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Bi-level Optimization of Inventory and Production

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***Abstract**: A quantitative approach to inventory integration and production management is applied. Quantifying the relationships between inventory and production allows for minimizing inventory costs and maximizing production. A general optimization problem is derived that integrates inventory and production decisions. The optimization is formalized as a bi-level problem. The inventory and production objective functions are hierarchically integrated. The solution to the bi-level problem provides both optimal arguments for inventory size and production volumes. The problem is applied to the inventory of agricultural products that meet the required nutritional content for animal feed. The inventory delivery costs are minimized and nutritional content is maximized to meet the given nutritional level of the animal feed. Empirical results provide advantages for the bi-level solutions obtained. A sensitive analysis of the inventory is performed and the corresponding changes in the feed content are evaluated.*

***Keywords**: Bi-level formalization, Inventory management, Integration between inventory and production tasks, Production optimization.*

1. Introduction

The production of goods must be supported by effective support with raw materials, which is a prerequisite for the quantity and quality of final products. The obligation to supply raw materials in production is assigned to the inventory management task. Effective linkages and integration of inventory and production are key prerequisites for successful enterprise management. Integrating inventory and related production cases such as production type planning, machine scheduling, meeting just-in-time requirements, demand management, and decisions based on quantitative decisions derive optimal solutions for the entire enterprise operations.

The inventory process is aimed at ensuring a smooth supply of raw materials needed to produce goods. Supply is concerned with attempts to reduce costs incurred in the process of supplying and transporting raw materials. Furthermore, inventory management is responsible for keeping these resources safe from loss, theft, or unauthorized use. These inventory process requirements are formalized in the costs of shipping, transportation, holding, and storage. The optimization process in inventory management is aimed at minimizing all these different sets of costs, which is beneficial for the overall management of the business entity. By its very nature, the inventory process incurs costs. This is the opposite economic direction of the production process, which is directed towards the flow of resources into the business organization. Therefore, an optimal inventory policy can reduce the costs incurred for [1]:

- increasing profits by reducing financial costs;
- to maintain no exceeded quantities of spare parts for production engines;
- timely delivery of resources favors the time for production of goods and flexible changes of the production plan according to the dynamic requirements of the market;
- reorganization of production technology by increasing the production of goods;
- assessment and maintenance of a minimum level of additional resources to cover losses from shortages.

For inventory management, an appropriate set of formal relations is used, which are applied to define and solve optimization problems. In particular, inventory optimization provides quantitative solutions for the optimal volumes of materials with minimal costs; reducing losses in warehouse storage, considering the warehouse limitation, and finding the sequences of new delivery [2].

Since production technology is strongly influenced by the results of inventory processes, overall business management must coordinate and integrate tasks related to production resource supply. Such integration should be formalized into a general optimization problem that will benefit the business entity's performance. By having a formally defined optimization problem, the business can respond appropriately to market changes and fluctuations in demand.

Inventory management is closely related to the requirements created by production technology. Therefore, it is worthwhile to integrate resource supply with commodity production into a general optimization problem, as market behavior implements dynamic changes in commodity demand.

The added value of the paper concerns the formalization of the integration of the two important tasks of business management: inventory and production. The formalization is performed with a bi-level optimization problem. The solutions to such problems give simultaneously optimal values of two-objective functions. The set of constraints satisfies the requirements of both optimization criteria. The bi-level problem is applied to the composition of nutrients in cow feed. This favors cow farming milk production. Empirical evaluation of the bi-level solutions and comparisons show superior results in minimizing inventory costs and increasing the nutritional value of feed.

The paper contains six sections. After the introduction, the second section presents an overview of potential solutions integrating the inventory and production task. An analysis of the applied formal modeling is performed. Most of the solutions belong to the definition and solution of a general optimization problem. Section three presents the formal definition of our bi-level problem. It contains two hierarchically ordered optimization problems for minimizing inventory costs and maximizing production. In section four, the bi-level problem is applied to the nutrient composition of cow feed with the inventory of agricultural products. Numerical results from the solution of the optimization problem are presented. In section five, a sensitivity analysis of the optimization solutions is presented. It assesses changes in stocks and nutrient content in response to increases in market prices of agricultural products. The final section 6 concludes that an increase in market prices of agricultural products can be quantitatively offset by an increase in another and this can preserve the nutritional content of the feed. Future developments are discussed for the application of bi-level optimization to simultaneously minimize the required power from the electrical grid and maximize the use and storage of photovoltaic generation.

2. Literature review of solutions for integration of inventory and production tasks

To obtain quantitative solutions for both stocks and production, a suitable numerical problem must be derived. The estimation of quantitative relationships between the parameters of the inventory and the production processes is a prerequisite for defining an optimization problem aimed at minimizing costs and/or maximizing returns. A formal definition of such a problem is presented in [3], where a linear integer optimization problem is described and solved.

The problem simultaneously estimates the values of resources and the resulting quantities of goods. A similar approach to integrating inventory and production activities is formalized in [4]. The result of such a formalization gives the optimal supply values of supplies for inventory.

The integration of inventory and production is formalized in [5] with the application of Markov chains. The defined formal model aims to maximize the quality of goods during their production. In [6], a kind of integration of several processes related to production is derived. These processes are sequentially linked and relate to the supply of inventory, logistics operations, and the production of goods. In [7], the intensity of good production following changes in market requirements is defined as an optimization problem. The formalization applied to this problem involves statistical estimates of the parameters of average demand and its standard deviation. Additionally, correlations between the required volume of resources and the number of goods produced are evaluated. The defined optimization problem contains statistical parameters and it is combinatorial by definition. A heuristic algorithm for its solution, based on the ant colony algorithm is presented in [7]. Further analysis of the supply of materials and the production of goods and the integration of these two tasks into a common economic chain is presented in [8]. This analysis is carried out for the production of predetermined goods: textile, food production, and flour products. The administrative processes of business management have an impact on inventory and production processes. In [9] explicit recommendations for administrative management to improve inventory and production are presented.

The importance of integrating the production and distribution planning model into the supply chain is discussed in [10]. The practical case that is solved is the production of a set of orders for different customers on a single machine. A bi-level formalization is applied that defines the planning parameters for the production of different orders. In [11], the same integration task between the production schedule and delivery schedule is formalized as an optimization problem. Integration is achieved at the production stage, as all parts of the products are stored at retail and then the set of different parts is delivered to the customers. The problem of production and distribution in a multitier supply chain network is considered in [12]. A bi-objective problem is presented that integrates the tasks of production planning and production allocation in a multi-level supply chain network with multiple product types and multiple periods. In [13], manufacturing is integrated with concurrent tasks in the customer supply chain. This study evaluates the interactions between manufacturing management and related decisions regarding supply chain strategy, requirements, and intensity. In [14], a bi-level formalization for resource support of a set of projects is implemented.

Manufacturing enterprise management policies aim to integrate production with inventory management by achieving just-in-time delivery resources. This can reduce material storage costs and reduce the amount of excess and residual materials from the production stage. The formalization of the inventory process is discussed in [15]. An optimization problem is defined and an iterative delivery algorithm is implemented. The peculiarity of the algorithm is that it gives a greater production gain as the iterative calculations increase. In [16] the inventory process is optimized for the case of constant production levels. In this way, inventories are closely related to the rate of production, which favors the retention of unnecessary resources.

Inventory can significantly affect the production process. This is the reason for deriving integration solutions between these two processes. An overview of the works dealing with the so-called integrated production of supplies can be found in [17]. Solutions for integration between inventory and production are derived for different cases and industries. In [18], an optimization problem for the textile industry was developed. In [19] such integration is applied to perishable goods that need a short time between delivery and preservation. The formalization of inventory rules is usually different depending on the applied production processes. The formalization of the production is related to the goods and the internal rules for the production of the specific goods. However, inventory relationships are common to this activity. An overview of the concepts that formalize inventory relations can be found in [20].

The analysis of the formal modeling and the definition of the problem related to production and/or inventory concludes that a formalization of optimization should be applied. Optimization can take the form of optimal control [21], and application of optimization models and methods [22]. Optimization can be supported by simulation [23] and the use of cluster shapes [24]. The extension of optimization with more than one objective function is addressed through the use of dual-step maximization [25] and with bi-level formalization and optimization [26].

The overview above illustrates that successful business process management requires optimizing the relationship between the inventory process and the production of goods. Such optimization requires deriving quantitative relationships that must be introduced as formal relations in an optimization problem. The solution to the optimization problem will give optimal inventory volumes along with the volumes of goods coming with production.

This study provides a formal description of the integration between inventory and production tasks. Integration refers to the supply of raw materials. Production is taken as the amount of required nutritional content for animal feed. The supply of the necessary products to ensure predetermined levels of nutrition is formalized in a general optimization problem. This problem is defined hierarchically and its formal relations are applied to bi-level optimization. Solutions to the bi-level problem estimate the volumes of optimal raw material stocks that will be used precisely to satisfy the requirements.

The inventory problem evaluates arguments that relate to the volume of resources that must be supplied for production. Production has arguments that affect the supply of inventory. The specific problem being defined estimates the minimum agricultural resources that are required to prepare a predetermined content of animal feed. The inventory in this case is agricultural products. Production goods are the nutrition in the food of the animals. Thus, the solution to the bi-level problem can benefit business outcomes such as minimizing inventory costs and maximizing nutritional content. Therefore, the formulation of the bi-level problem is performed through a hierarchical integration of two sub-problems: the upper one aims at minimizing inventory costs, and the lower one performs production maximization. Both subproblems are related to general constraints that formalize the fractions of resources that are required to prepare the contents of animal feed.

The solution to the bi-level problem extends the results of the classical inventory problem since its arguments give the optimal production values for the expected inventory volumes. Therefore, the bi-level problem estimates the optimal supply volumes and its output as a nutrition content.

3. Materials and methods

3.1. Definition of the bi-level optimization for integration of the production and inventory

The inventory process explicitly considers that raw materials must be supplied from the market and the resource costs are independent of the supply inventory process [27]. Inventory optimization aims to minimize the costs of delivery requests, transportation, and storage in the warehouse. The parameters to be considered for the inventory process are recommended in [28] as:

- Initial price for delivery order K [BGN/per 1 order];
- Purchase costs for inventory;
- Transport costs to the storage location;
- Costs for storing inventory in warehouse h [BGN/per 1 resource quantity];
- Potential costs from loss and spoilage of materials.

Purchasing and transportation costs depend on market behavior. They do not depend on management decisions about inventory. Therefore, these costs are not taken into account in inventory management. Potential losses from warehouse storage can be considered as part of the total storage costs h.

Given the delivery costs K and the storage cost h, the inventory optimization problem aims to estimate the optimal volume of the resource y that gives the minimum cost per delivery set. This volume y must satisfy the desired quantity of demand D for that type of raw material.

The classic form of inventory formalization problem applies Economic Order Quantity (EOQ) modeling [27-28]. The input parameters for the inventory are:

- the delivery value K for the inventory request;
- the storage costs h for supplies to be stored in a warehouse.

Therefore, the total inventory cost is calculated by the sum

$$\text{Total\_costs} = K + hy,$$

and y is the volume of the resource provided for the inventory.

The solution to the optimization problem is the volume y that minimizes the total delivery cost but must satisfy the requested quantity of resources for the demand D. Since the warehouse for storing inventory has capacity limitations, the inventory process must be performed repeatedly over time, illustrated in Fig. 1. On the vertical axis is the volume of inventory y, which changes with consumption from production.

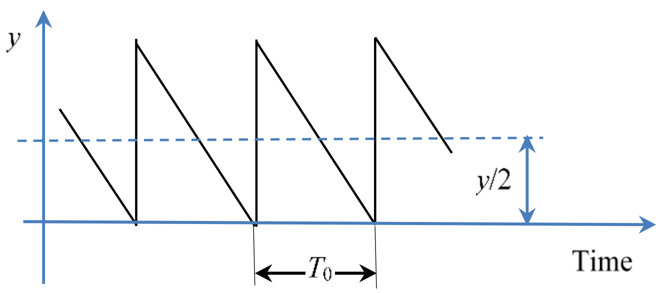

Fig. 1. Consistent execution of inventory over time

The volume of inventory y decreases linearly, assuming constant consumption from the production process. The slope of the decline depends on the value of demand D. Higher demand makes the slope of the decline steeper.

Economic Order Quantity (EOQ) modeling assumes that a new inventory order of y will be placed when the current value of y reaches the horizontal axis or $y = 0$. An ideal inventory process then occurs and immediately the new inventory volume y is introduced into the warehouse and the production process.

The period T_0 between two sequential processes of inventory can be formalized in terms of demand D and volume of the stock y or T_0:

(1) $$T_0 = D/y.$$

This formalization of the inventory process is a static EOQ because demand D is assumed to be constant over time.

The value of the holding cost calculated for one delivery period T_0 is the product h. Here it is assumed that the average volume of the inventory resource for a period T_0 is $y/2$ (Fig. 1).

The rate of stock reduction y is defined by the demand D for the production. A new inventory is required when the current level is $y = 0$. The average holding cost is $h\frac{y}{2}$ as y varies between the value 0 and y. The sequence of the inventory series has a period T_0. Therefore, the cost per delivery for inventory RC(y) per cycle T_0 is

(2) $$\mathrm{RC}(y) = \frac{K + h\left(\frac{y}{2}\right)T_0}{T_0}.$$

The inventory costs for one cycle are formally found by substituting the relationship (1) into (2) or

(3) $$\mathrm{RC}(y) = \frac{KD}{y} + h\frac{y}{2}.$$

This relationship is used to estimate the optimal inventory value ***opt*** for one period T_0 that minimizes RC(y):

(4) $$y^{\mathrm{opt}} = \arg\left\{\frac{d\mathrm{RC}(y)}{dy} = \ 0 = -\frac{KD}{y^2} + \ \frac{h}{2}\right\},$$

or

(5) $$y^{\mathrm{opt}} = \ \sqrt{\frac{2\,KD}{h}}.$$

The relationship (5) is the value of the optimal inventory volume. It is derived from the assumptions of the EOQ model for a constant demand value D. The value $\boldsymbol{y}^{\mathrm{opt}}$ gives the volume of the resource that must be supplied for one inventory period T_0.

To extend the inventory modeling to supply m types of resources $y_j,\ j = 1, \dots, m,$ the optimization relation (4) must be decomposed into m independent equations. Since the production of one type of good generally requires several types of resources, the inventory warehouse must accommodate all of these resources.

Since the warehouse is technologically limited in capacity, it is necessary to add constraints on the volume of stored materials to the inventory problem (4). This complicates the original problem (4) and the inventory optimization takes the form

(6) $$\min_{y}\left\{\mathrm{RC}(y) = \ \sum_{j=1}^{m}\left(\frac{K_j D_j}{y_j} + \ h_j\ \frac{x_y}{2}\right)\right\},$$
$$\sum_{j=1}^{m} a_j y_j\ \le d.$$

The solution of (6) is a vector $\mathbf{y} = (y_1, \dots, y_m)$, with each component corresponding to the type of resource. The values of the coefficients $a_j, j = 1, \dots, m,$ give the relative space required in the warehouse per unit volume of resource j.

Problem (6) is a nonlinear optimization problem and must be solved with appropriate nonlinear algorithms, which increases the computational time for its solution.

For the integration of the production with the inventory, problem (6) should include additional relations, coming from the production process. A simple form of such relations can be taken in linear form describing what part of inventory resource is used for the production of one good or

(7) $$\sum_{i}^{N} a_{ij} x_i\ \ge\ y_j\ ,\ j = 1, \dots, m,$$

where the coefficients a_{ij} give the volume of the resource y_i that is needed for the product x_j, where j=1,…, m, is the number of product.

Relations (7) should formalize the characteristics of production in terms of the fractions of resources used for the different products.

This study aims to derive relationships between the required amounts of inventory resources that must satisfy the production of goods, according to a certain demand for goods. Since a product from production requires a set of inventory resources, integrating inventory and production can benefit business management by reducing inventory costs and increasing production returns.

The optimization problem that addresses the integration of inventory and production in this study is defined as a bi-level optimization problem. This problem creates a hierarchical relationship between two subproblems. The low-level problem is defined to maximize the return of production, given the inventory resources provided. The upper-level subproblem estimates the optimal inventory volumes required by the low-level problem. Therefore, the two optimization subproblems are formally interrelated. The solution of each depends on the solutions of the other.

3.2. Definition of the production optimization sub-problem

The objective function in production can be chosen in content for minimization of the production costs or to maximize the production income. The formal relations for these two cases can be described with the sums

(8) $$\min_{y_j} \sum_j^m c_{1_j} y_j \ \text{ or } \ \max_{y_j} \sum_j^m c_{2_j} y_j,$$

where c_{1_j} and c_{2_j} are the respective costs of producing and costs for selling the product. Additional constraints can be added for the Lower Bound (LB) and Upper Bound (UB) for the type of inventory, required for each product or

(9) $$\text{LB}_j \le y_j \le \text{UB}_j,\ j = 1,\dots, m.$$

The production problem can be complicated by additional constraints, which can correspond to the technological process of the production of a good. In this research, our problem for the integration of the inventory and the production is addressed for the preparation of food for the cows in a dairy farm. The cow food should have such nutritional elements, to stimulate the milk supply. For this case, the inventory problem has to evaluate the volumes of appropriate agricultural products. The nutrition content of these agricultures must give the needed nutrition content. The target of the optimization is to have minimal costs of the inventory and to maximize the content of nutrition elements in the food.

3.3. Integration of inventory and production subproblems in common bi-level optimization problem

The bi-level optimization problem is defined as the interconnection between two optimization problems. The solution of the upper-level problem changes the solutions of the lower-level problem and vice-versa, the low-level problem makes changes to the upper problem, Fig. 2.

The modifications made to the two problems can be formalized by changes to their objective functions and/or to their set of constraints. The low-level problem makes changes to the upper-level by its arguments *x*. Accordingly, the low-level solutions are influenced by the upper solutions *y*. The benefit from the bi-level optimization comes from the extended space of problem arguments (x, y) that have corresponding optimal values. The optimization is performed by satisfying two objective functions of the upper and lower problems in hierarchical order. This makes the optimization richer towards optimal requirements. Finally, the constraints of the bi-level problem are the intersection of the two constraints of the upper and lower problems, which means that more constraints are taken into consideration for the bi-level problem in comparison with its constituent parts. The formal description of the bi-level problem has the form (10) and we make its interpretation in the meaning of integration of inventory and production.

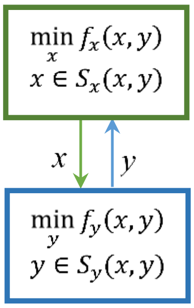

Fig. 2. Bi-level optimization as two interrelated subproblems

The production problem is aimed at maximizing an objective function, formally defined as a function $F(\mathbf{x})$, where the vector $\mathbf{x} = (x_1, \dots, x_n)$ affects the n nutrients in the food. The objective function $F(\mathbf{x})$ estimates the profit from including these nutrients in the food. In this study, a linear objective function $F = \mathbf{c}^{\mathrm{T}}\mathbf{x}$ is used for the case of simplifying the computational estimates in a bi-level problem. The value of the components $\mathbf{c} = (c_1, \dots, c_n)$ gives the relative gain from the nutrient x_j, $j = 1, \dots, n$. Additional constraints for the production problem are the inequalities that give the relationship between the required amount of agricultural product $b_{j,i}$ to obtain a nutrient with volume x_i as

(10) $$\sum_{j=1}^{m} b_{j,i} y_j \geq x_i, \quad j = 1, \dots, m, \quad i = 1, \dots, n.$$

For the storage of the food $\mathbf{x}$ in a limited space, similar inequalities can be added as in the inventory problem in (6) for the production one as

(11) $$\sum_{i=1}^{n} g_i x_i \leq f,$$

where g_i is the relative space requirement for nutrient x_i and f is the available holding space.

Therefore, the production problem takes the form

(12) $$\max_x \{F(\mathbf{x}(\boldsymbol{y}))\},$$

with constraints (10) and (11). The bi-level optimization problem targets the joint solution of the inventory and production subproblems (6) and (8). The bi-level optimization contains a common set of inequalities (10), which have to be satisfied by both subproblems (6) and (8). The explicit formal definition of the bi-level problem, which integrates the inventory and the production is

(13) $$\min_y \left\{ \mathrm{RC}(\boldsymbol{y}(\boldsymbol{x})) = \sum_{j=1}^{m} \left(\frac{K_j D_j}{y_j} + h_j \frac{y_j}{2} \right) \right\},$$

$$\sum_{j=1}^{m} a_j y \ \le d \ ,$$
$$\sum_{j=1}^{m} b_{j,i} x_j \ \ge \ y_i, \ j = 1, \dots, m, i = 1, \dots, n,$$
subject to
$$\max_x \{\mathrm{F} = \mathbf{c}^{\mathbf{T}}\mathbf{x}\},$$
$$\sum_{i=1}^{n} g_i y_i \ \le f,$$
$$\sum_{j=1}^{m} b_{j,i} x \ge \ y_i \ , \ j = 1, \dots, m, \ i = 1, \dots, n.$$

The arguments of the bi-level problem (13) are both production arguments **x** and inventory arguments **y.** Inventory resources **y** minimize inventory costs, but their volumes yield food volumes **x**, which maximizes production profit.

4. Numerical simulation with bi-level optimization

The optimal nutrition elements for the cow food **x** and the optimal volumes of agricultural resources **y** are given as solutions to the bi-level optimization problem.

The production process takes place for the preparation of the content of the food for the animals in a livestock farm. The feed must contain the necessary nutritional components that the animal must have in its diet. However, the nutritional components do not exist in pure form and must be taken from various agricultural products. Thus, our interpretation of inventory refers to the volumes of products to be delivered. However, their supply must correspond to the required content of nutritional elements. In this way, production targets the achievement of the optimal level of nutritional components. The inventory optimization targets the minimization of the costs of product delivery. However, their quantity should maximize the level of nutritional elements of the feed prepared from the supplied inventory.

For the definition of the bi-level problem, the inputs used are taken from publicly available data on cow husbandry requirements. Three types of vegetables and fruits are considered. Nutrient items are selected for calories and carbohydrates. This data is taken from virtual sources [29-31]. The numerical values are given in Table 1.

Table 1. Input data for the parameters of the inventory-production optimization problem

Arguments	Vegetables	Calories	Calor value, 100 g	Carbohydrate	Carbo-value, mg	Inventory prices	Values, BGN per 1 kg
x_1	carrots	b_1	41	a_1	9.58	c_1	0.23
x_2	apple	b_2	52	a_2	13.81	c_2	0.34
x_3	orange	b_3	55	a_3	11	c_3	0.28

The nutritional requirements insist that calories y_1 must be four times the carbohydrate content y.

Numerically, the inventory problem takes an analytical form as
$$\min_x \left\{ \sum_{i=1}^{N} \left(\frac{K_i D_i}{y_i} + \ h_i \ \frac{y_i}{2} \right) \right\},$$
$$\sum_i^N a_{ij} y_i \ \ge \ x_j, \ j = 1, \dots, m,$$
or
$$\min_{y_1, y_2, y_3} \left\{ |0.023 \ \ 0.034 \ 0.028|. \begin{vmatrix} 1/y_1 \\ 1/y_2 \\ 1/y_3 \end{vmatrix} + |0.23 \ \ 0.34 \ \ 0.28 \ |. \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} \right\},$$

$$|b_1\ \ b_2\ b_3\ \ |.\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} \geq x_1, \quad |a_1\ \ a_2\ a_3\ \ |.\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} \geq x_2.$$

The component of the delivery cost K_iD_i are taken equal to inventory prices c_i or $K_iD_i = c_i,\ i = 1, \dots, N = 3.$ Operating costs are estimated at 10% of inventory costs, $h_i = 0.1\ K_i,\ i = 1, \dots, 3$.

The low-level problem targets the maximization of $x_1^2 + x_2^2, i = 1, 2$, with the production goal function. The last is defined in a square form: $x_1^2 + x_2^2$. Therefore the numerical form of the low-level problem is

$$\min_{x_1, x_2}\{-(x_1^2 + x_2^2)\},$$

$$x_1 + \frac{1}{4}\ xy_2 \leq g = 10\ ,\ x_1\ \leq 10\ ,\ x_2\ \leq 10\ .$$

The value g=10 mg is selected according to the requirements for the preparation of one set of cow feed.

The solution of the bi-level problems gives **y**=(0.0669 kg; 0.0722 kg; 0.637 kg) vegetables that will provide; **x**=(10 cal; 3.44 mg) per feed dose. The inventory value is 12.55. The production value is calculated with the sum $(x_1^2 + x_2^2)$ as 175.19.

This solution meets the requirements of minimization of the inventory costs and at the same time gives the maximal content of possible nutrition elements. The benefit of bi-level optimization is that it gives simultaneously results for the inventory and the production tasks. These solutions are evaluated considering the interconnections between the two problems of inventory and production. By varying the parameters of the bi-level problem, we can evaluate the sensitivity of the solutions to nutrient content requirements and assess the inventory cost when the prices on the market change.

5. Assessment and sensitivity analysis of the bi-level problem

The sensitivity analysis was performed to estimate the changes in inventory volumes y_1, y_2, y_3 according to the changes in the nutrient content. Formally, the nutrient content is changed by several values of the coefficient g. Then solving the bi-level problem the inventory and production costs are valued for g. The nutritional value g is changed from g=0.5 to g=3. This means that the solutions to the bi-level problem (10) are solved for the case of half the required food concentration (g=10), up to a 3-fold increase in this concentration (g=3×10=30). In Fig. 3 the corresponding volumes of supplies y_1, y_2, y_3 for the increase of g.

It can be seen that the increase in the nutritional content x_1, x_2 requires a corresponding increase in the initial supplies y_1, y_2, y_3. Accordingly, the amount of production given in Fig. 4 also increases.

The inventory cost follows a decreasing tendency. Although the supplies y_1, y_2, y_3 increase, the total inventory costs do not follow an increasing tendency. This reason comes from the component $\frac{K_iD_i}{y_i}$ of the inventory goal function. With the increase in the volume $y_{i,}$ this gives a decrease in the cost's components $\frac{K_iD_i}{y_i}$ and respectively decreasing impact on total cost inventory. This can be explained from practical considerations that the inventory cycles are reduced, due to the higher inventory delivery ***y***. Inventory costs related to the increasing nutrient concentration *g* are given in Fig. 5.

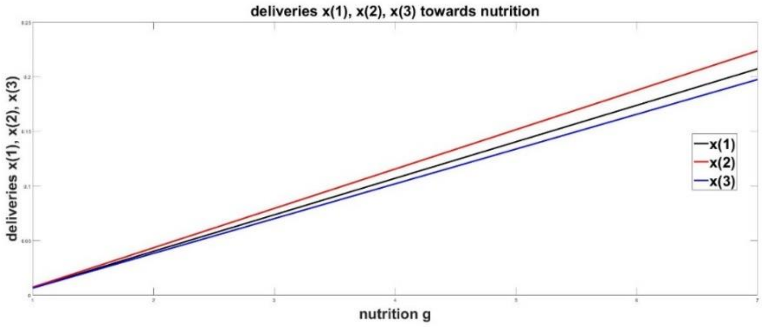

Fig. 3. Increase in inventory in line with increasing nutrient concentration *g*

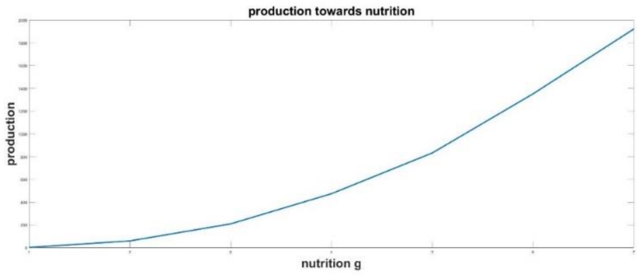

Fig. 4. Increase in production level as nutrition increases *g*

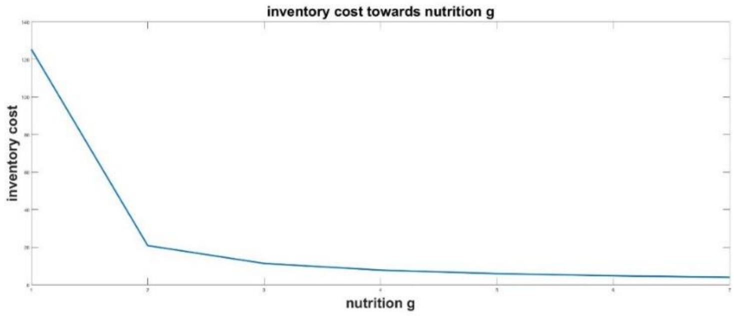

Fig. 5. Inventory costs to increase nutrition *g*

An additional type of sensitivity analysis is made by the delivery price of the product. The price of y_1 is changed from 0.8 up to 1.2 from its current value of *c*(1)=0.23. The corresponding behavior of the inventories y_1, y_2, y_3 is given in Fig. 6. It follows that when the cost price *c*(1) is lower than its nominal value, each raw material y_1, y_2, y_3 has an increasing character for its stock. But when the price *c*(1) exceeds its nominal value, all supplies decrease in amount. The corresponding production value graph is given in Fig. 7. The behavior of the output of food components follows the supply dynamics y_1, y_2,. At a lower price of *c*(1) below the nominal value, the production components increase. But when the costs $c(1)$ are higher their nominal level of production falls.

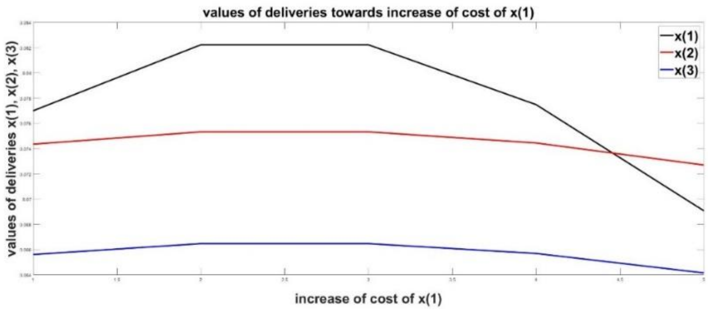

Fig. 6. Changes in supply quantities y_1, y_2, y_3 relative to cost change $c(1)$

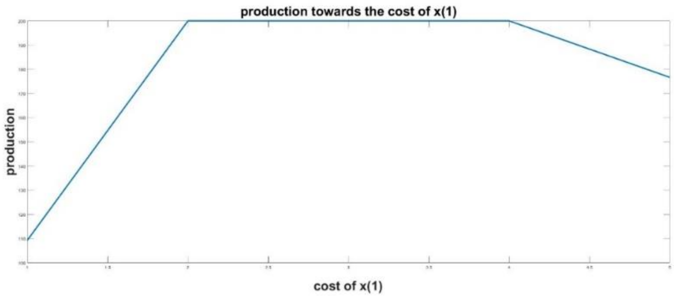

Fig. 7. Value of production with change in cost price $c(1)$

The value of inventories is inversely proportional to changes in cost $c(1)$, Fig.8. If the $c(1)$ is lower than its nominal level, the stock price declines. In the other case, because the cost price $c(1)$ increases, the total inventory cost also increases.

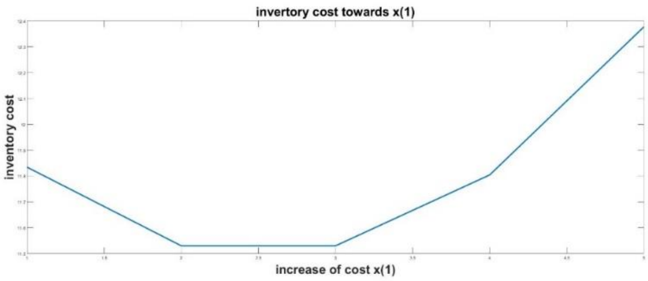

Fig. 8. Inventory value versus price variation $c(1)$

Therefore, the inventory costs are positively correlated with the dynamical behavior of the market for its increase or decrease level.

The results of the sensitive analysis provide evidence that bi-level optimization gives useful quantitative results according to changes in market parameters for inventory and production. However, the benefit of the bi-level problem comes from the evaluation of quantitative solutions for both arguments about inventory and production management.

6. Conclusions and future work

This study derives an optimization problem that integrates inventory requirements and objectives for the production task. The optimization problem is in a hierarchical form at two levels. At each level, a corresponding inventory delivery problem with cost minimization and a production problem with output maximization is solved. Both optimization problems are related to general constraints formalizing the necessary inventory resources for the production of a good. The application of this bi-level optimization problem is applied to the case of supplying agricultural products with minimal costs for the nutritional containing of the feed to cows on a dairy farm. The production good is the nutrient that is needed for the food. The bi-level problem integrates inventory and production, allowing to minimize the cost of inventory products but maximizing the nutrients needed for the cows' food. The inventory problem is defined based on EOQ modeling. The production problem is aimed at maximizing the nutritional elements. The solution of the bi-level problem has a positive effect from an economic point of view in terms of simultaneous optimization of inventory and production.

The added value of the paper is the definition of a bi-level optimization problem, which integrates two important business management tasks: inventory and production. The benefit of the bi-level problem is that it quantifies solutions and provides quantitative recommendations for inventory and production. This formalization gives minimum inventory costs, but at the same time maximizes production output. The advantage of this optimization is the mutual integration of the obtained optimal solutions. The resulting problem has been applied to the assessment of food resources for animal feed that maintain an optimal level of nutrition. A potential complication of this formalization could be assuming additional constraints on the stochastic nature of inventory in terms of lead times and the random nature of production requests.

The applied approach is not limited to the example considered here but can be used in various application areas such as optimal charging of electric vehicles, which is also the subject of further research. A future direction for research and application is the simultaneous minimization of the required power from the electrical grid and the maximum use and storage from photovoltaic generation, which will minimize the operating costs of charging electric vehicles.

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