BULGARIAN ACADEMY OF SCIENCES

CYBERNETICS AND INFORMATION TECHNOLOGIES • Volume **23**, No 3
Sofia • 2023 Print ISSN: 1311-9702; Online ISSN: 1314-4081
DOI: 10.2478/cait-2023-0021

Involutory Negator of Basic Belief Assignments

Jean Dezert[1], *Albena Tchamova*[2]

[1]*The French Aerospace Lab, Palaiseau, France, France*
[2]*Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, Sofia, Bulgaria*
E-mails: jean.dezert@onera.fr albena.tchamova@iict.bas.bg

Dedication. This paper is dedicated to the memory of our colleague and friend Ludmil Bojilov (1944-2023).

Abstract**:** *This paper analyzes the different definitions of a negator of a probability mass function* (*pmf*) *and a Basic Belief Assignment* (*BBA*) *available in the literature. To overcome their limitations we propose an involutory negator of BBA, and we present a new indirect information fusion method based on this negator which can simplify the conflict management problem. The direct and indirect information fusion strategies are analyzed for three interesting examples of fusion of two BBAs. We also propose two methods for using the whole available information* (*the original BBAs and their negators*) *for decision-making support. The first method is based on the combination of the direct and indirect fusion strategies, and the second method selects the most reasonable fusion strategy to apply* (*direct, or indirect*) *based on the maximum entropy principle.*

Keywords**:** *Belief functions, BBA negator, Information fusion, Measure of uncertainty, Entropy.*

1. Introduction

This paper deals with Basic Belief Assignments (BBAs) introduced in Dempster-Shafer Theory (DST) [1]. We propose an involutory negator of a BBA, and its application for information fusion. The concept of a complement of a body of evidence (i.e., negator) has been introduced by D u b o i s and P r a d e [2] in 1986, and re-examined by Y a g e r [3]. The main disadvantage with these negators (and of the most recent proposals) is that they are not involutory (An involutory function (or involution) is a function f that is its own inverse, that is $f(f(x))=x$ for all x in the domain of *f*. This means that applying f twice produces the original value) in general so that the information content of the negator of a negator of a BBA is not equal to the information content of the original BBA. This is problematic from the informational standpoint because we naturally expect that working with negator of negator of evidence should be equivalent to working with original evidence. The problem we address in this paper can be stated as follows: let's consider a Frame of Discernment (FoD) of a problem under concern. Knowing a first expert providing a BBA $m(.)$ defined on the power set of the FoD, is it possible to find a second expert with a BBA $\bar{m}(.)$ defined on the power set of FoD that expresses the opposite (or negation) assessment of the first expert? How can this be done effectively? Based on which principle and justifications? The second problem we address is the use of negator of BBAs for the information fusion for decision-making support.

2. Belief functions and entropy

The Belief Functions (BF) have been introduced by Shafer [1] for modeling epistemic uncertainty, reasoning about uncertainty and combining distinct sources of evidence (SoEs). The answer of the problem under concern is assumed to belong to a known finite discrete frame of discernment (FoD) $\Theta=\{\theta_1,\theta_2,\dots,\theta_N\}$ where all elements (i.e., members) of Θ are exhaustive and exclusive. The set of all subsets of Θ is the power-set of Θ denoted by 2^Θ. A normalized BBA (also referred as a normal BBA or a proper BBA in the literature), or mass function, is a mapping $m^\Theta(.): 2^\Theta \to [0,1]$ such that $m^\Theta(\emptyset)=0$ and $\sum_{X\in 2^\Theta} m^\Theta(X)=1$. We omit the superscript Θ in $m^\Theta(.)$ notation if there is no ambiguity on the FoD we work with. The element $X\in 2^\Theta$ is called a Focal Element (FE) of $m(.)$ if $m(X)>0$. The belief and plausibility of X are defined by [1]

(1) $$\mathrm{Bel}(X)=\sum_{Y\in 2^\Theta | Y\subseteq X} m(Y),$$

(2) $$\mathrm{Pl}(X)=\sum_{Y\in 2^\Theta | X\cap Y\neq\emptyset} m(Y)=1-\mathrm{Bel}(\bar{X}),$$

where $\bar{X}=\Theta\setminus\{X\}$ is the complement of X in Θ.

To quantify the uncertainty (i.e., the imprecision) of $P(X)\in[\mathrm{Bel}(X),\mathrm{Pl}(X)]$, we use $u(X)\in[0,1]$ defined by

(3) $$u(X)=\mathrm{Pl}(X)-\mathrm{Bel}(X).$$

If all focal elements of $m(.)$ are singletons of 2^Θ, the BBA $m(.)$ is a Bayesian BBA because $\forall X\in 2^\Theta$ one has $\mathrm{Bel}(X)=\mathrm{Pl}(X)=P(X)$, and $u(X)=0$. The vacuous BBA for a totally ignorant SoE [4] is defined by $m_v(X)=1$ for $X=\Theta$, and $m_v(X)=0$ for all $X\in 2^\Theta, X\neq\Theta$.

In [5] we have analyzed in details forty-eight Measures of Uncertainty (MoU) of BBAs by covering 40 years of research works on this topic. Some of these MoUs capture only a particular aspect of the uncertainty inherent to a BBA (typically, the non-specificity and the conflict). Most of these MoUs fail to satisfy four very simple reasonable and essential desiderata and they are not effective. Only six MoUs can be considered as effective from the mathematical sense presented next, but they appear as conceptually defective and disputable [5]. That is why, a better effective measure of uncertainty, i.e., generalized entropy of BBAs has been developed in [4]. It is defined by

(4) $$U(m) = \sum_{X \in 2^{\Theta}} s(X),$$

with

(5) $$s(X) = -m(X)(1-u(X))\log(m(X)) + u(X)(1-m(X)),$$

$s(X)$ is the uncertainty contribution related to X named the *entropiece* of X. This entropy $U(m)$ is effective [4] because is verifies the four essential desiderata:

1) $U(m)=0$ for any BBA $m(.)$ focused on a singleton X of 2^{Θ};

2) $U(m_v^{\Theta}) < U(m_v^{\Theta'})$ *if* $|\Theta| < |\Theta'|$;

3) $U(m) = -\sum_{X\in\Theta} m(X)\log(m(X))$ if the BBA $m(.)$ is a Bayesian BBA. Hence, $U(m)$ reduces to Shannon entropy [6] in this case;

4) $U(m) < U(m_v)$ for any non-vacuous BBA $m(.)$ and for the vacuous BBA $m_v(.)$ defined with respect to the same FoD.

The maximum of entropy $U(m_v^{\Theta}) = 2^{|\Theta|} - 2$ (see derivation in [4]) obtained for the vacuous BBA m_v over a FoD Θ, because m_v characterizes a SoE with a full lack of information. It is worth mentioning that one has always $U(m_v^{\Theta}) > \log(|\Theta|)$. Hence, the vacuous BBA has always an entropy greater than the maximum of Shannon entropy $\log(|\Theta|)$ obtained with the uniform probability mass function (pmf) on the frame of discernment Θ.

3. Negators of pmf and BBA in the literature

In this section we present several negators proposed in the literature with some examples, and we comment them.

3.1. Dubois and Prade non-involutory negator of a BBA (1986)

In 1986, D u b o i s and P r a d e (DP) [2] introduced in (pp. 202-203) for the first time the concept of negation of a BBA. This negator for any $X \subseteq \Theta$ is defined as

(6) $$\bar{m}(X) = m(\bar{X}),$$

where $\bar{X} = \Theta \setminus \{X\}$ is the complement of X in the FoD Θ.

This simple definition is quite natural except that it does not satisfy the involution property because $\bar{\bar{m}} \neq m$ in general. Because we consider that the involution property must be a very natural property to satisfy by an effective negator, we do not consider Dubois and Prade (DP) negator as effective. Moreover, it is clear that the DP negator of the vacuous BBA $\bar{m}_v(\emptyset)=m_v(\Theta)=1$ is not a proper BBA.

3.2. Yager's non-involutory negator of a pmf (2015)

Yager [3] has introduced the concept of the negation of a probability distribution P in, which has been raised by Zadeh in his Berkeley Initiative in Soft Computing (BISC) blog. By the term *negation* Yager means the representation of the knowledge we use if we have the statement *not* P. The negation of a pmf $P(.)$ over a reference set $\Theta=\{\theta_1,\theta_2,\ldots,\theta_n\}$ is defined by

(7) $$\bar{P}(\theta_i)=(1/\lambda)\cdot P(\bar{\theta}_i),$$

where $\bar{\theta}_i=\Theta\setminus\{\theta_i\}$ is the complement of θ_i in the set Θ, $P(\bar{\theta}_i)=1-P(\theta_i)$, and λ is a normalization factor given by

(8) $$\lambda=\sum_{i=1}^{n}P(\bar{\theta}_i)=\sum_{i=1}^{n}(1-P(\theta_i))=n-1.$$

In the literature [8] definition (7) is called Yager's negator. Yager's justification for this definition is based on a maximal entropy of the weights associated with each focal element. As Yager [3] pointed out in the definition (7) does not satisfy the double negation property in general (when $|\Theta|>2$ and $P(.)$ is not the uniform pmf), that is $\bar{\bar{P}}(.)\neq P(.)$. Yager's negator is the one that provides the maximum entropy among all possible negator definitions. The iterative application of Yager's negator converges towards the uniform pmf for which the entropy is maximal in the framework of the probability theory. The uniform pmf is the fixed point of Yager's negator. Note that in the very particular case where $\Theta=\{\theta_1\}$ (i.e., there is only one element in the reference set), we have $n=1$ and necessarily $P(\theta_1)=1$. Hence, we obtain $\bar{P}(\theta_1)=(1-P(\theta_1))/(n-1)=0/0$, which is indeterminate.

A generalization of Yager's negator has been proposed in [8], which is still not involutory. The analysis of the new properties of Yager's negator has been done by Srivastava and Maheshwari [9], and Srivastava and Kaur [10]. Other non-involutory extensions of Yager's negator have been proposed in [11, 12].

3.3. Yin's non-involutory negator of a BBA (2019)

In 2019, L. Yin, X. Deng and Y. Deng [13] proposed a definition of the negator of a BBA as a three steps procedure concisely given here by

(9) $$\bar{m}(X)=(1/\lambda)(1-m(X)),$$

where λ is the normalization constant defined by

(10) $$\lambda=\sum_{X\in 2^{\Theta}|m(X)>0}(1-m(X))=N-1,$$

where N is the number of focal elements of the BBA $m(.)$.

Clearly, Yin's negator imitates Yager's negator, but it works with BBA instead of pmf. Yin's negator is disputable because it is non-involutory. More problematic, Yin's negator is indeterminate for the vacuous BBA $m_v(.)$ for which $m_v(\Theta)=1$, because in this case one has only one focal element equal to Θ so that $N=1$ and from (9) we get $\bar{m}(\Theta)=(1-m(\Theta))/(N-1)=0/0$, which is indeterminate. Actually, this is a very serious problem for any BBA focused on only one focal element. Hence, Yin's negator is not a good proposal for a negator of a BBA. We mention that Yin's negator has been used by Gao and Deng [14] with incorrect claims and results. In example 1 of [14] authors consider $\Theta=\{a,b\}$ with $m(a)=m(b)=0.5$ (i.e., a Bayesian BBA with $N=2$ focal elements). Applying (9) we must obtain $\bar{m}(a)=(1-0.5)/(2-1)=0.5, \ \bar{m}(b)=(1-0.5)/(2-1)=0.5$, and not $\bar{m}(a)=0.25, \ \bar{m}(b)=0.25, \ \bar{m}(a\cup b)=0.5$ as the authors claim. This casts doubts on the correctness of the whole technical content of [14].

3.4. Xie-Xiao non-involutory negator of a BBA (2019)

Xie and Xiao [15] have defined a new non-involutory negator by

$$\bar{\boldsymbol{m}}=\boldsymbol{E}\cdot\boldsymbol{m}, \tag{11}$$

where $\boldsymbol{m}$ is the BBA $m(.)$ expressed as a vertical vector of size $2^{|\Theta|}$, and $\bar{\boldsymbol{m}}$ negation vector of the BBA vector $\boldsymbol{m}$ which characterizes the negation of $m(.)$. The matrix $\boldsymbol{E}$ is a negation symmetrical matrix $\boldsymbol{E}=[e_{ij}]$ of size $2^{|\Theta|}\times 2^{|\Theta|}$ defined in [15]. This negator is based on redistribution factors which appears ad-hoc and counter-intuitive. For instance, consider $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and the BBA entirely focused on θ_1 for which vector $m(\theta_1)=1$. The negation matrix $\boldsymbol{E}=[e_{ij}]$ is explicitly given in the Example 3 in [15] and the negation of the BBA $m(.)$ is

$$\bar{m}(\theta_1)=1/6,\ \bar{m}(\theta_2)=1/12,\ \bar{m}(\theta_3)=1/12,$$
$$\bar{m}(\theta_1\cup\theta_2)=1/4,\ \bar{m}(\theta_1\cup\theta_3)=1/4,\ \bar{m}(\theta_2\cup\theta_3)=1/6.$$

The result is counter-intuitive because this negator commits some mass of belief to elements that have non-empty intersection with θ_1. This behavior is not appropriate because the complement of θ_1 must have an empty intersection with θ_1 so the mass of θ_1 must be redistributed only to elements of the power set that have an empty intersection with θ_1 (or to their disjunction). Moreover, the authors present the analysis of their negator using Deng's entropy known to be non-effective [5].

3.5. Deng-Jiang non-involutory negator of a BBA (2020)

Deng and Jiang [16] have proposed a new negator for any BBA defined as follows:

(12) $$\bar{m}(X)=\sum_{Y\in 2^{\Theta}|\cup_{\theta_i\in Y}(\Theta\setminus\{\theta_i\})=X} m(Y).$$

As explained in [16] (p. 348) authors consider that the negation of a singleton focal element $X=\theta_i$ is $X=\bar{\theta}_i=\Theta\setminus\{\theta_i\}$, and if a focal element X is not a singleton its negation is equal to $\bar{X}=\cup_{\theta_i\in X}(\Theta\setminus\{\theta_i\})=\Theta$. This complementation principle is ad-hoc and very counter-intuitive because the negation of all non-singleton focal elements will correspond to the same complement element Θ which is the whole FoD. This principle is actually inappropriate. Besides its weird complementation principle, Deng-Jiang's negator is not involutory in general. For instance, consider the FoD $\Theta=\{\theta_1,\theta_2,\theta_3\}$ and the BBA entirely focused on θ_1 (i.e., $m(\theta_1)=1$) if we apply this negator on the negator $\bar{m}(.)$ we obtain $\bar{\bar{m}}(\theta_1\cup\theta_2\cup\theta_3)=1$ which is the vacuous BBA and not the original BBA. In the case where $\Theta=\{\theta_1\}$ and $m(\theta_1)=1$ we have $\bar{\theta}_1=\Theta\setminus\{\theta_1\}=\varnothing$ and applying Deng-Jiang's formula (12) we will get $\bar{m}(\bar{\theta}_1)=\bar{m}(\varnothing)=m(\theta_1)=1$ which is not a proper BBA according to Shafer's definition [1]. Deng-Jiang negator is not acceptable and not effective.

3.6. Batyrshin's involutory negator of a pmf (2021)

In [17, 18] Batyrshin has proposed an involutory negator of a pmf $P(.)$ defined over a reference set (with $\mathrm{MP}=\max P+\min P$) as follows

(13) $$\bar{P}(\theta_i)=(\mathrm{MP}-P(\theta_i))/(n\times\mathrm{MP}-1).$$

The uniform pmf is a fixed point for this negator, see [17] for details. Batyrshin's negator of $P(.)$ equals $P(.)$ in the very special case where $\Theta=\{\theta_1\}$, because one has $n=1$ and necessarily $P(\theta_1)=1$. This negator works only for probabilities but its real usefulness has to be shown in real applications. Batyrshin's negator has not yet been extended for the framework of the theory of belief functions, and it may be interesting to extend it (if possible) for the theory of evidence.

3.7. Liu's non-involutory negator of a BBA (2023)

Liu, Deng and Li [19] propose a new negator defined by

(14) $$\bar{m}(X)=(1/\lambda)\Big((2^{|X|}-1)\Big/\sum_{Y\in 2^{\Theta}|Y\neq X}2^{|Y|}-1\Big)(1-m(X)),$$

where $X\in 2^{\Theta}$ and λ is the normalization constant defined by

(15) $$\lambda=\sum_{X\in 2^{\Theta}}\Big((2^{|X|}-1)\Big/\sum_{Y\in 2^{\Theta}|Y\neq X}2^{|Y|}-1\Big)(1-m(X)).$$

This new negator is unfortunately not involutory as proved by the authors in [19]. They justify this negator based on Deng's entropy concept, which is known to be non-effective [4, 5]. It is obvious that the concept of complementation used by Liu et al. is inappropriate. Indeed, let's consider $\Theta=\{AB\}$ and the vacuous BBA $m_v(.)$

defined on this FoD by $m_v(\emptyset)=0, m_v(A)=0, m_v(B)=0, m_v(A\cup B)=1$. By applying (14) we will obtain the following Liu's negator

$$\bar{m}_v(\emptyset)=0,\ \bar{m}_v(A)=0.5,\ \bar{m}_v(B)=0.5,\ \bar{m}_v(A\cup B)=0$$

One can see that this negator is flawed because A and B cannot be considered as valid complements of $A\cup B$ because $A\cap(A\cup B)\neq\emptyset$ and $B\cap(A\cup B)\neq\emptyset$.

4. A new involutory negator for BBAs

In this section we present a new simple definition for an involutory negator of any BBA $m(.):2^\Theta\to[0,1]$ which expresses the opposite evidence of any source of evidence characterized by $m(.)$. The opposite (i.e., negator) of the BBA $m(.)$ is denoted by $\bar{m}(.)$, and it is simply defined by

(16) $$\bar{m}(X)=\begin{cases} 0 \text{ if } X=\emptyset, \\ m(\bar{X})\ \forall X\neq\emptyset\subset\Theta, \\ m(\Theta) \text{ if } X=\Theta, \end{cases}$$

where $\bar{X}$ is the complement of the subset X in the FoD Θ, that is $\bar{X}=\Theta\setminus\{X\}$.

This new negator defined by (16) is actually a revised definition of Dubois and Prade negator (6). At a first glance for some readers the conditions $\bar{m}(\emptyset)=0$ and $\bar{m}(\Theta)=m(\Theta)$ may appear strange. Some readers may dispute why the mass of belief committed to the whole ignorance proposition Θ is kept unchanged in the expression (16) of the negator of the source of evidence. This is a legitimate question because the (classical) complement $\bar{\Theta}$ of Θ in Θ is equal to the empty set, and because the (classical) complement $\bar{\emptyset}$ of the empty set in Θ is equal to Θ. As Dubois and Prade did, we could consider a priori taking $\bar{m}(\emptyset)=m(\Theta)$ and $\bar{m}(\Theta)=m(\emptyset)$. We think however that this option is actually not very reasonable because it would mean that the negation of a BBA will not be a proper BBA (as defined by Shafer in [1]). In fact, we would have $\bar{m}(\emptyset)>0$ when $m(\Theta)>0$, and we would always have $\bar{m}(\Theta)=0$ because $m(\emptyset)=0$, which is very restrictive. We consider that the most reasonable solution is to consider that the negation of the BBA $m(.)$ is better defined by (16). This new very simple definition for the negator of a BBA presents the great advantage to preserve the involutory property of the negator concept of a BBA so that $\bar{\bar{m}}(.)=m(.)$. Note that the negator of any BBA $m(.)$ defined by $\bar{m}(.)$ in (16) is a proper BBA because $\bar{m}(X)\in[0,1], \bar{m}(\emptyset)=0$, and $\sum_{X\in 2^\Theta}\bar{m}(X)=1$ because the focal elements of $\bar{m}(.)$ belong to 2^Θ and they correspond to the complement of the focal elements of , $m(.)$ which is a proper BBA.

We mention that our negator of a Bayesian is not a Bayesian BBA in general as soon as the FoD Θ has more than two elements. Also, the negator of the vacuous BBA $m_v(.)$ is equal to itself. Hence, the vacuous SoE has a neutral role with respect to this new negator concept. This is not very surprising because from no useful information (i.e., the vacuous BBA) we cannot draw any conclusion for making a decision in favor of one hypothesis or its opposite. This makes the definition (16) coherent with the intuition when working with vacuous BBA and the negator concept. Of course, it is always possible to approximate any non-Bayesian BBA (or any non-Bayesian negator of a BBA) by a pmf (if we want) using different techniques of approximation.

5. Direct and indirect fusion approaches

5.1. Direct fusion approach

To make this presentation simple, we present the main formulas for the direct combination of two BBAs only. General formulas for more than two BBAs can be found in [1, 20, and 21]. More fusion rules are listed in [22].

- Conjunctive rule of combination: $\forall X \in 2^\Theta$,

(17) $$m_{1,2}^{\cap}(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2).$$

- Disjunctive rule of combination: $\forall X \in 2^\Theta$,

(18) $$m_{1,2}^{\cup}(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cup X_2 = X}} m_1(X_1) m_2(X_2).$$

- Dempster-Shafer rule of combination [1]: $m_{1,2}^{\mathrm{DS}}(\varnothing) = 0$, and $\forall X \in 2^\Theta \setminus \{\varnothing\}$,

(19) $$m_{1,2}^{\mathrm{DS}}(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) \Big/ \left(1 - m_{1,2}^{\cap}(\varnothing)\right).$$

- Proportional Conflict Redistribution Rule No 6 [21]: $m_{1,2}^{\mathrm{PCR6}}(\varnothing) = 0, \quad \forall X \in 2^\Theta \setminus \{\varnothing\}$

(20) $$m_{1,2}^{\mathrm{PCR6}}(X) = m_{1,2}^{\cap} + \sum_{\substack{Y \in 2^\Theta \\ X \cap Y = \varnothing}} \left(m_1(X)^2 m_2(Y)\right) \Big/ \left(m_1(X) + m_2(Y)\right) + \left(m_2(X)^2 m_1(Y)\right) \Big/ \left(m_2(X) + m_1(Y)\right).$$

Here we consider only PCR6 rule because we use examples for the fusion of only two BBAs to keep the presentation as simple as possible. If one needs to combine three BBAs (or more) altogether, we recommend the improved PCR6 rule (denoted by PCR6+) presented in details (note that PCR6+ and PCR6 rules coincide for the fusion of two BBAs) in [21]. If the SoEs are considered fully reliable the conjunctive fusion rule applies. Because the SoEs are often conflicting, the conjunctive fusion result $m_{1,2}^{\cap}(.)$ is not a proper BBA. To overcome this problem, Dempster-Shafer (DS) rule of combination or PCR6 fusion rule can be used to obtain a normalized and combined proper BBA. DS rule offers the main advantage of being associative making its use quite easy for the applications, and DS preserves the neutrality (the vacuous BBA m_v has no impact on the result when combined to a

BBA $m \neq m_v$ with DS rule) of the vacuous BBA m_v, which is generally considered as a good property for a fusion rule. Unfortunately, DS rule exhibits counter-intuitive dictatorial behavior in high and low conflict situations as well [23, 24]. This is one of the main reasons (the second reason is that Shafer's conditioning based on DS rule is not consistent with lower and upper bounds of conditional probability [25]) why DS rule has been abandoned by many researchers and engineers working with belief functions. If the two sources are in total conflict (i.e., $m_{1,2}^{\cap}(\varnothing)=1$), DS rule does not work because of the division by zero in (19). PCR6 rule provides a more reasonable fusion result and it works in low and high conflicting situations as well. PCR6 does not behave dictatorially. The main disadvantage of PCR6 rule is its high complexity because it is not associative that is why all the SoEs must be combined altogether (not sequentially) with this rule. PCR6 does not preserve the neutrality of the vacuous BBA m_v when combining more than two BBAs altogether, but an improved version denoted by PCR6+ preserves the neutrality of m_v, see [21]. If one of the SoEs is not reliable and we do not know which one, the disjunctive fusion rule applies. The direct fusion approach of S BBAs $m_1, m_2, \ldots, m_S$ defined over the same FoD Θ is denoted symbolically by

(21) $$m_{1,2,\ldots,S}^{\mathrm{DF}} = F(m_1, m_2, \ldots, m_S),$$

where DF means the chosen Direct Fusion (DF) rule used for the combination of the S sources of evidence.

5.2. Indirect fusion approach using the involutory negator of BBAs

The fusion of BBAs $m_1, m_2, \ldots, m_S$ (with $S > 1$) is often problematic if there are some conflicts between the SoEs. This means that $m_1(X_1)m_2(X_2)\ldots m_S(X_S) > 0$ when $X_1 \cap X_2 \cap \ldots \cap X_S = \varnothing$ for some focal elements $X_1, X_2, \ldots, X_S$. When conflicts occur the simple conjunctive rule of combination (17) is not able to provide an acceptable fusion result because it commits a strictly positive mass of belief to the impossible event (i.e., to the empty set), that is $m_{1,2,\ldots,S}^{\cap} > 0$. Therefore, it is generally necessary to manage the existing conflict between the sources to obtain what we consider a reasonable fusion result for decision-making support. That is the reason why many fusion rules of combination have been developed and proposed in the literature during the last decades [20, 22]. Here we propose a new generic approach to combine the sources of evidence thanks to their involutory negator of the BBAs, which is what we call the Indirect Fusion (IF) approach. The idea behind the IF approach is rather simple. Instead of combining directly the original BBAs by some fusion rules (typically by DS rule [1], PCR6 rule [20, 21], DP rule [2], etc.), we propose to compute the fusion result indirectly using the negators of BBAs thanks to the following three simple steps:

- **Step 1.** BBAs negators.

  Calculate the involutory negators of BBAs $m_1(.), m_2(.), \ldots, m_S(.)$ using (16).

- **Step 2.** Fusion of negators.

Combine (i.e., fuse) the $S>1$ BBAs $\bar{m}_1(.),\bar{m}_2(.),...,\bar{m}_S(.)$ by a chosen fusion rule denoted symbolically by DF to get the direct fusion of negators, that is

(22) $$\bar{m}^{\mathrm{DF}}_{1,2,...,S}=F(\bar{m}_1,\bar{m}_2,...,\bar{m}_S).$$

The choice of the direct fusion rule DF for combining the negators is left to the fusion system designer. Proponents of DST will prefer DS rule (19), while opponents of DS rule will use other fusion rules (typically PCR6 rule (20), etc.).

- **Step-3.** Negator of the fused negators.

Once the fusion result $\bar{m}^{\mathrm{DF}}_{1,2,...,S}(.)$ is obtained, one calculates its negator to get the final Indirect Fusion (IF) result of the original BBAs thanks to definition (16) where $m(.)$ is replaced by $\bar{m}^{\mathrm{DF}}_{1,2,...,S}(.)$, that is

(23) $$m^{\mathrm{IF}}_{1,2,...,S}(X)=\begin{cases}0 \text{ if } X=\varnothing,\\ \bar{m}^{\mathrm{DF}}_{1,2,...,S}(\bar{X}),\ \forall X\neq\varnothing\subset\Theta,\\ \bar{m}^{\mathrm{DF}}_{1,2,...,S}(\Theta) \text{ if } X=\Theta.\end{cases}$$

More concisely, we will write steps 1, 2 and 3 by the symbolic expression

(24) $$m^{\mathrm{IF}}_{1,2,...,S}=\bar{\bar{m}}^{\mathrm{DF}}_{1,2,...,S},$$

where the negator operator used in (24) (represented by a bar symbol) is the involutory negator defined in (16). As it will be discussed in Section 8, in general we have $m^{\mathrm{IF}}_{1,2,...,S}(.)\neq m^{\mathrm{DF}}_{1,2,...,S}(.)$ which means that the direct and indirect fusion methods provide different results depending on the fusion rule chosen and the distribution of masses of belief to focal elements. This is because the fusion rules do not satisfy De Morgan's law when a conflict exists between the sources. Only in the case when $S=2$ and $m_1(.)=m_v$, or $m_2(.)=m_v$ one has $m^{\mathrm{IF}}_{1,2}(.)=m^{\mathrm{DF}}_{1,2}(.)$ because there is no conflict between the two SoEs to deal with in this very particular case.

6. Some interesting examples

Here we examine three interesting examples where a conflict exists between two SoEs, and we compare the results of direct and indirect fusion methods.

6.1. Example 1 – Zadeh's example (two Bayesian BBAs)

Consider Zadeh's example [23] where $\Theta=\{A,B,C\}$, and two Bayesian BBAs $m_1(.)$ and $m_2(.)$ as follows

$$m_1(A)=0.9,\ m_1(C)=0.1 \text{ and } m_2(B)=0.9,\ m_2(C)=0.1.$$

- Direct fusion with DS rule

Using DS rule (19), we obtain the Bayesian BBA $m^{\mathrm{DS}}_{1,2}(C)=1$ which is considered as a counter-intuitive result by Zadeh and by many authors because this result means that the hypothesis C is declared for sure even if both SoEs agree in committing a low belief to the origin C. This example is important because it has served as a starting point to question the validity of DS rule by Zadeh. Some proponents of DST of argue that DS rule should not be applied without preprocessing (i.e., discounting) the SoEs in this high conflict situation and other propoenents argue that DS result makes perfectly sense. Both types of proponents defend DS rule based on very different contradictory arguments, which amplify the suspicion about the validity of DS rule. Actually, these two types of arguments are flawed because DS rule behaves dictatorially even in low conflict situation as shown in [24] and in the next example.

- Direct fusion with PCR6 rule

By applying PCR6 fusion rule (20) to combine $m_1(.)$ and $m_2(.)$ we get

$$m_{1,2}^{\text{PCR6}}(A)=0.486,\ m_{1,2}^{\text{PCR6}}(B)=0.486,\ m_{1,2}^{\text{PCR6}}(C)=0.028\ .$$

This Bayesian PCR6 result is more reasonable than DS result because it clearly points out the difficulty to make a choice between hypotheses A and B because of the disagreement of two SoEs while rejecting both the third hypothesis C.

- Indirect fusion approach

By applying the indirect fusion approach, after step 1 we get the following BBAs negators $\bar{m}_1(B\cup C)=0.9,\ \bar{m}_1(A\cup B)=0.1$ and $\bar{m}_2(A\cup C)=0.9,\ \bar{m}_2(A\cup B)=0.1$.

We observe that there is no conflict between these two negators so that the conjunctive fusion rule can be used, and there is no need to adopt a specific management of conflicting masses either by DS rule, or by PCR6 rule because results from both rules are equal to the result computed with the conjunctive rule, when no conflict occurs. At step-2, we use the conjunctive fusion of $\bar{m}_1$ and $\bar{m}_2$ because there is no conflict between these negators, and we get

$$\bar{m}_{1,2}^{\cap}(A)=0.09,\ \bar{m}_{1,2}^{\cap}(B)=0.09,\ \bar{m}_{1,2}^{\cap}(C)=0.81,\ \bar{m}_{1,2}^{\cap}(A\cup B)=0.01\ .$$

At step 3, we take $\bar{m}_{1,2}^{\cap}(.)$ as the final indirect fusion result. We obtain (we use the notation $m_{1,2}^{\text{IF}-\cap}(.)$ to explicitly specify that the Indirect Fusion (IF) has been done with the conjunctive rule symbolized by the $\cap$ symbol) $m_{1,2}^{\text{IF}-\cap}(A\cup B)=0.81,\ m_{1,2}^{\text{IF}-\cap}(A\cup C)=0.09,\ m_{1,2}^{\text{IF}-\cap}(B\cup C)=0.09,\ m_{1,2}^{\text{IF}-\cap}(C)=0.01$.

This non-Bayesian indirect fusion result is more acceptable than DS result because it reveals clearly the uncertainty between hypotheses A and B, while reinforcing the disbelief of hypothesis C as we intuitively expect. We observe that $m_{1,2}^{\text{IF}-\cap}\neq m_{1,2}^{\text{DS}}$ and $m_{1,2}^{\text{IF}-\cap}\neq m_{1,2}^{\text{PCR6}}$. The distribution obtained from this indirect fusion result coincides with the simple averaging fusion rule which is a common fusion rule adopted by users not familiar with belief functions. This behavior is another argument against the direct fusion result provided by DS rule.

6.2. Example 2 – Dezert-Tchamova example (two non-Bayesian BBAs)

Here we consider another problematic example presented by D e z e r t, W a n g and T c h a m o v a in [24] to show the dictatorial behavior of DS rule of combination in high and low conflicting situations as well. An infinity of problematic examples as this one can be built, see [27] for more examples. We consider the FoD $\Theta=\{A,B,C\}$ with the following two (generic) non-Bayesian BBAs with $0<a,b_1,b_2<1$ and $b_1+b_2<1$.

$$m_1(A)=a, m_1(A\cup B)=1-a,$$
$$m_2(A\cup B)=b_1, m_2(C)=1-b_1-b_2, m_2(A\cup B\cup C)=b_2.$$

The conflict of these two BBAs is actually independent of the BBA $m_1(.)$ because $m_{1,2}^{\cap}(\varnothing)=1-b_1-b_2$ .One can easily verify that DS fusion of these two BBAs gives $m_{1,2}^{\mathrm{DS}}(A)=m_1(A)=a$ and $m_{1,2}^{\mathrm{DS}}(A\cup B)=m_1(A\cup B)=1-a$ which indicates that the fusion result is actually independent of the BBA $m_2(.)$ even if $m_2(.)$ is not the vacuous BBA and the conflict degree can be taken as high or as low as we want. This behavior of DS rule is of course counter-intuitive and dictatorial, and that is why we do not recommend DS rule in applications. As a numerical example, we take the parameters $a=0.3, b_1=0.2, b_2=0.3$, and we have

$$m_1(A)=0.3, m_1(A\cup B)=0.7,$$
$$m_2(A\cup B)=0.2, m_2(C)=0.5, m_2(A\cup B\cup C)=0.3.$$

Using the conjunctive fusion rule, we obtain

$$m_{1,2}^{\cap}(\varnothing)=0.5,\ m_{1,2}^{\cap}(A)=0.15,\ m_{1,2}^{\cap}(A\cup B)=0.35\ .$$

One sees that there exists a positive conflict $m_{1,2}^{\cap}(\varnothing)$ between these two sources of evidence that needs to be redistributed in order to obtain a proper resulting BBA.

- Direct fusion with DS rule: By applying DS rule (19) we obtain

$$m_{1,2}^{\mathrm{DS}}(\varnothing)=0,\ m_{1,2}^{\mathrm{DS}}(A)=0.3=m_1(A),\ m_{1,2}^{\mathrm{DS}}(A\cup B)=0.7=m_1(A\cup B)\ .$$

The same dictatorial DS fusion result may be obtained for other numerical values of positive parameters b_1 and b_2 with $b_1+b_2<1$.

- Direct fusion with PCR6 rule: By applying PCR6 rule (20), we obtain

$$m_{1,2}^{\mathrm{PCR6}}(\varnothing)=0,\ m_{1,2}^{\mathrm{PCR6}}(A)=0.2062,\ m_{1,2}^{\mathrm{PCR6}}(A\cup B)=0.5542,\ m_{1,2}^{\mathrm{PCR6}}(C)=0.2396\ .$$

We see that the PCR6 fusion rule does not behave dictatorially, and the PCR6 fusion result changes with different values of the parameters b_1 and b_2 .

- Indirect fusion with DS rule

If we apply the indirect fusion approach, the negators of $m_1(.)$ and $m_2(.)$ are

$$\bar{m}_1(B\cup C)=0.3,\ \bar{m}_1(C)=0.7,$$
$$\bar{m}_2(C)=0.2,\ \bar{m}_2(A\cup B)=0.5, \bar{m}_2(A\cup B\cup C)=0.3.$$

Hence the conjunctive fusion of negators gives

$$\bar{m}_{1,2}^{\cap}(\varnothing)=0.35,\ \bar{m}_{1,2}^{\cap}(B)=0.15,\ \bar{m}_{1,2}^{\cap}(C)=0.41,\ \bar{m}_{1,2}^{\cap}(B\cup C)=0.09\ .$$

Note that $\bar{\bar{m}}_{1,2}^{\cap}(.)\neq m_{1,2}^{\cap}(.)$. Applying DS rule of combination of these negators we obtain $\bar{m}_{1,2}^{\mathrm{DS}}(\varnothing)=0,\ \bar{m}_{1,2}^{\mathrm{DS}}(B)=0.23,\ \bar{m}_{1,2}^{\mathrm{DS}}(C)=0.63,\ \bar{m}_{1,2}^{\mathrm{DS}}(B\cup C)=0.14$.

After taking the negator of $\bar{m}_{1,2}^{\mathrm{DS}}(.)$, using the indirect DS (i.e., IF-DS) fusion approach we obtain the following final result

$$m_{1,2}^{\mathrm{IF-DS}}(\varnothing)=0,\ m_{1,2}^{\mathrm{IF-DS}}(A\cup B)=0.63,\ m_{1,2}^{\mathrm{IF-DS}}(A\cup C)=0.23,\ m_{1,2}^{\mathrm{IF-DS}}(A)=0.14\ .$$

This result appears a bit more acceptable than the direct DS fusion result without being dictatorial because $m_{1,2}^{\mathrm{IF-DS}}(.)\neq m_1(.)$ and $m_{1,2}^{\mathrm{IF-DS}}(.)\neq m_2(.)$. It is worth mentioning that this new indirect DS fusion approach does not always circumvent the bad dictatorial behavior of DS rule in general thanks to the negators and their DS fusion. To emphasize this remark, it is easy to build another (dual) Dezert-Tchamova example where the fusion of negators of BBAs really provides a dictatorial behavior instead of the direct DS fusion. For instance, consider $\Theta=\{A,B,C\}$ and the following BBAs

$$m_1(B\cup C)=a, m_1(C)=1-a,$$

$$m_2(C)=b_1,\ m_2(A\cup B)=1-b_1-b_2,\ m_2(A\cup B\cup C)=b_2.$$

It can be verified that the fusion result based on our negators of these BBAs and the indirect DS fusion is dictatorial and gives $m_{1,2}^{\mathrm{IF-DS}}(B\cup C)=a=m_1(B\cup C)$ and $m_{1,2}^{\mathrm{IF-DS}}(C)=1-a=m_1(C)$. So, the DS rule based on the negators of BBA remains also disputable in this case. That is why in any strategy of fusion chosen (direct and indirect) we do not recommend DS rule because of its potential dictatorial behavior.

- Indirect fusion with PCR6 rule

The indirect PCR6 fusion approach of the negators of $m_1(.)$ and $m_2(.)$ yields

$$\bar{m}_{1,2}^{\mathrm{PCR6}}(\varnothing)=0,\ \bar{m}_{1,2}^{\mathrm{PCR6}}(B)=0.15,\ \bar{m}_{1,2}^{\mathrm{PCR6}}(C)=0.6142,\ \bar{m}_{1,2}^{\mathrm{PCR6}}(B\cup C)=0.09,\ \bar{m}_{1,2}^{\mathrm{PCR6}}(A\cup B)=0.1458$$

After taking the negator of $\bar{m}_{1,2}^{\mathrm{PCR6}}(.)$ using Indirect PCR6 (IF-PCR6) fusion approach we obtain the following final result

$$m_{1,2}^{\mathrm{IF-PCR6}}(\varnothing)=0,\ m_{1,2}^{\mathrm{IF-PCR6}}(A\cup B)=0.6142,$$

$$m_{1,2}^{\mathrm{IF-PCR6}}(A\cup C)=0.15,\ m_{1,2}^{\mathrm{IF-PCR6}}(A)=0.09,\ m_{1,2}^{\mathrm{IF-PCR6}}(C)=0.1458.$$

We see that direct and indirect PCR6-based fusion methods give distinct results because $m_{1,2}^{\mathrm{IF-PCR6}}(.)\neq m_{1,2}^{\mathrm{PCR6}}(.)$. The indirect fusion results based on DS rule and PCR6 rule provide similar maximal mass value for the same focal element $A\cup B$ because $m_{1,2}^{\mathrm{IF-DS}}(A\cup B)\approx 0.63, m_{1,2}^{\mathrm{IF-PCR6}}(A\cup B)=0.6142$. We see that the set of focal elements of $m_{1,2}^{\mathrm{IF-DS}}(.)$ and $m_{1,2}^{\mathrm{IF-PCR6}}(.)$ are different because IF-PCR6 commits a mass specifically to the element C which is not a focal element of $m_{1,2}^{\mathrm{IF-DS}}(.)$. Actually, the sets of focal elements of BBAs $m_{1,2}^{\mathrm{IF-DS}}(.)$ and $m_{1,2}^{\mathrm{IF-PCR6}}(.)$ are different.

6.3. Example 3 – Blackman's example (Bayesian and non-Bayesian BBAs)

This simple example has been introduced by B l a c k m a n and P o p o l i in [28] and analyzed by the authors in [29]. We consider the FoD $\Theta=\{A,B\}$ and the following two BBAs

$$m_1(A)=0.5, m_1(B)=0.5, m_1(A\cup B)=0,$$
$$m_2(A)=0.1, m_2(B)=0.1, m_2(A\cup B)=0.8.$$

We see that there is no way to decide either A or B in this example because each SoE does not bring useful information to help for decision-making. Each BBA $m_1(.)$ and $m_2(.)$ is completely symmetrical to A and B. So intuitively, there is no reason to expect an improvement in the decision-making based on the fusion of these two BBAs. The conjunctive fusion of $m_1(.)$ and $m_2(.)$ yields

$$m_{1,2}^{\cap}(\varnothing)=0.1,\ m_{1,2}^{\cap}(A)=0.45,\ m_{1,2}^{\cap}(B)=0.45\ .$$

We see that the conflicting mass $m_{1,2}^{\cap}(\varnothing)=0.10$ must be redistributed to some elements of $2^{\Theta}\setminus\{\varnothing\}$ in order to get a proper fused BBA.

- Direct fusion with DS rule

By applying DS rule (19), we obtain $m_{1,2}^{\mathrm{DS}}(\varnothing)=0,\ m_{1,2}^{\mathrm{DS}}(A)=0.5,\ m_{1,2}^{\mathrm{DS}}(B)=0.5.$

- Direct fusion with PCR6 rule

By applying PCR6 rule (20), we obtain

$$m_{1,2}^{\mathrm{PCR6}}(\varnothing)=0,\ m_{1,2}^{\mathrm{PCR6}}(A)=0.5,\ m_{1,2}^{\mathrm{PCR6}}(B)=0.5\ .$$

As intuitively expected, the direct fusion results based on DS rule and on PCR6 rule do not help to make a rational decision in favor of A or B.

- Indirect fusion with DS and PCR6 rules

Applying BBA negator defined by (16), we obtain

$$\bar{m}_1(B)=0.5, \bar{m}_1(A)=0.5, \bar{m}_1(A\cup B)=0,$$
$$\bar{m}_2(B)=0.1, \bar{m}_2(A)=0.1, \bar{m}_2(A\cup B)=0.8.$$

Because $|\Theta|=2$, we see that we have $\bar{m}_1(.)=m_1(.),\quad \bar{m}_2(.)=m_2(.)$. Therefore, we will get the same result with the conjunctive fusion of $\bar{m}_1(.)$ and $\bar{m}_2(.)$ as for the direct conjunctive fusion of $m_1(.)$ and $m_2(.)$. The direct or indirect fusion methods based on DS and PCR6 rules will yield actually to the same fusion result, that is

$$m_{1,2}^{\mathrm{DS}}(A)=m_{1,2}^{\mathrm{IF-DS}}(A)=m_{1,2}^{\mathrm{PCR6}}(A)=m_{1,2}^{\mathrm{IF-PCR6}}(A)=0.5,$$
$$m_{1,2}^{\mathrm{DS}}(B)=m_{1,2}^{\mathrm{IF-DS}}(B)=m_{1,2}^{\mathrm{PCR6}}(B)=m_{1,2}^{\mathrm{IF-PCR6}}(B)=0.5.$$

This example is interesting because it is a situation where there is no advantage of using direct fusion with respect to indirect fusion and vice-versa.

7. Two important remarks

Remark 1. As shown in Zadeh's example (Section 6.1) the indirect fusion gives

$$m_{1,2}^{\mathrm{IF-\cap}}(A\cup B)=0.81,\ m_{1,2}^{\mathrm{IF-\cap}}(A\cup C)=0.09,\ m_{1,2}^{\mathrm{IF-\cap}}(B\cup C)=0.09,\ m_{1,2}^{\mathrm{IF-\cap}}(C)=0.01\ .$$

It is interesting to observe that this result coincides with the fusion result obtained with the disjunctive rule of combination (18). Indeed, we have

$$m_{1,2}^{\cup}(A\cup B)=0.81,\ m_{1,2}^{\cup}(A\cup C)=0.09,\ m_{1,2}^{\cup}(B\cup C)=0.09,\ m_{1,2}^{\cup}(C)=0.01\ .$$

We may question if the equality $m_{1,2}^{\text{IF}-\cap}(.)=m_{1,2}^{\cup}(.)$ is a general property satisfied or only just a coincidence. In fact, it is clear that this is just a simple coincidence due to the particular structure of focal elements of the BBAs of Zadeh's example. This property does not hold in general even if there is no conflict between the negators. As a simple counter-example, consider the extended Blackman's example with $\Theta=\{A,B,C\}$ with two BBAs

$$m_1(A)=m_1(B)=m_1(C)=1/3,$$
$$m_2(A)=m_2(B)=m_2(C)=0.1, m_2(A\cup B\cup C)=0.7.$$

In this case, no conflict exists between the negators $\bar{m}_1(.)$ and $\bar{m}_2(.)$.The indirect fusion approach gives the final result

$$m_{1,2}^{\cup}(\varnothing)=0,\ m_{1,2}^{\cup}(A)=m_{1,2}^{\cup}(B)=m_{1,2}^{\cup}(C)=0.1/3,$$
$$m_{1,2}^{\cup}(A\cup B)=m_{1,2}^{\cup}(A\cup C)=m_{1,2}^{\cup}(B\cup C)=0.2/3,\ m_{1,2}^{\cup}(A\cup B\cup C)=0.7.$$

The fusion result obtained with the disjunctive rule of combination (18) for this extended Blackman's example is

$$m_{1,2}^{\cup}(\varnothing)=0,\ m_{1,2}^{\cup}(A)=m_{1,2}^{\cup}(B)=m_{1,2}^{\cup}(C)=0.1/3,$$
$$m_{1,2}^{\cup}(A\cup B)=m_{1,2}^{\cup}(A\cup C)=m_{1,2}^{\cup}(B\cup C)=0.2/3,\ m_{1,2}^{\cup}(A\cup B\cup C)=0.7.$$

We see that $m_{1,2}^{\text{IF}-\cap}(.)\neq m_{1,2}^{\cup}(.)$ in this example, so the property $m_{1,2}^{\text{IF}-\cap}(.)=m_{1,2}^{\cup}(.)$ is not always satisfied. This means that De Morgan's law does not hold in general in information fusion of BFs. More precisely, the direct disjunctive fusion of BBAs is not necessarily equivalent to the negator of the conjunctive fusion of negators. Similarly, the direct conjunctive fusion of BBAs is not necessarily equivalent to the negator of the disjunctive fusion of the negators.

Remark 2. The negation of a BBA does not necessarily increase the entropy contrary to what is claimed in the literature in some papers cited in Section 3. To prove this claim, just consider the FoD $\Theta=\{A,B,C\}$ and the BBA $m(.)$ given by

$$m(A\cup B)=0.7, m(A\cup C)=0.2, m(A\cup B\cup C)=0.1.$$

This verifies that the entropy of $m(.)$ obtained by the formula (4) is $U(m)\approx 4.299$ nats. Our negator of $m(.)$ is $\bar{m}(C)=0.7, \bar{m}(B)=0.2, \bar{m}(A\cup B\cup C)=0.1$, whose entropy is $U(\bar{m})\approx 1.254$ nats. One sees $U(\bar{m})<U(m)$ in this simple example. Therefore, the negation of a BBA $m(.)$ does not necessarily increase the entropy. It really depends on the mass of belief committed to focal elements of BBA $m(.)$.

8. Management of direct and indirect fusions

As shown in the examples of Section 6 the results obtained with direct fusion approach and indirect fusion approach do not coincide except but in very particular cases. In general, we have $m^{\mathrm{IF}}_{1,2,\ldots,S} \neq m^{\mathrm{DF}}_{1,2,\ldots,S}$. Therefore, at this stage of our research work, we are facing a new problem: what to do with these two fusion results $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ for decision-making support. This section provides two possible answers to this important question.

Answer 1. Fuse $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ with $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$

The first intuitive answer to the aforementioned question would consist in fusing (i.e., combining) the two fusion results $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ with $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ by some chosen appropriate rule of combination, typically the PCR6 rule (or the PCR6+ rule if $S > 2$, see [21]). This first answer is unfortunately not very satisfactory and not recommended from a theoretical point of view, because $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ are actually based on exactly the same original inputs corresponding to BBAs $m_1(.), m_2(.), \ldots, m_S(.)$. Therefore, $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ cannot be considered as (cognitively) independent and their fusion is not recommended because of redundant information, which may generate some biases in the final result and decision-making mistakes. If this approach is however used in applications by some users, we suggest at least to take into account the quality of each source $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$, characterized by their entropies $U(m^{\mathrm{DF}}_{1,2,\ldots,S})$ and $U(m^{\mathrm{IF}}_{1,2,\ldots,S})$.

A very simple fusion method would consist for instance to apply the weighted averaging fusion of $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ with $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ defined for any $X \in 2^{\Theta}$ by

(25) $$m(X) = \varpi^{\mathrm{DF}} m^{\mathrm{DF}}_{1,2,\ldots,S}(X) + \varpi^{\mathrm{IF}} m^{\mathrm{IF}}_{1,2,\ldots,S}(X),$$

where the importance weighting factors $\varpi^{\mathrm{DF}}, \varpi^{\mathrm{DF}} \in [0,1]$ and $\varpi^{\mathrm{DF}} + \varpi^{\mathrm{IF}} = 1$.

Other fusion methods based on discounting techniques and entropies could be eventually developed also, but fundamentally we do not recommend to combine $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ with $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ for the aforementioned reason of underlying dependency of original BBAs that have been used to generate direct and indirect fusion results $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$.

Answer 2. Select either $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ or $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$

As we consider that the intuitive previous answer is not satisfactory, we need to seriously consider a second option of management of direct and indirect fusion results $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$. This second option consists in selecting only one BBA or $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ for decision-making support. However, which one to select? How?

For selecting the BBA $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ or $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ we propose to adopt the maximum entropy principle which states we should select the BBA which leaves us the largest remaining uncertainty. More precisely, we will select $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ if $U(m^{\mathrm{DF}}_{1,2,\ldots,S}) > U(m^{\mathrm{IF}}_{1,2,\ldots,S})$, and we will select $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ if $U(m^{\mathrm{IF}}_{1,2,\ldots,S}) > U(m^{\mathrm{DF}}_{1,2,\ldots,S})$. In the very rare cases where $m^{\mathrm{DF}}_{1,2,\ldots,S}(.) = m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ no selection is needed because the two BBAs $m^{\mathrm{DF}}_{1,2,\ldots,S}(.)$ and $m^{\mathrm{IF}}_{1,2,\ldots,S}(.)$ coincide. This maximum entropy principle is rather simple to use in practice because we need only to calculate the entropies $U\left(m^{\mathrm{DF}}_{1,2,\ldots,S}\right)$ and $U\left(m^{\mathrm{IF}}_{1,2,\ldots,S}\right)$.

We now provide more details on how to proceed in the interesting examples considered in Section 6.

8.1. For Example 1 – Zadeh's example (two Bayesian BBAs)

- With direct fusion using DS rule

We obtain the Bayesian BBA $m^{\mathrm{DS}}_{1,2}(C)=1$. The entropy of $m^{\mathrm{DS}}_{1,2}$ is $U\left(m^{\mathrm{DS}}_{1,2}\right)=0$ nats. This stipulates that there is no uncertainty carried by this very specific BBA which is a counter-intuitive result as explained in [23].

- With direct fusion using PCR6 rule

We obtain $m^{\mathrm{PCR6}}_{1,2}(A)=m^{\mathrm{PCR6}}_{1,2}(B)=0.486, m^{\mathrm{PCR6}}_{1,2}(C)=0.028$ .The entropy of this Bayesian $m^{\mathrm{PCR6}}_{1,2}$ based on the formula (4) is $U\left(m^{\mathrm{PCR6}}_{1,2}\right)\approx 0.8014$ nats.

- With indirect fusion approach

We obtain (see Section 6.1)

$$m^{\mathrm{IF}-\cap}_{1,2}(A\cup B)=0.81,\ m^{\mathrm{IF}-\cap}_{1,2}(A\cup C)=0.09,\ m^{\mathrm{IF}-\cap}_{1,2}(B\cup C)=0.09,\ m^{\mathrm{IF}-\cap}_{1,2}(C)=0.01 .$$

The entropy of this non-Bayesian BBA $m^{\mathrm{IF}-\cap}_{1,2}$ based on formula (4) is $U\left(m^{\mathrm{IF}-\cap}_{1,2}\right)\approx 3.8714$ nats. Clearly, the BBA to use for decision-making support corresponds to the indirect fusion result $m^{\mathrm{IF}-\cap}_{1,2}$ because $U\left(m^{\mathrm{IF}-\cap}_{1,2}\right)>U\left(m^{\mathrm{PCR6}}_{1,2}\right)$. From the selected BBA $m^{\mathrm{IF}-\cap}_{1,2}$ the final decision can be done thanks to different techniques that are detailed in [30]. In Zadeh's example the hypothesis C will be rejected whereas even if there is a tie between A and B that can be eliminated arbitrarily (if we want) by a uniform random draw (i.e., perfect coin tossing) between A and B.

8.2. For Example 2 – Dezert-Tchamova example (two non-Bayesian BBAs)

We consider the Example 2 given in Section 6.2.

- For direct fusion with DS rule

$$m^{\mathrm{DS}}_{1,2}(A)=0.3=m_1(A),\ m^{\mathrm{DS}}_{1,2}(A\cup B)=0.7=m_1(A\cup B).$$

The entropy of $m^{\mathrm{DS}}_{1,2}$ is $U\left(m^{\mathrm{DS}}_{1,2}\right)\approx 0.6108$ nats.

- For direct fusion with PCR6 rule

$$m^{\mathrm{PCR6}}_{1,2}(A)=0.2062,\ m^{\mathrm{PCR6}}_{1,2}(A\cup B)=0.5542,\ m^{\mathrm{PCR6}}_{1,2}(C)=0.2396 .$$

The entropy of $m^{\mathrm{PCR6}}_{1,2}$ is $U\left(m^{\mathrm{PCR6}}_{1,2}\right)\approx 2.917$ nats.

- For indirect fusion with DS rule

$$m^{\mathrm{IF-DS}}_{1,2}(A)\approx 0.14,\ m^{\mathrm{IF-DS}}_{1,2}(A\cup B)\approx 0.63,\ m^{\mathrm{IF-DS}}_{1,2}(A\cup C)\approx 0.23 .$$

The entropy of $m^{\mathrm{IF-DS}}_{1,2}$ is $U\left(m^{\mathrm{IF-DS}}_{1,2}\right)\approx 3.4175$ nats.

- For indirect fusion with PCR6 rule

$m_{1,2}^{\text{IF-PCR6}}(A)=0.09$, $m_{1,2}^{\text{IF-PCR6}}(C)=0.1458$, $m_{1,2}^{\text{IF-PCR6}}(A\cup B)=0.6142$, $m_{1,2}^{\text{IF-PCR6}}(A\cup C)=0.15$.

The entropy of $m_{1,2}^{\text{IF-PCR6}}$ is $U\left(m_{1,2}^{\text{IF-PCR6}}\right)\approx 3.4358$ nats.

One sees that if DS rule is used and preferred by the user (for his own reason) and because $U\left(m_{1,2}^{\text{IF-DS}}\right)>U\left(m_{1,2}^{\text{DS}}\right)$ it will be more reasonable for the user to select $m_{1,2}^{\text{IF-DS}}$ rather than $m_{1,2}^{\text{DS}}$ to draw the final decision. Because we do not recommend DS fusion rule in general due to its bad dictatorial behavior, we will actually select $m_{1,2}^{\text{IF-PCR6}}$ for decision-making because $U\left(m_{1,2}^{\text{IF-PCR6}}\right)>U\left(m_{1,2}^{\text{PCR6}}\right)$. For this example and based on $m_{1,2}^{\text{IF-PCR6}}$ we will finally decide A because $m_{1,2}^{\text{IF-PCR6}}$ is closest to the sure BBA defined by $m_A(A)=1$ than to the sure BBAs defined by $m_B(B)=1$ and by $m_C(C)=1$. More precisely, for this numerical example, we get $d_{BI}\left(m_{1,2}^{\text{IF-PCR6}},m_A\right)=0.5019$, $d_{\text{BI}}\left(m_{1,2}^{\text{IF-PCR6}},m_B\right)=0.6456$ and $d_{\text{BI}}\left(m_{1,2}^{\text{IF-PCR6}},m_C\right)=0.7093$, where $d_{\text{BI}}(.,.)$ is the Euclidean belief interval distance between two BBAs, see [30] for details. Note that the same decision A will be drawn incidentally from $m_{1,2}^{\text{IF-DS}}$.

8.3. For Example 3 – Blackman's example (Bayesian and non-Bayesian BBAs)

For the simple Blackman's example of Section 6.3 we have

$$m_{1,2}^{\text{DS}}(A)=m_{1,2}^{\text{IF-DS}}(A)=m_{1,2}^{\text{PCR6}}(A)=m_{1,2}^{\text{IF-PCR6}}(A)=0.5,$$

$$m_{1,2}^{\text{DS}}(B)=m_{1,2}^{\text{IF-DS}}(B)=m_{1,2}^{\text{PCR6}}(B)=m_{1,2}^{\text{IF-PCR6}}(B)=0.5.$$

Therefore, there is no BBA selection to do because all coincide and we have $U\left(m_{1,2}^{\text{DS}}\right)=U\left(m_{1,2}^{\text{IF-DS}}\right)=0.6931$ nats and $U\left(m_{1,2}^{\text{PCR6}}\right)=U\left(m_{1,2}^{\text{IF-PCR6}}\right)=0.6931$ nats.

Because all the masses of belief of A and B are equal there is no way to make a rational decision towards A or towards B. The final decision-making in this situation (where there is a tie) can be done based either on an arbitrary choice between A and B, or by a (uniform) random choice between A and B based on a perfect coin tossing experiment. Eventually in a given practical fusion problem (for instance in a tracking application) where a tie occurs, we would estimate the main consequences generated by the arbitrary (or random) decision chosen (in term of costs and benefits for instance) to select the best one. This tie elimination method needs of course extra knowledge about the problem under concern. This goes beyond the scope of this paper.

9. Conclusion

In this paper we have analyzed the different definitions of a negator of a probability mass function (pmf) and a Basic Belief Assignment (BBA) existing so far in the literature. In order to overcome their limitations we have introduced a new involutory negator of BBA. Based on it, a new indirect information fusion method has been proposed which can circumvent the conflict management problem in difficult fusion situations. The classical direct and the new indirect information fusion strategies are analyzed for three interesting examples of fusion of two BBAs. In order to manage properly these two types of fusion, two methods for using the whole available information (the original BBAs and their negators) for decision-making support are presented. The first method is based on the combination of the direct and indirect fusion strategies. The second one selects the most reasonable fusion strategy (direct or indirect) to apply based on the maximum entropy principle. A deep analysis of the advantages and drawbacks of these two methods has been made. We will evaluate these new fusion approaches in different fields of applications (multi-sensor data association for tracking, multi-criteria decision-making under uncertainty, perception in robotics, risk assessment, etc.) in our future research works. We also invite the users of belief functions and the fusion system designers to share and report their evaluation of this new approach on their own applications in future publications

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Received: 25.04.2023; Accepted: 05.06.2023