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The Probabilistic Risk Measure VaR as Constraint in Portfolio Optimization Problem

Todor Stoilov[1], *Krasimira Stoilova*[1], *Miroslav Vladimirov*[2]

[1]*Institute of Information and Communication Technologies, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria*
[2]*Varna University of Economics, 9002 Varna, Bulgaria*
E-mails: todor@hsi.iccs.bas.bg k.stoilova@hsi.iccs.bas.bg vladimirov@ue-varna.bg

Abstract**:** *The paper realizes inclusion of probabilistic measure for risk, VaR* (*Value at Risk*)*, into a portfolio optimization problem. The formal analysis of the portfolio problem illustrates the evolution of the portfolio theory in sequentially inclusion of different market characteristics into the problem. They make modifications and complications of the portfolio problem by adding various constraints to consider requirements for taxes, boundaries for assets, cardinality constraints, and allocation of the investment resources. All these characteristics and parameters of the investment participate in the portfolio problem by analytical algebraic relations. The VaR definition of the portfolio risk is formalized in a probabilistic manner. The paper applies approximation of such probabilistic constraint in algebraic form. Geometrical interpretation is given for explaining the influence of the VaR constraint to the portfolio solution. Numerical simulation with data of the Bulgarian Stock Exchange illustrates the influence of the VaR constraint into the portfolio optimization problem.*

Keywords**:** *Portfolio optimization problem, VaR, approximation of probabilistic constraint, graphical interpretation of VaR.*

1. Introduction

The portfolio theory progressively evolves and complicates its formal background. This results in continuously increase in the set of parameters and variables, which are considered for the portfolio management, optimization of resources and investments, the definition and solution of portfolio optimization problems [18]. This paper pays general attention to the inclusion of new risk parameter Value at Risk (VaR) [12, 20] for the portfolio problem. This parameter is defined in probabilistic manner to formalize in other way the risk of the portfolio management. The paper makes contribution to the usage of this probabilistic relation for VaR in portfolio optimization problem. An algebraic approximation is worked out for VaR relation, which transforms VaR parameter in deterministic form. The resulting analytical inequality is included in the portfolio problem and the influence of this new constraint on the solutions of the portfolio problem has been analyzed. The analysis is given in graphical forms, which helps the understanding of the relations, where VaR influences to the portfolio solutions.

2. Evolution of the formal relations and variables in the portfolio optimization problem

The modern portfolio theory starts its flourish with quantitative estimation and evaluation of assets characteristics. The initial parameters, which have been mainly evaluated were the mean assets returns, the risk of each asset and the mutual influence between the assets returns, numerically assessed by values of the components of the covariation matrix [6, 17]. Having these initially estimated asset characteristics the portfolio optimization has been formalized as a static optimization problem defined by [12] in the well-known forms

(1) $$\min_{\mathbf{w}} \begin{bmatrix} \text{Risk}(\mathbf{w}) \\ \text{Return}(\mathbf{w}) \quad \geq \text{Return}_{\min} \\ \mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1,\ \mathbf{w}^{\mathrm{T}} \geq 0 \end{bmatrix} \quad \text{and/or} \quad \max_{\mathbf{w}} \begin{bmatrix} \text{Return}(\mathbf{w}) \\ \text{Risk}(\mathbf{w}) \leq \text{Risk}_{\max} \\ \mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1,\ \mathbf{w}^{\mathrm{T}} \geq 0 \end{bmatrix},$$

where:

$\mathbf{w}^{\mathrm{T}} = (w_1,\dots,w_N)$ is the vector of weights, which gives the relative value of the investment, allocated to the asset i, i=1, N is the number of assets in the portfolio;

relation $\mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1$ gives limitations about the investment amount to be totally allocated to the portfolio;

$\mathbf{w}^{\mathrm{T}} \geq 0$ means that the assets must be bought for the portfolio;

Risk($\mathbf{w}$) and Return($\mathbf{w}$) are analytical relations, describing these portfolio characteristics as functions of the arguments $\mathbf{w}$;

$\text{Return}_{\min}$ and $\text{Risk}_{\max}$ are predefined values, needed to be achieved by the portfolio problem.

Analytically, the portfolio return is defined as weighted sum of asset returns

(2) $$\text{Return}(\mathbf{w}) = E_{\mathrm{p}} = \sum_{i=1}^{N} w_i E_i,$$

where E_i is the mean asset return, estimated according to the historical data R_i^t, $i = 1, \dots, N,\ \ t \in [0, T]$ of the return values in period $t \in [0, T]$. The simplest relation between E_i and R_i^t is given by the average calculation $E_i = \frac{1}{T}\sum_{t\in[0,T]} R_i^t$.

The portfolio risk σ_p is defined as standard deviation of the portfolio return E_{p}. Analytically it is expressed by the asset risks σ_i and covariance coefficients $\sigma_{ij},\ i, j = 1, \dots, N$, as quadratic relation

(3) $$\sigma_{\mathrm{p}}^2 = \sum_{i=1}^{N}\sum_{j=1}^{n} c_{ij} w_i w_j \text{ or in matrix form } \text{Risk}(\mathbf{w}) = \sigma_{\mathrm{p}}^2 = \mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w},$$

where $\mathbf{\Sigma}_{N\times N} = \begin{vmatrix} \sigma_{11}^2 & \sigma_{12}^2 \dots & \sigma_{1N}^2 \\ & \dots & \dots \\ \sigma_{N1}^2 & \sigma_{N2}^2 \dots & \sigma_{NN}^2 \end{vmatrix}$ is a symmetric matrix and its main diagonal contains the values of the assets risk σ_{ij}^2 (i=j) and the covariation coefficients σ_{ij}^2 (i≠j) are evaluated according to the relation

(4) $\sigma_{ij} = \frac{1}{T}\sum_{t\in[0,T]}\sum_{t\in[0,T]}\left(R_i^{(t)} - E_i\right)\left(R_j^{(t)} - E_j\right),\ i,j = 1,\dots,N.$

Having the asset characteristics E_i, σ_{ij} , $i,j \in [1,N]$, the portfolio problem can be modified in the form

(5) $$\max_{\mathbf{w}}[\mathbf{E}^{\mathrm{T}}\mathbf{w} - \lambda\mathbf{w}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{w}],\ \mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1,\ \mathbf{w}^{\mathrm{T}} \geq 0,$$

where by changing the coefficient $\lambda \in [0,\infty]$ the solution of (5) gives points from the "efficient frontier" of the portfolio.

Considerable efforts for the development of the modern portfolio theory have been allocated for the correct estimation of the mean assets returns [13]. The accuracy of this estimation is a prerequisite for successful forecast of the future level of the portfolio return [9]. The principal difficulties of the portfolio optimization is that at time of the investment the assets characteristics are known, but in the end of the investment period their future values have to be forecast for the estimation of the result of the portfolio investment [19, 21]. The main assumption applied for the forecast of the assets characteristics is the precise identification now and assuming same level of the asset characteristics in the future [4]. That is why the precise identification now has been under the scope and development of several identification procedures [10]. Here we are going to mention the ARMA and GARCH approaches for the estimation of the assets returns.

The ARMA process is constituted with two subprocesses: Auto Regressive (AR) and Moving Average (MA). The first one, AR describes the value of the current return of an asset as a function of its past values

$$R^{(k)} = R_0 + \varphi_1 R^{k-1} + \cdots + \varphi_n R^{k-n} + \varepsilon_{\mathrm{r}},$$

where R_0 is the basic level of return and $R^{(k-n)}$ are its previous values, *n* is the length of the historical period, *k* is the current discrete time, φ_n are weighted coefficients for the past values, ε_{r} is the market noise.

This relation is used to forecast the asset return for the end of the investment horizon. Such regresive estimation of AR process requires keeping long list of historical data for the estimation of the weighted coefficients φ_n.

For the MA subprocess the investor has to consider previous errors in the historical series of return values

$$R^k = \Theta_0 + \varepsilon^k + \eta_1\varepsilon^{k-1} + \cdots + \eta_n\varepsilon^{k-n},$$

where same estimation difficulties take place for the evaluation of the weighted coefficient η_n. The practical implementation of ARMA methodology is constrained up to ARMA(1, 1), which considers only current and previous value of the asset return and estimation error.

The GARCH methodology for estimation and prediction of the asset risk σ_i and/or covariances $\sigma_{i,j}$ is applied with usage of the square errors in previous steps of the identification. GARCH notation means General Auto Regressive Conditional Heteroskedasticity process [3]

$$\sigma^{2(k)} = \sigma_0 + \alpha_1\sigma^{2(k-1)} + \cdots + \alpha_p\sigma^{2(k-p)} + \beta_1\varepsilon^{2(k-1)} + \cdots + \beta_1\varepsilon^{2(k-q)},$$

where the notation σ concerns both asset risk σ_i and covariance components $\sigma_{i,j}$. The parameters σ_0, α_p, β_q are unknown and they must be estimated in advance. The practical implementation of GARCH is also constrained till *p*=*q*=1, GARCH(1, 1)

[14]. Fig. 1 in graphical way illustrates the influence of the different estimation methodology to the portfolio parameters, defining the optimization problem.

A sequential step ahead for the progress of the modern portfolio theory is the definition of the Capital Asset Pricing Model (CAPM) [16]. It introduces new portfolio parameters, which are related to the portfolio optimization problem. The CAPM defines new market parameters: E_M as Market return and σ_M as Market risk. Respectively, several analytical relations has been derived:

- Relation between the portfolio mean return E_p, portfolio risk σ_p and the market characteristics E_M, σ_M, and the risk free return r_f. This new relation $E_p = \sigma_p(E_M, \sigma_M, r_f)$ is named Capital Market Line (CML);
- Relation between the current values of the market return R_M and the asset return R_i, $R_i = R_i(R_M, r_f)$, named "cHaracteristic Line" (HL);
- Relation between the mean asset return E_i and the market characteristics E_M, σ_M, r_f, named Security Market Line (SML).

These relations allow the values of the portfolio return and risk for particular market behavior (CML) to be estimated; the asset mean return for the particular market (SML) to be assessed and/or forecast; to have analytical relation for forecasting the current value of asset return according to the market return. Having in mind that the market parameters are approximated by the market indices, the CAPM gives additional opportunities to forecast the parameters of the portfolio optimization problem.

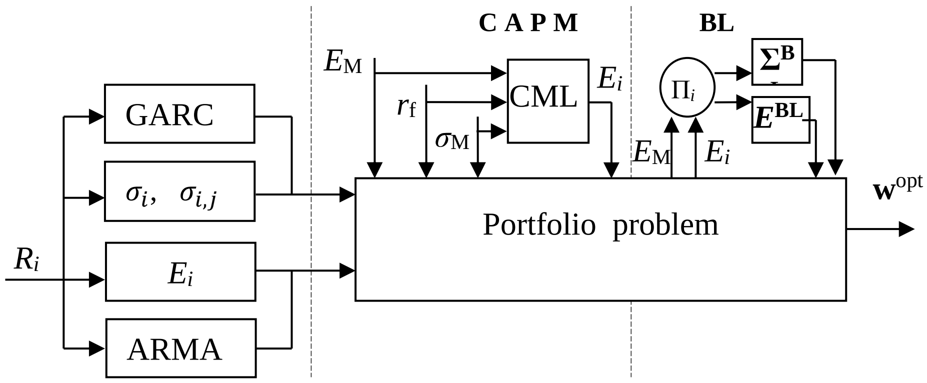

Fig. 1. Formal development of the portfolio characteristics

The classical portfolio problem and the CAPM they both rely on the historical trends of the assets returns as initial input data. The Black-Litterman (BL) model gives additional power to the portfolio theory by adding new information from experts for the assessment and estimation of the asset's parameters. The BL model provides integration between historical data and subjective expert assessment. The idea of the BL model takes into consideration that the usage only of historical data of asset returns does not provide enough information for their future behavior [22]. The usage of expert views is a successful approach for a forecast of assets returns. The formal development of such inclusion of expert views results in definition of new assets characteristics, named “implied returns”, $\Pi_i,\ i = 1, \dots, N$. The values of Π_i are defined according to the market characteristics E_{M}, σ_{M}. Then, the new “implied returns” together with the expert views, matrix **P,** modify the mean assets returns $E_i^{\mathrm{BL}} = E_i(\Pi_i,\ \mathbf{P})$ and the covariance matrix $\mathbf{\Sigma}^{\mathrm{BL}} = \mathbf{\Sigma}(\Pi_i,\ \mathbf{P})$. Thus, the parameters of the portfolio problem are modified and provide a prerequisite for accurate forecast of the future assets parameters.

Complication of the portfolio optimization problem has been made by inclusion of additional constraints to the portfolio problem (5). The incorporation of additional constraints allows additional requirements and peculiarities of the portfolio investment process to be tackled. The set of additional constraints can consider cases like:

- Transactional costs for implementation of the portfolio;
- Value constraints, which restrict the small quantities of assets in the portfolio;
- Position restriction defining bounds on asset position;
- Cardinality constraints for keeping integer set of asset quantities [5].

This paper is going to introduce in the constraint of the portfolio problem the parameter “Value at Risk”, which is a new form of presentation of the investment risk. This constrain has applications for restricting the maximal portfolio losses for a predefined future period. The inclusion of VaR in constraints of the portfolio problem will give optimal solutions for the portfolio weights **w,** which will respect the required level of risk, given by the value of VaR.

3. Value at Risk as a new portfolio risk parameter

The history of VaR development comes from a request of the JP Morgan chairman [2]. He insisted to receive from his staff a report, indicating risk about portfolio losses for the next working day of the trading portfolio of the bank. The risk measure has been later named Value at Risk. This parameter gives value about the maximum likely loss for the next trading day. The VaR value has been evaluated using the portfolio theory according to the standard deviations and correlations between the assets returns [11]. The VaR evaluation has been accepted as new risk management system with major positive effect [1]. VaR was developed and can be regarded as a component of the Portfolio theory. But it gives quantitative value of the risk in terms of maximum likely loss [7]. The formal description of VaR is explained on Fig. 2, where density function of stochastic profit and loss variable of the portfolio is presented.

The positive value of the portfolio return is the profit, while the negative value is the loss [15]. The γ value is the amount of the portfolio loss, while β is the probability for having γ losses. VaR is the integral value of the density function $f(\mathbf{X})$ of a stochastic variable $\mathbf{X}$ and VaR can take values from $-\infty$ to β [8],

$$\mathrm{VaR} = \int_{-\infty}^{\beta} f(y)dy = \gamma.$$

Using the cumulative probability function $F(x)$ of the stochastic variable x, the VaR value is given on Fig. 3.

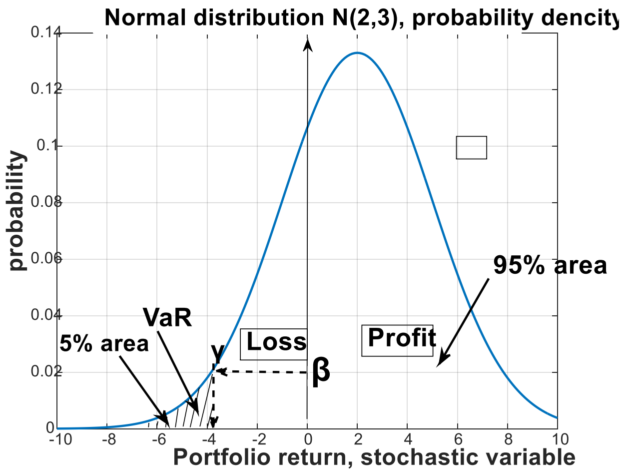

Fig. 2. Density function of the stochastic variable of portfolio profit/loss

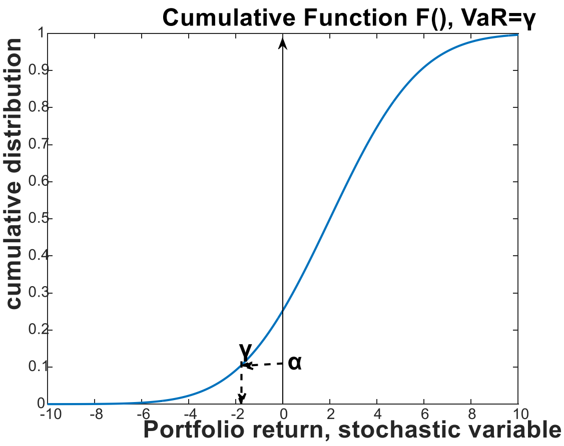

Fig. 3. Graphical presentation of VaR

Analytically, the value γ can be expressed with the probability relation

(6) $$F_x(\gamma) = P_x(x \leq \gamma) > \alpha,$$

because the function $F_x()$ can take values from $(0, 1)$ the opposite probability inequality holds

(7) $$1 - F_x(\gamma) = P_x(x > \gamma) \leq 1 - \alpha.$$

From (6) and (7) the value of γ can be expressed as

(8) $$\text{VaR}_\alpha = \inf\{\gamma : P(x > \gamma) \leq 1 - \alpha\} = \inf\{\gamma : F_x(\gamma) = \alpha\}.$$

The expression defines that VaR is the limit of losses for the chosen confidence level $\alpha \in (0, 1)$. Thus, $\text{VaR}_\alpha = \gamma$ is a value of the portfolio loss, for the confidence level α, which loss is the smallest value γ. The probability that the loss x exceeds γ is no larger than $1 - \alpha$. If **X** has normal distributions for its components x with mean μ_p

and variance σ_{p}, the analytical evaluation of VaR at α (%) confidence level mathematically is found by the integral of Equation (8)

$$\int_{-\infty}^{\mathrm{VaR}_\alpha} \frac{1}{\sigma_{\mathrm{p}}\sqrt{2\pi}} e^{\frac{-(x-\mu_{\mathrm{p}})^2}{2\sigma_p^2}} dx = \frac{\sigma}{100}.$$

The value VaR_α is the value of losses only for the confidential interval α (%) of the defined time period.

Using the inverse cumulative function it holds $F^{-1}(1-\alpha) = \gamma$. Hence, the value of VaR can be expressed also in the form

(9) $$\mathrm{VaR}_\alpha = \min\{\gamma : F(\gamma) = \alpha\} = F^{-1}(1-\alpha).$$

Relations (6)-(9) are probabilistic ones. They are not expressed explicitly with the portfolio parameters of mean asset returns E_i, risks σ_i, correlations σ_{ij} and portfolio weights w_i. It is necessary to derive analytical relations about VaR_α with the portfolio parameters, which is formal requirement to include VaR as constraint in the portfolio problem.

4. Approximation of the probabilistic VaR inequality to algebraic relation

The initial probabilistic inequality comes from the definition of the *cumulative distribution function* $F(.)$ of a stochastic variable **X** and its expression with the probability of the *probability density function* $P(\mathbf{X})$ or

(10) $$F(x) = P(\mathbf{X} < x) \geq \alpha \text{ or } 1 - F(x) = P(\mathbf{X} > x) \leq \alpha,$$

where x is a value of the stochastic variable **X**, α is a value of the probability P.

Let's assume the stochastic variable **Y** is the loss of the portfolio, γ is required VaR value, defining upper bound for the portfolio losses and α is the confidential probability, constraining the loss amount of γ. Hence from (7) it follows

(11) $$\mathrm{VaR}_\alpha = \min\{\gamma : P(\mathbf{Y} > \gamma) \leq 1 - \alpha\}.$$

Relation (11) can be expressed in form (10) as

(12) $$P(\mathbf{Y} < \gamma) \geq \alpha,$$

because **Y** has meaning of portfolio losses, which is expressed as negative portfolio return relation (12) becomes

(13) $$P(\mathbf{Y} < -\gamma, (-1)) \geq \alpha = P(-\mathbf{Y} > -\gamma) \geq \alpha,$$

where the negative **Y** has new meaning as portfolio return and according to its definition it follows $-\mathbf{Y}=\mathbf{R}^{\mathrm{T}}\mathbf{w}$ and (13) is rewritten as

(14) $$P\left(\mathbf{R}^{\mathrm{T}}\mathbf{w} > -\gamma\right) \geq \alpha,$$

where $\gamma=\mathrm{VaR}_\alpha$. To include the probabilistic inequality (14) in the portfolio problem (5) it is needed all parameters of (14) to be expressed with portfolio arguments **w** and parameters $\boldsymbol{\Sigma}$ and **E**. Respectively, (14) from probabilistic it must be presented as analytical constraint. This transformation is made as follows.

The stochastic variables of the current assets returns **R** makes the portfolio current return process $\mathbf{R}^{\mathrm{T}}\mathbf{w}$ to be also stochastic. Latter is normalized to stochastic process with normal distribution with zero mean value and standard deviation equals 1. The normalization of $\mathbf{R}^{\mathrm{T}}\mathbf{w}$ is made by substitution the mean $\mathbf{E}^{\mathrm{T}}\mathbf{w}$ from the left and right parts of the inequality in (14) and dividing both parts by the portfolio risk $\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}$ or it holds

(15) $$P\left(\mathbf{R}^{\mathrm{T}}\mathbf{w} > -\gamma\right) = P\left(\frac{\mathbf{R}^{\mathrm{T}}\mathbf{w}-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}} \geq \frac{-\gamma-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}}\right) \geq \alpha.$$

Considering the equivalent relation (10) which gives $1 - F(x) = P(\mathbf{X} > x)$, relation (15) becomes

$$P\left(\frac{\mathbf{R}^{\mathrm{T}}\mathbf{w}-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}} \geq \frac{-\gamma-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}}\right) = 1 - F\left(\frac{-\gamma-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}}\right) \geq \alpha,$$

or $$F\left(\frac{-\gamma-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}}\right) \leq 1 - \alpha,$$

or $$\left(\frac{-\gamma-\mathbf{E}^{\mathrm{T}}\mathbf{w}}{\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}}}\right) \leq F^{-1}[(1-\alpha)],$$ or

(16) $$\mathbf{E}^{\mathrm{T}}\mathbf{w} + F^{-1}[(1-\alpha)]\sqrt{\mathbf{w}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{w}} \geq -\gamma.$$

The value $F^{-1}[(1-\alpha)]$ is taken from the *z*-score tables about the inverse cumulative function for stochastic process with zero mean and standard deviation equals 1 [23]. Relation (16) considers the parameter VaR_{α} and its inclusion in the portfolio problem (5) for satisfaction of the constraint $\mathrm{VaR}_{\alpha} = \gamma$.

5. Graphical interpretation of VaR constraint

In this section a portfolio problem is defined with real market data. The initial assets returns are taken from the Bulgarian Stock Exchange for its Premium Segment for the period May 2019 till January 2020 [23]. To make graphical interpretations only two assets must be considered. The average monthly costs of assets 5 F4 (CB First Investment Bank AD Sofia) and 5 MB (Monbat AD Sofia) are given in Table 1.

The individual assets returns are calculated according to the relative changes of their prices

Return(*t*)= (Price(*t*+1) – Price(*t*))/(Price(*t*)),

where *t* is the number of the corresponding month for the period May 2019 – January 2020.

Table 1. Average monthly costs of assets 5 F4 and 5 MB

Date	28.5.19	24.6.19	19.7.19	15.8.19	12.9.19	10.10.19	6.11.19	3.12.19	6.1.20
5 F4	3.4	3.44	3.0305	3.2321	3.1768	2.9084	2.92	2.8891	3.3984
5 MB	6.4054	6.5504	6.7591	6.6	6.7	6.3	6.2186	6.2104	6.5

The calculated assets returns per month and their mean returns are given in Table 2.

Table 2. Monthly asset returns and mean returns E_i

Month	5.19	6.19	7.19	8.19	9.19	10.19	11.19	12.19	E_i
5 F4	1.1765	–0.1190	0.06652	–0.0171	–0. 0844	0.0039	–0.0106	0.176324	**0.003418**
5 MB	2.2637	0.0318	–0.0235	0.0151	–0.0597	–0.0129	–0.0013	0.0466	**0.00235**

The components of the correlation matrix $\boldsymbol{\Sigma}$ are evaluated according to (4)

$$\mathbf{\Sigma} = \begin{vmatrix} 0.0455 & 0.0182 \\ 0.0182 & 0.0360 \end{vmatrix}.$$

These initial data about $E_i, i = 1, 2$, and $\mathbf{\Sigma}$ are used for the definition of two portfolio problems: problem (5) without constraint for VaR and problem (5) with VaR constraint of form (16)

$$(17) \qquad \text{P1:} \quad \max_{\mathbf{w}} [\mathbf{E}^{\mathrm{T}}\mathbf{w} - \lambda \mathbf{w}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{w}] \ , \quad \mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1, \ \mathbf{w}^{\mathrm{T}} \geq 0,$$

$$(18) \qquad \text{P2:} \quad \max_{\mathbf{w}} [\mathbf{E}^{\mathrm{T}}\mathbf{w} - \lambda \mathbf{w}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{w}], \mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1, \ \mathbf{w}^{\mathrm{T}} \geq 0,$$

$$\mathbf{E}^{\mathrm{T}}\mathbf{w} + F^{-1}[(1-\alpha)]\sqrt{\mathbf{w}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{w}} \ \geq \ -\gamma.$$

The VaR parameters were chosen: γ=0.3, α=95%. These values, following (8) define that the probability for portfolio losses bigger than 0.3% will be less than 1 – 95% = 5%. The *z*-score value of $F^{-1}[(1-\alpha)]$ is taken from [23] which gives $F^{-1}[(1-\alpha)] = F^{-1}(5\%) = -1.645$. The analytical form of the VaR constraint in problem (17) becomes

$$\mathbf{E}^{\mathrm{T}}\mathbf{w} - 1.645\sqrt{\mathbf{w}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{w}} \ \geq -0.3.$$

Solving problems (17) and (18) for different values of the parameter $\lambda \in (0, \infty)$, points of the corresponding efficient frontier are evaluated. From these sets of portfolio, for comparison reasons, it has been chosen the portfolios from (17) and (18), which have maximal Sharpe ratio = Portfolio return/Portfolio risk. In Fig. 4 the efficient frontier and the position of the corresponding portfolio with maximal Sharpe ratio is presented. For the current values of both problems (17) and (18) the portfolio solutions are the same and the resulting efficient frontiers and the Sharpe portfolios are overlapped. The Sharpe portfolio has parameters $\mathbf{w}^{\mathrm{T}}$=(0.5180 0.4820), Risk=0.0296, Return=0.0029.

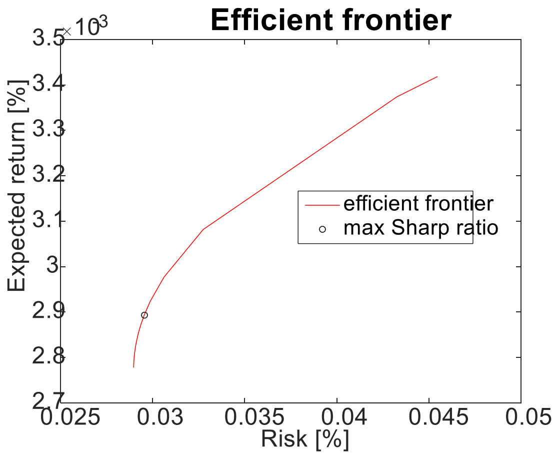

Fig. 4. The “Efficient frontier” is the same for problems (17) and (18)

To find the reason why problems (17) and (18) have equal solutions here it is graphically presented the influence of the VaR constraint. The form of constraint (16) is quadratic one and it makes and elliptic curve in the space of the assets weights $w_2(w_1)$. This elliptic curve restricts the set of points on the line, $\mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1$, where problem (18) can have a solution. For appropriate value of γ the relation (16) can define only one feasible point between (16) and $\mathbf{w}^T|\mathbf{1}| = 1$. On Fig. 5 this case is illustrated and the tangent point between the two constraints is obtained for the value γ=0.2772.

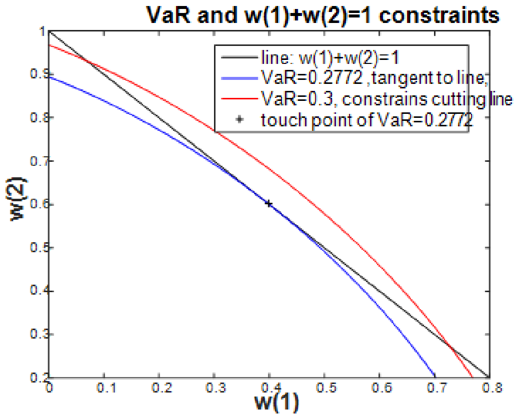

Fig. 5. Graphical presentation of relations (16) and $\mathbf{w}^T|\mathbf{1}| = 1$

For greater values of γ relation (16) makes more feasible points from the line $\mathbf{w}^T|\mathbf{1}| = 1$ where the portfolio problem (18) can have solutions. On Fig. 3 it is illustrated the cases γ=0.2772, γ=0.3 and the graphical view of constraints (16).

On Fig.6 it has been illustrated the case for the equal solutions of problems (17) and (18) even for the value γ=0.3. Graphically the solution of (17) is presented with the tangent point between the objective function of (17) and the line $\mathbf{w}^T|\mathbf{1}| = 1$. The objective function also is an elliptic curve, which makes a tangent point with $\mathbf{w}^T|\mathbf{1}|$ =1.

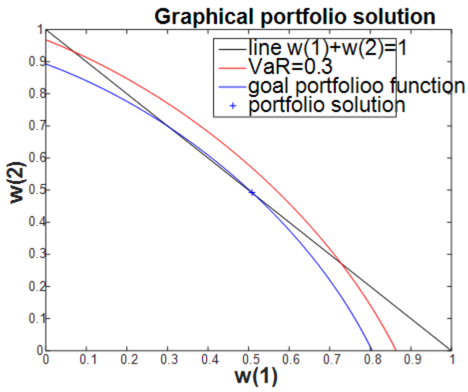

Fig. 6. Graphical solutions of problems (17) and (18)

For the case of problem (18) the objective function is the same as for (17). The VaR_α constraint (16) restricts the feasible points on line $\mathbf{w}^T|\mathbf{1}| = 1$, but the solution of (17) falls down in this feasible area. Hence problem (18) gives the same solution as (17). The graphical interpretation of the influence of the VaR_α constraint (16) is that it cuts off parts of the feasible region of problem (17). But if the solution of (17) belongs to this feasible area, the solution of (18) is the same as this one of (17). For the current numerical case the parameters γ=0.3, $F^{-1}(5\%) = -1.645$ they don't define the solution of (18) to be different from (17). If the requirements for VaR_α are more restrictive like bigger α and less γ, the solutions of (17) and (18) can be different.

6. Conclusions

The contribution of this paper illustrates the influence of the VaR parameter to a portfolio problem. The case when VaR is included as additional constraint to the portfolio problem has been considered. Because the VaR parameter is defined as a probabilistic variable, its approximation is applied by means to derive algebraic relation, which can be added as a constraint to the portfolio problem. The approximation of VaR results in a quadratic form. Thus, this constraint imposes limitations on the feasible area of the portfolio problem over the line $\mathbf{w}^{\mathrm{T}}|\mathbf{1}| = 1$. The graphical interpretation of VaR relation illustrates the limitations in the feasible portfolio set of solutions. The graphical interpretation shows that for particular values of the parameters of VaR, this constraint cannot change the portfolio solutions, based on problem without VaR constraint. This research assumes that the stochastic process of changes of the assets returns has normal distribution. This allows the value of $F^{-1}(1-\alpha)$ as parameter in constraint (16) to be evaluated from *z*-score table. For the case when the stochastic process of assets returns differs from normal distribution, relation (16) cannot be used for approximation of the VaR description of the portfolio risk.

The paper illustrates the transformation of VaR probabilistic relation to an algebraic approximation which has a quadratic form. Thus, the portfolio problem is complicated and has to be solved with methods of convex programming. The particular case of a portfolio with two assets allows the portfolio problem to be solved graphically and gives picture about the influence of the VaR constraint to the portfolio problem. This has been illustrated with real data from the Bulgarian Stock Exchange. The case of complication the portfolio problem with VaR constraint results in finding such portfolio solutions **w,** which will satisfy the predefined level of risk, according VaR value. A potential extension of this research is to consider the VaR as objective function of the portfolio problem. This formulation will give not only optimal solutions of the portfolio weights but the minimal value of VaR for the current behaviour of the market, defined by the mean returns of assets and their covariance. A definition of portfolio problem, which targets minimization of risk and maximization of return, requires satisfaction of min and max objectives, which cannot be achieved in single optimization problem. A potential solution could be the application of hierarchical and/or bi-level portfolio definition, which is a future trend of this research.

As a potential for future researches and elaboration of the portfolio problem the current findings and results in this paper gives opportunities for definition a new complicated form of formalization of the portfolio problem. Due to the internal power of the hierarchical optimization concept, the authors find that the bi-level portfolio definition can improve the content of the portfolio problem with additional arguments like the probabilistic form of risk estimation and its derivatives.

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