_{ij})$ for forward arcs of the cycle and $\varepsilon_2 = \min\ x_{ji}$ for backward arcs and the value $\varepsilon = \min\ (\varepsilon_1, \varepsilon_2)$ are determined. We change the flow along the cycle by ε; T_p is changed to T_{p+1} by determining the leaving arc and pivoting (p, l). The potentials of the nodes of the network are adapted.

The first basic P.o. solution is defined solving the problem of minimal flow in a network with an objective function $(1 - \alpha)g_1 + \alpha g_2$, where α approximates 0 and is a positive number.

Statement 2

Another property that will be used by the future considerations is the theorem proved in [3], which states:

Let for a multicriteria problem with k objectives $g_i(x)$, $i \in I_k$, the numbers λ_i, $i \in I_{k-1}$ be arbitrary nonnegative numbers, the sum of which is 1. Then a P.o. solution of the bicriteria network flow problem with two objectives $\sum_{i \in I_{k-1}} \lambda_i g_i$ and g_k is a P.o. solution of the given k-criteria problem and the reverse.

Statement 3

In [2] the properties of a flow in a network are discussed, when besides the constraints on the arcs capacity, κ in number additional linear constraints are added, their left sides being linear combinations of the flows on arcs sets and let the matrix A of these constraints be of rank k. Let f be a flow, which is a basic solution of the problem set and $f(m,\sigma)$ is its decomposition in paths μ and cycles σ of the network. In the case when the arc (t,s) is included in the basis, the flow is decomposed only with respect to the cycles $\sigma = \{\sigma_i : i=1,\ldots, m_1\}$. Let $A(\sigma) = |\alpha(j,\sigma_i)|$ denotes the constraints matrix when the flow of this type is formulated in terms paths-cycles. The element $a(j,\sigma_i)$ is a sum of the coefficients in function g_j of the arcs of this cycle, multiplied by 1 or -1, depending on the orientation of each one of these arcs with respect to the cycle orientation.

In the case when the arc (t, s) is included in the basis, the flow is decomposed with respect to cycles only. It is proved that $A(\mu,\sigma)$ has a rank k, i.e. a spanning tree corresponds to the basic P.o. solution f plus some arcs of the network the number of which is k, or k cycles formed by those k arcs and the spanning tree.

Statement 4

Let x be a P.o. solution for a BCF problem. Then in the network there is not cycles with negative cost with respect to the second objective which cost is zero for the first objective.

Description of the algorithm

Deriving from **statements 1–3**, an interactive algorithm is proposed, with the following general description: basis P.o. solution. is determined at the first iteration, for which the first and the second objective functions satisfy the conditions set by the DM.

At each following iteration p such basic P.o. solution is determined, for which the criteria (p+1) satisfies the constraint on its value solving a bicriteria problem of a network flow. The first criterion of this problem is a linear combination of the linear functions of the first p criteria, located within the required bounds, and the second one – p+1-st criterion.

Algorithm description:

Iteration1. The following BCF problem is solved with the help of the algorithm described in the **Statement**:

$$\text{Min } \left(\sum_{(i,j)\in U} a_{ij}^1 x_{ij} , \sum_{(i,j)\in U} a_{ij}^2 x_{ij} \right),$$

$$\text{s.t } x\in X.$$

As a result of the solution basic P.o. solutions are obtained, for which the values of the two objective functions are ranked as above:

$$g_1(x^1) < g_1(x^2) < \ldots g_1(x^p) < \ldots,$$

$$g_2(x^1) > g_2(x^2) > \ldots g_2(x^p) > \ldots$$

Let T_i denotes the spanning trees corresponding to the P.o. x^i.

1. If x^p is reached, for which the conditions

(7) $$g_1(x^p) \le b_1 \text{ and } g_2(x^p) \le b_2,$$

are satisfied, the numbers λ_1^1 and λ_2^1 are defined from (6**)** and the objective function G_1, which is minimized by $x^p\in X$ is formulated as a linear combination of $g_1 + x_1(s,t)_1 = b_1$ and $g_2 + x_1(s,t)_2 = b_2$ with coefficients λ_1^1 and λ_2^1; x_1 denotes the flow, corresponding to the basic solution. **Iteration 2** is executed.

2. In case x^p is obtained, for which the conditions $g_1(x^p) > b_1$ and $g_2(x^p) > b_2$ are satisfied under the condition that for $g_1(x^{p-1}) \le b_1$ and $g_2(x^{p-1}) > b_2$, a nonbasic arc (i, j), is searched for the spanning tree T_p, for which

(8) $$CR^2(i, j) < 0 \text{ и } CR^1(i, j) = 0.$$

The flow along the cycle formed by the arc (i, j) and spanning tree T_{p-1} is altered. The procedure continues till a nonbasic arc satisfying (8) is obtained. The flow corresponding to the basic solution is denoted by x_1. If conditions (7) are satisfied for the current solution, the algorithm goes to step 1. Otherwise, go to step 2 or if all the nonbasic arcs are tested the problem of finding P.o. solution, for which the objective functions are within the searched bounds, has no solution. It is necessary the DM to change his/her requirements.

Iteration q. 1. If $q=k$, end of the algorithm. Otherwise the bicriteria flow problem BCF is considered:

$$\text{Min } (G_q, g_{q+1}) \text{ s.t. } x\in X,$$

where $G_q = \sum_{i\in I_{q-1}} \lambda_i^{q-1} g_i$.

If $g_{q+1}(x_{q-1}) \le b_{q+1}$, the next objective function is regarded and so on, until such one is reached exceeds its upper limit and it becomes the second criterion. Otherwise – the algorithm ends.

The algorithm for solving BCF is applied. When passing from one to another P.o. solution an approach different from the described in **statement 1** is applied to alter the flow and hence way of determining the new basis.

Let the arc (i_0, j_0) be the nonbasic arc for which condition (5) is satisfied. This arc is determined to be pivoted into a basis the spanning tree of which is T_p.

2. With the purpose to pass to new basic P.o. solution of the bicriteria problem, not altering the values of the first q criteria already determined, a system of q in number linear equations $\sigma_1,\ldots,\sigma_q$ are solved, the left sides of which are defined on cycles created by the current basis of the problem. Let the arc (i_0, j_0) and T_p form the cycle σ_{q+1}. Then the system acquires the following form:

$$a(1,\sigma_1)y_1+\ldots+a(1,\sigma_q)y_{\backslash q}=a(1,\sigma_{q+1}),$$

$$\text{-}$$

$$a(q,\sigma_1)y_1+\ldots+a(q,\ s_q)y_q=a(q,\sigma_{q+1}).$$

The solution of the system is unique, because det $|A(i,\sigma)|\neq 0$.

Each basic arc (i,j) is assigned the value $y_r(i,\ j)=y_r$, if $(i,j)\in\sigma_r$. We define $y_{k+1}(i_0,\ j_0)=1$ and change the flow along the arcs $(i,\ j)$ of the cycles $\sigma_1,\ldots,\sigma_{q+1}$–

$$x_{ij}:=x_{ij}+\theta\,\Sigma y_r(i,\ j),$$

where θ is determined by the correlations computed on the basic arcs $(i,\ j)$:

$$\theta_1=\min\Big(c_{i0,j0}-x_{i0,j0},\ \min_{\Sigma y_r(i,j)>0}(c_{ij}-x_{ij})/\Sigma y_r(i,j),\ \min_{\Sigma y_r(i,j)<0}(-x/\Sigma y_r(i,j))\Big),$$

if the arc $(i_0,\ j_0)$ is a forward arc for the cycle σ_{q+1} .

$$\theta_2=\min\Big(x_{i0,\ j0},\ \min_{\Sigma y_r(i,j)>0}(c_{ij}-x_{ij})/\Sigma y_r(i,j),\ \min_{\Sigma y_r(i,j)<0}(-x/\Sigma y_r(i,j))\Big),$$

if the arc $(i_0,\ j_0)$ is a backward arc for the cycle σ_{q+1}.

$$\theta=\min\,(\theta_1,\theta_2).$$

The arc $(i_*,\ j_*)$, for which θ is reached, leaves the basis, and the arc $(i_0,\ j_0)$ is introduced in it.

If there is a nonbasic arc for which $a(q,\sigma_{q+1})<0$, where σ_{q+1} is the cycle formed by this arc denoted by $(i_0,\ j_0)$ and the new spanning tree, go to the step 2.

In case x^p is reached, for which the conditions

$$g_{q+1}(x^p)\leq b_{q+1}, \tag{9}$$

are satisfied, the numbers $\lambda_1^{\ q}$ and $\lambda_2^{\ q}$ are defined from (6) and the objective function G_q minimized by $x^p\in X$ is stated as a linear combination of G_{q-1} and g_{q+1} and with coefficients $\lambda_1^{\ q}$ and $\lambda_2^{\ q}$. The flow corresponding to the basic solution is denoted by x_q. The potentials of the nodes of the network are altered. $q:=q+1$ is set and the **Iteration q** is executed.

3. Conclusion

The paper makes a survey of the approaches that seek for P.o. solutions of multicriteria problems of flows in a network considered by the author. An algorithm is proposed to find such solutions when the DM sets the requirement for upper limit on the criteria. The main idea of the approach suggested is to determine a sequence of basic P.o. solutions of bicriteria problems for a network flow. The first one of the criteria is a linear combination of the criteria that are already within the limits required, the second one is the next criterion that is to be improved. The P.o. solutions found are not integer at integer data of the problem studied.

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Подход за намиране на Парето-оптимални решения при многокритериални задачи за потоци в мрежа

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(Р е з ю м е)

В статията е представен обзор на Парето оптимални решения при многокритериални задачи за потоци в мрежа. Предложен е алгоритъм, при който лицето, вземащо решение, поставя изискване за горната граница на критериите. Направен е анализ на предложения подход и на получените резултати.