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A New Class of "Growth Functions" with Polynomial Variable Transfer Generated by Real Reaction Networks

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Abstract**:** *In* [4, 5], *two classes of growth models with "exponentially variable transfer" and "correcting amendments of Bateman-Gompertz-Makeham-type" based on a specific extended reaction network have been studied* [1]. *In this article we will look at the new scheme with "polynomial variable transfer". The consideration of such a dynamic model in the present article is dictated by our passionate desire to offer an adequate model with which to well approximate specific data in the field of computer viruses propagation, characterized by rapid growth in the initial time interval. Some numerical examples, using CAS Mathematica illustrating our results are given.*

Keywords**:** *Reaction networks, Generalized growth model, Exponentially and polynomial variable transfers.*

1. Introduction and preliminaries

In [1] the following class of growth-decay model formulated in terms that include various types of evolution of the resource species:

$$\begin{gathered} S \xrightarrow{k_1} P \xrightarrow{k_2} Q, \\ S \xrightarrow{k_3} P_1 \xrightarrow{k_4} Q_1, \\ \dots \\ S \xrightarrow{k_{2n-1}} P_{n-1} \xrightarrow{k_{2n}} Q_{n-1}, \\ P_{n-1} + X \xrightarrow{k} nX + P_{n-1} \end{gathered} \tag{1}$$

is considered.

The general model can be written for the growth function in the form [1]:

$$\begin{cases} x'(t) = kxP_{n-1}(t), \\ x(0) = x_0, \end{cases} \tag{2}$$

where for $k_i \neq k_j$, $i, j = 1, 2, \dots, 2n$,

$$P_{n-1}(t) = \frac{k_{2n-1}s_0}{k_{2n} - \sum_{i=1}^{n} k_{2i-1}} \left(e^{-\sum_{i=1}^{n} k_{2i-1}t} - e^{-k_{2n}t} \right) + p_{n-1,0}e^{-k_{2n}t}. \tag{3}$$

We will explicitly note that for $n \geq 2$ the model (1)-(3) summarizes Markov's research [2, 3].

A number of basic results in this direction can be found in [6-11].

It is of interest to observe the new growth model based on this reaction networks in the case where, for example, $k_1 = k_1(t)$, $k_3 = k_3(t)$, …, $k_{2n-1} = k_{2n-1}(t)$.

In this article we will get a generalized class of growth curves with polynomial variable transfer based on this reaction scheme.

2. Main results

Let us consider the following reaction network:

$$\begin{gathered} S \xrightarrow{k_1(t)} P \xrightarrow{k_2} Q, \\ S \xrightarrow{k_3(t)} P_1 \xrightarrow{k_4} Q_1, \\ \cdots \\ S \xrightarrow{k_{2n-1}(t)} P_{n-1} \xrightarrow{k_{2n}} Q_{n-1}, \\ P_{n-1} + X \xrightarrow{k} nX + P_{n-1}, \end{gathered} \tag{4}$$

where

$$k_j(t) = \sum_{i=1}^{n} a_i^{(j)} t^i, \ \ j = 1, 3, \ldots, 2n-1,$$

and k_r, $r = 2, 4, \ldots, 2n$, and k are constants.

In the general case, the detailed study of the scheme (4) and the corresponding "reaction system of differential equations", as well as the important question of the stability of the component of the solution $x(t)$ is an extremely difficult task.

For this reason, we will study the dynamics of this model in the following particular case.

2.1. The case $k_1(t) = 1 + t$, and $k_2 = 1$

Consequently the following differential equations can be formulated

$$\begin{cases} \dfrac{ds}{dt} = -(1+t)s, \\ \dfrac{dp}{dt} = (1+t)s - p, \\ \dfrac{dx}{dt} = kxp, \end{cases} \tag{5}$$

with $s(0) = s_0$; $p(0) = p_0$; $x(0) = x_0$.

For the solution of the first equation in (5) we obtain

$$s(t) = Ce^{-\left(t+\frac{1}{2}t^2\right)},$$

and from $s(0) = s_0$ we get $C = s_0$ and

$$s(t) = s_0 e^{-\left(t+\frac{1}{2}t^2\right)}. \tag{6}$$

From the second equation of the system (5) we have

$$\frac{dp}{dt} + p(t) = s_0(1+t)e^{-\left(t+\frac{1}{2}t^2\right)}.$$

The solution to this equation is

$$\begin{aligned}
p(t) &= s_0 e^{-t}\int_0^t (1+t)e^{-\left(t+\frac{1}{2}t^2\right)}e^t.dt + Re^{-t} = \\
&= s_0 e^{-t}\int_0^t (1+t)e^{-\frac{1}{2}t^2}.dt + Re^{-t} = \\
&= s_0 e^{-t}\left(-e^{-\frac{1}{2}t^2} + \frac{\sqrt{\pi}}{2}\operatorname{Erf}\left(\frac{t}{\sqrt{2}}\right)\right)\Big|_0^t + Re^{-t} = \\
&= s_0 e^{-t}\left(1-e^{-\frac{1}{2}t^2} + \frac{\sqrt{\pi}}{2}\operatorname{Erf}\left(\frac{t}{\sqrt{2}}\right)\right) + Re^{-t},
\end{aligned}$$

where

$$\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}.dt.$$

From $p(0) = p_0$ we get $R = p_0$.

So, finally, for the solution $p(t)$ we get

$$p(t) = s_0 e^{-t}\left(1-e^{-\frac{1}{2}t^2} + \frac{\sqrt{\pi}}{2}\operatorname{Erf}\left(\frac{t}{\sqrt{2}}\right)\right) + p_0 e^{-t}. \tag{7}$$

More importantly, the solution $x(t)$ of the last equation of the differential system

$$x(t) = x_0 e^{k\left(-\frac{1}{2}e^{-t}\left(2(p_0+s_0)+\sqrt{2\pi}s_0\operatorname{Erf}\left(\frac{t}{\sqrt{2}}\right)\right)+p_0+s_0\right)}, \tag{8}$$

generates a new growth model not encountered to have been described in the literature.

We illustrate our new model for fixed s_0, p_0 and various parameters k and x_0 (Fig. 1).

Remark. It is important to study the characteristic – "super saturation" of the model to the horizontal asymptote.

In this connection, the following Proposition is valid.

Proposition. For the new model (8) it is fulfilled that

$$\lim_{t\to\infty} x(t) = x_0 e^{k(s_0+p_0)}.$$

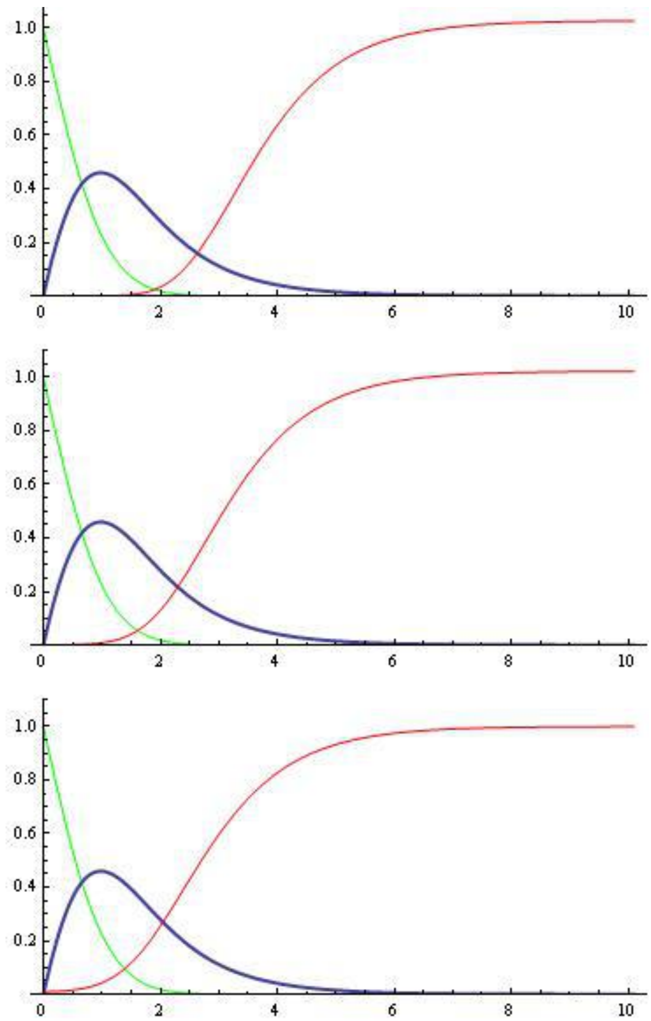

Fig. 1. The functions $x(t)$ (red), $p(t)$ (blue), $s(t)$ (green) for fixed $s_0 = 1$, $p_0 = 0$: a) $k = 11.54$, $x_0 = 0.00001$; b) $k = 6.93$, $x_0 = 0.001$; c) $k = 4.605$, $x_0 = 0.01$

2.2. Some applications

The consideration of such a dynamic model in the present article is dictated by our passionate desire to offer an adequate model with which to well approximate some specific data (including datasets from computer viruses propagation), characterized by rapid growth in the initial time interval.

Example 1. We consider the following specific data [12]:

data_CDF_1:= {{1, 0.6762}, {2, 0.8286}, {3, 0.8667}, {4, 0.9143}, {5, 0.9333}, {6, 0.9429}, {7, 0.9524}, {8, 0.9571}, {9, 0.9667}, {10, 0.9714}, {11, 0.9733}, {14, 0.9810}, {20, 0.9829}, {23, 0.9857}, {25, 0.9885}, {55, 0.9905}}.

The model (8) for $p_0 = 0$, $s_0 = 1$, $x_0 = 0.5$, $k = 0.660972$ is visualized on Fig. 2.

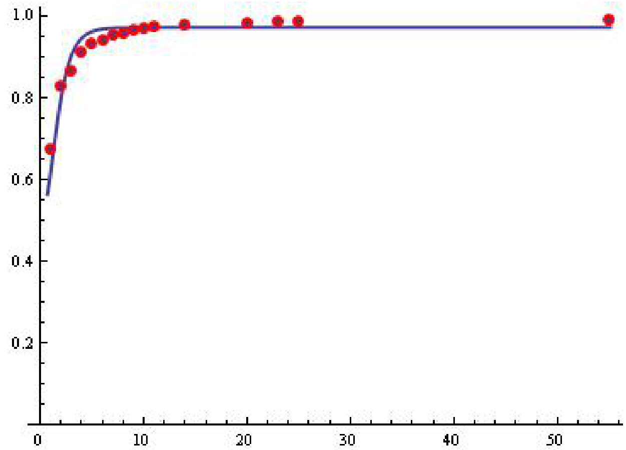

Fig. 2. The fitted model (8) for approximation of the data: "data _CDF_1" [12]

Example 2. Storm worm was one of the most biggest cyber threats of 2008 [13]. We consider the following data:

data_Storm_IDs

:= {{1, 0.843}, {4, 0.926}, {5, 0.954}, {6, 0.967}, {7, 0.976}, {8, 0.981}, {9, 0.985}, {10, 0.991}, {22, 0.995}, {38, 0.997}, {51, 0.998}, {64, 0.9985}, {74, 0.999}, {83, 1}, {100, 1}}.

The model (8) for $p_0 = 0$, $s_0 = 1$, $x_0 = 0.5$, $k = 0.683003$ is visualized on Fig. 3.

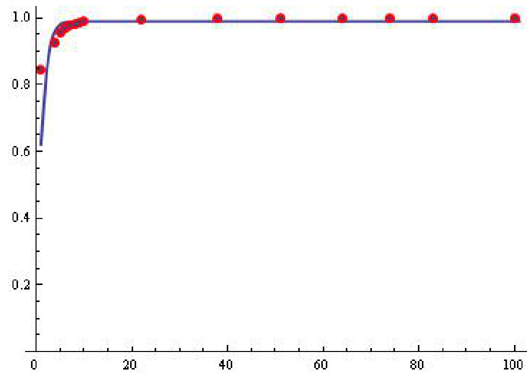

Fig. 3. The fitted model (8) for approximation of the data "data-Storm"

Example 3. For the data (collected from the 140 telecommunication systems that manage the radio access part of wireless systems by week (see, for example [14])) the model (8) for

$$k = 3.44078,\ s_0 = 1,\ p_0 = 0.65,\ x_0 = 0.17$$

is depicted on Fig. 4.

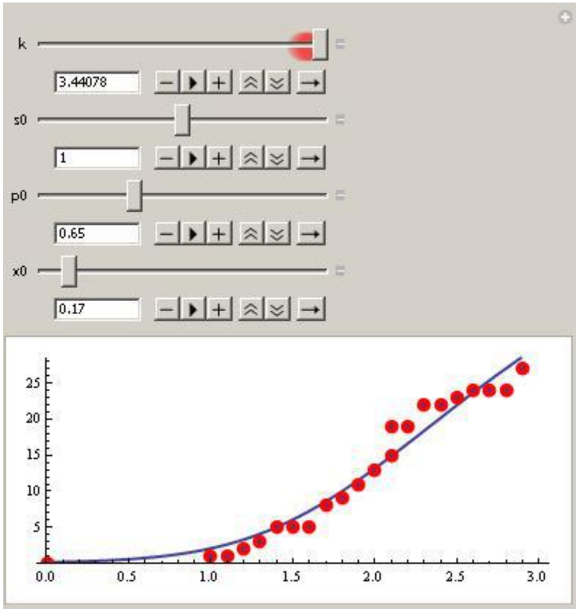

Fig. 4. The model (8) (Example 3)

Example 4. The mean value functions of nine models for the "Failure data" from the Debugging Theory are considered by L e e, C h a n g and P h a m [14], (Fig. 5)

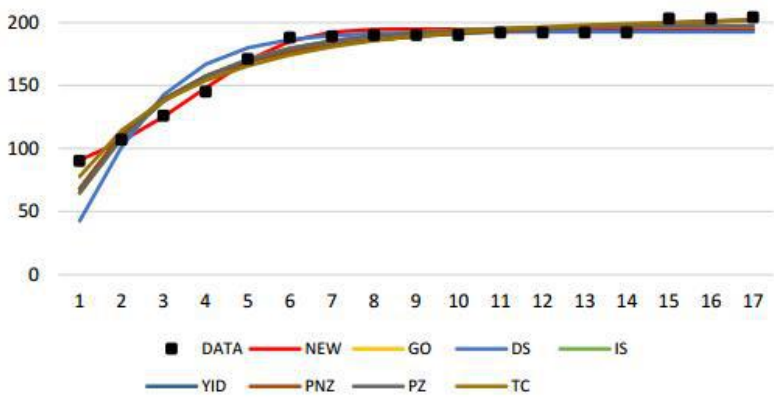

Fig. 5. The mean value functions of nine models (including GO (Goel-Okumoto), NEW (Le-Chang-Pham), DS (Delayed S-shaped), PZ (Pham-Zhang) and other models for this data [14])

For the data from [14] the model (8) for $k = 14.3787$, $s_0 = -0.521269$, $p_0 = 1.1$, $x_0 = 0.0441936$, is depicted on Fig. 6. (We have adopted a scale on the horizontal axis: 0.1 division corresponds to one time interval).

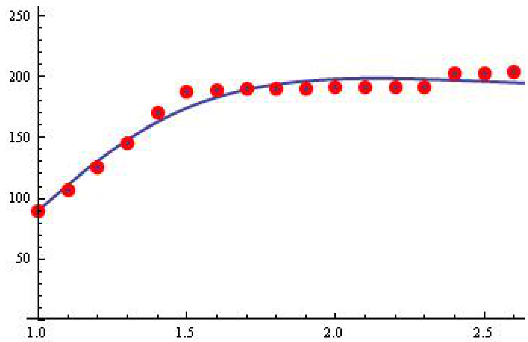

Fig. 6. The model (8) for the "Failure data"

The experiments show that in some cases the use of the growth model proposed in this article is satisfactory.

Specialists working in this scientific field have a say.

For other results, see [15-19].

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