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New Uniform Subregular Parallelisms of PG(3, 4) Invariant under an Automorphism of Order 2

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Abstract**:** *A spread in PG*(*n*, *q*) *is a set of lines which partition the point set. A parallelism is a partition of the set of lines by spreads. A parallelism is uniform if all its spreads are isomorphic. Up to isomorphism, there are three spreads of PG*(*3, 4*) *– regular, subregular and aregular. Therefore, three types of uniform parallelisms are possible. In this work, we consider uniform parallelisms of PG*(*3*, *4*) *which possess an automorphism of order 2. We establish that there are no regular parallelisms, and that there are 8253 nonisomorphic subregular parallelisms. Together with the parallelisms known before this work, this yields a total of 8623 known subregular parallelisms of PG*(*3, 4*).

Keywords**:** *Finite Geometry, Combinatorics, Classification, Isomorphism, Parallel Computing, Spread, Parallelism, MPI.*

1. Introduction

For the basic concepts and notations concerning spreads and parallelisms of projective spaces, refer, for instance, to [15, 22].

A *spread* in PG(*n*, *q*) is a set of lines which partition the point set. A *parallelism* (line parallelisms) is a partition of the set of lines by spreads. Line spreads exist if and only if *n* is odd.

Spreads and parallelisms have various relations and applications. Recently one of the most intensively investigated relation is that to subspace codes [13], which are important in random network coding [16]. Different applications of parallelisms in this area are already known [12, 14]. Parallelisms are also related to translation planes [15, 17, 27]. Every regular parallelism in PG(3, *q*) corresponds to a spread in PG(7, *q*) and therefore to a translation plane of order q^4 [17].

Two parallelisms are *isomorphic* if there exists an automorphism of the projective space which maps each spread of the first parallelism to a spread of the second one. A subgroup of the automorphism group of the projective space which maps each spread of the parallelism to a spread of the same parallelism is called *automorphism group* of the parallelism.

A dual spread in PG(3, q) is a set of lines which have the property that each plane contains exactly one line. The lines of a spread in PG(3, q) are lines of a dual spread too [15]. Consequently, each parallelism defines a dual parallelism. Dual parallelisms in PG(3, q) are parallelisms. A parallelism is *self-dual* if it is isomorphic to its dual.

A spread S of PG(3, q) has $q^2 + 1$ lines and a parallelism has $q^2 + q + 1$ spreads. There are some general constructions of parallelisms: in PG(n, 2) by Zaicev, Zinoviev and Semakov [28] and independently by Baker [2], and in PG(2^m-1, q) by Beutelspacher [8]. Several constructions are known in PG(3, q) due to Denniston [10], Johnson [15], Penttila and Williams [19].

A regulus of PG(3, q) is a set R of $q + 1$ mutually skew lines such that any line intersecting three elements of R intersects all elements of R. The lines that meet every line of R themselves form a new regulus R' which is called the *opposite* regulus of R. A spread S of PG(3, q) is *regular* if for every three distinct elements of S, the unique regulus determined by them is a subset of S. It is *aregular* if for every three distinct elements of S, the unique regulus determined by them is not a subset of S. A regulus R and its opposite regulus R' cover exactly the same set of points. For $R \in S$, a new spread S' can be obtained by replacing the lines of R with the lines of R' (i.e., by *reversing* R). A spread obtained from a regular spread by reversing is called *subregular* [22]. Orr in [18] showed that any spread of PG(3, q) which is neither regular, nor aregular, can be obtained in this manner. The theory of subregular spreads of PG(3, q) was developed by Bruck [9].

Up to isomorphism, there are three spreads in PG(3, 4): regular, subregular and aregular. The regular spread has an automorphism group of order 81,600 and the other two spreads have groups of orders 1200 and 576, respectively. The subregular spread is also known as the Hall spread.

A parallelism is *uniform* if it consists entirely of isomorphic spreads. So, three types of uniform parallelisms can exist in PG(3, q) – regular, subregular, and aregular. A parallelism is *cyclic* if there is an automorphism which permutes its spreads in one cycle. Cyclic parallelisms are uniform.

There is an infinite class of cyclic regular parallelisms due to Penttila and Williams [19], which include all parallelisms of PG(3, 2), the regular parallelisms in PG(3, 5) discovered by Prince [20], and the regular parallelisms in PG(3, 8) found by Denniston [11]. Recently, six noncyclic regular parallelisms of PG(3, 5) invariant under a group of order three were found in [24]. Bamberg [3] establishes by computer search that there are no regular parallelisms in PG(3, 4) and conjectures that a regular parallelism of PG(3, q) exists if and only if $q \equiv 2 \bmod 3$.

Uniform subregular parallelisms were first considered by Prince [21], where the existence of at least 54 uniform subregular parallelisms in PG(3, 3) and the nonexistence of regular ones is shown. Recently, Betten [5] classified all 73,343 parallelisms of PG(3, 3) and established that 2860 of them are uniform *and* subregular. As there are only two nonisomorphic spreads in this projective space, this comprises all uniform parallelisms in PG(3, 3).

Parallelisms of PG(3, 4) with automorphisms of odd orders have been classified [25, 26] and there are no regular ones among them. There are 27,762 uniform parallelism in PG(3, 4) known up to the present paper, 27,138 aregular and 624 subregular (Table 1, where Aut P is the order of the full automorphism group). The present paper adds 7999 new subregular parallelisms with a full automorphism group of order 2.

Table 1. The order of the full automorphism group of the uniform parallelisms of PG(3, 4)

Aut P	2	4	5	7	10	20	All
Subregular	**7999**	244 [7]	346 [25]	24 [26]	2 [25]	8 [25]	8623
Aregular	≥22,144 [6]	≥4816 [7]	–	178 [26]	–	–	≥ 27,138

In this paper, we consider the possibilities to construct uniform parallelisms which are regular or subregular and possess an automorphism of order 2. We do not find regular parallelisms. We construct all subregular parallelisms with automorphisms of order 2.

Section 2 describes the specifics of the problem – the projective space PG(3, 4), the action of the predefined automorphism group on the lines, the types of spread orbits that the parallelisms should contain, and the partial solutions of all fixed spreads. Section 3 is devoted to the construction methods, and Section 4 is a comment on the results and the problems that remain open.

Our method is different from the one used by Prince [21] who calculates a special parameter for each subregular spread. In fact the three authors use slightly different algorithms and different software based on MPI and the C^{++} language.

Part of the results are obtained on RMACC Summit based on Intel Xeon “Haswell” CPU (**https://www.colorado.edu/rc/resources/summit**) with Intel Xeon E5-2680 v3 @2.50 GHz by the first autor while the other autors use the HPC system Avitohol (**http://nchdc.acad.bg/en/resources/iict/avitohol/**) with Intel Xeon E5-2650 v2 @2.60 GHz processors.

2. Line and spread orbits under the automorphism group of order 2 with 5 fixed points and 21 fixed lines ($G_{2_{21}}$)

We consider the projective space PG(3, *q*). The points of PG(3, *q*) are the one-dimensional subspaces of $V(4, q)$, the four dimensional vector space over GF(*q*). They correspond to the nonzero vectors (v_1, v_2, v_3, v_4) over GF(*q*) such that the rightmost nonzero coordinate is one. Any invertible matrix $A=(a_{i,j})_{4\times4}$ over GF(*q*) defines an automorphism of PG(3, *q*) by sending (v_1, v_2, v_3, v_4) to (w_1, w_2, w_3, w_4), where $w_i= v_1a_{i,1}+ v_2a_{i,2}+ v_3a_{i,3}+ v_4a_{i,4}$. This assignment induces a map of the one-dimensional subspaces which is an automorphism of PG(3, *q*). These automorphisms are called *projectivities*. They form the subgroup PGL(4, *q*), called the *projectivity group*. It is contained in the *collineation group* G = PΓL(4, *q*), which is the full automorphism group of the projective space. Besides the projectivities, it contains the *automorphic collineations*, which map a point (v_1, v_2, v_3, v_4) to (w_1, w_2, w_3, w_4), where $w_i = v_i\phi$ is the image of v_i under a field automorphism ϕ of GF(q) over GF(p), with GF(p) being the prime field associated with GF(q).

A spread of PG(3, 4) contains 17 mutually disjoint lines. A parallelism of PG(3, 4) contains 21 mutually line-disjoint spreads. The elements of the finite field GF(4) can be represented as 0, 1, ω and ω^2, where ω is a root of the generating polynomial $x^2 + x + 1 = 0$. In PG(3, 4), there are 85 points. We order them lexicographically and assign a number to each point so that (1, 0, 0, 0) is number 1, and (ω^2, ω^2, ω^2, 1) is number 85. We also arrange the 357 lines of PG(3, 4) based on the lexicographic ordering induced by the points they contain.

Like many authors before us, we assume a group of symmetries H to be a known subgroup of the automorphism group of the parallelisms constructed. Because previous work has dealt with all groups of order three or more, only groups of order 2 are considered here. Isomorphic parallelisms have conjugate stabilizers, and therefore it suffices to consider the assumed symmetry group H up to conjugacy. Since H is a group of prime order, we need to consider the conjugacy classes of elements of that order. We let N be the normalizer of H in G, i.e., $N=N_G(H) = \{\gamma \in G \mid \gamma H \gamma^{-1} = H\}$. The group G = PΓL(4, 4) has three conjugacy classes of elements of order two [Lemma 2]. One is a Baer involution and fixes a Baer subgeometry PG(3, 2). The second class fixes a hyperplane pointwise. The third class fixes a line pointwise. Of these three types of elements, only the second and third are projectivities. The parallelisms admitting a Baer involution have been classified in an earlier work [6]. No parallelism can admit an element of order two from the second conjugacy class [Lemma 2]. This leaves the parallelisms admitting an element of order two from the third class to be determined. The present note is a contribution to this problem. We will determine all uniform parallelisms admitting a symmetry group H of order two generated by an element of the third conjugacy class. In the notation of [Table 3], this group is denoted by $G_{2_{21}}$. Let $H=G_{2_{21}}$. It fixes 5 points and 21 lines and is defined by the following invertible matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The normalizer of H in G has order 30,720.

2.1. Spread orbits

The nontrivial line orbits can be of two different types with respect to the way they can be used in the construction of parallelisms with predefined automorphisms. Orbits whose lines are mutually disjoint (the lines in one orbit contain each point at most once) are called *spread-like*. A whole orbit of this type can be part of a spread which is fixed by the assumed group. Lines from orbits of the second type (*non-spread-like*) can only be used in nonfixed spreads. Since H fixes 21 lines, any parallelism invariant under H must contain spreads which are fixed. The 336 non-fixed lines fall into 168 orbits of length 2, of which 128 are spread-like.

We distinguish the following types of *H*-orbits on spreads:

- Type F_f: such an orbit consist of a single spread fixed by *H*. It contains *f* fixed lines and (17 – *f*)/2 spread-like orbits of *H* on lines.
- Type *L*: this orbit consists of two spreads. There are no fixed lines in these spreads. The line-orbits of length two of *H* contained in *L* intersect each spread in exactly one element.

Lemma 1. Let *P* be a parallelism of PG(3, 4) invariant under *H*. The *H*-orbits on spreads contained in *P* are of type F_1 or F_5 or *L*. Let *#T* be the number of orbits of type *T* in *P*. Then ($\#F_1$, $\#F_5$, #*L*) = (1, 4, 8).

Proof: Recall that *H* fixes 5 points and 21 lines. The 5 fixed points comprise one line which is fixed pointwise. The remaining 20 fixed lines are incident with one of the fixed points each. Every spread *S* of *P* must cover the 5 fixed points. Suppose that the orbit of *S* is not of type *L* and hence *S* is fixed under *H*. The spread *S* either contains the line which is fixed pointwise or it contains exactly one fixed line through each of the fixed points. The remaining lines of *S* come in orbits of length two, forming 8 or 6 spread-like pairs, respectively. We have shown that any *H*-orbit on *P* has type either *L* or F_1 or F_5. The only way to cover the line which is fixed pointwise is by having one orbit of type F_1. The other 20 fixed lines can only be covered by using additional 4 orbits of type F_5. The remaining lines must be covered by the remaining 8 orbits of type *L*.

2.2. Fixed structure (partial solution of all fixed spreads)

Let *P* be a parallelism. The *fixed structure* of *P* is the set of spreads of *P* of type F_1 or F_5 with respect to the action of *H*. In this section, we will discuss the possibilities for fixed structures of a putative parallelism *P*.

Lemma 2. Let *#T* be the number of *H*-orbits of type *T* on the set of spreads of one isomorphism type which are line-disjoint. Then ($\#F_1$, $\#F_5$, #*L*) = = (128, 2304, 213,760) for the subregular spread, (96, 128, 4608) for the regular spread, and (800, 1920, 473,600) for the aregular spread.

The type of a fixed structure is listed as a triple (*s*, *r*, *a*) where *s* is the number of subregular (Hall) spreads, *r* is the number of regular spreads, and *a* is the number of aregular spreads contained in it. We divide cases according to the isomorphism type of spreads they contain.

Lemma 3. Under the action of the normalizer $N=N_G(H)$ of *H* in *G*, there are exactly 776,928 equivalence classes of fixed structures of putative parallelisms (Table 2). The last column lists the order of the stabilizers of the fixed structures in the group *N*.

Unfortunately, the parallelisms obtained from the cases in Lemma 3 are not necessarily nonisomorphic. Nevertheless, such a case distinction helps to approach the problem of constructing and classifying parallelisms invariant under the assumed group.

Since we are interested in uniform parallelisms, we only need to look at fixed structures consisting of five subregular spreads or of five regular spreads. We are interested in fixed structures of type (5, 0, 0) or (0, 5, 0). The *aregular spreads* case (0, 0, 5) will be left for future work.

Table 2. Type of fixed structures of the partial parallelisms of five fixed spreads

Type	#	Stabilizer orders[#]
(5, 0, 0)	**8268**	$\mathbf{4^{67}; 2^{8201}}$
(4, 1, 0)	7804	$64; 48^2; 16^{25}; 12^4; 8^{81}; 6^{28}; 4^{1180}; 2^{6483}$
(3, 2, 0)	1390	2^{1390}
(2, 3, 0)	241	$4^{65}; 2^{176}$
(1, 4, 0)	28	2^{28}
(0, 5, 0)	**7**	$\mathbf{8^4; 4^2; 2}$
(4, 0, 1)	74,763	$8^4; 4^{2637}; 2^{72122}$
(3, 1, 1)	34,337	$6^{26}; 4^{102}; 2^{34209}$
(2, 2, 1)	5052	$4^{281}; 2^{4771}$
(1, 3, 1)	387	2^{387}
(0, 4, 1)	27	$4^{16}; 2^{11}$
(3, 0, 2)	194,306	$4^{28}; 2^{194278}$
(2, 1, 2)	57,381	$8^2; 4^{879}; 2^{56500}$
(1, 2, 2)	4910	2^{4910}
(0, 3, 2)	182	$4^{60}; 2^{122}$
(2, 0, 3)	218,495	$4^{4110}; 2^{214385}$
(1, 1, 3)	37,553	$6^{32}; 4^{72}; 2^{37449}$
(0, 2, 3)	2379	$4^{309}; 2^{2070}$
(1, 0, 4)	97,037	2^{97037}
(0, 1, 4)	10278	$8^{26}; 6^{29}; 4^{352}; 2^{9871}$
(0, 0, 5)	22,103	$4^{1466}; 2^{20637}$
Total	776,928	$64; 48^2; 16^{25}; 12^4; 8^{117}; 6^{115}; 4^{11626}; 2^{765038}$

2.3. Extension of a fixed structure to parallelisms

Once the fixed structure has been selected, the next step is to extend it to a parallelism. To this end, we need to select eight spread orbits of type *L*, each of which has to be line-disjoint from the spreads in the fixed structure. The following computational result summarizes the situation for fixed spreads of type (5, 0, 0) and (0, 5, 0). This corresponds to Table 2, row 1 and row 6. The proof of this result is contained in the next section.

Lemma 4.

(a) The number of isomorphism types of uniform subregular parallelisms invariant under *H* and containing a fixed structure of type (5, 0, 0) is exactly 8253. The parallelisms have the following automorphism group orders: *8* times 20; 2 times 10; 244 times 4, and 7999 times 2. There are 119 self-dual parallelisms (with a full automorphism group of order 2).

(b) The number of isomorphism types of parallelisms invariant under *H* and containing a fixed structure of type (0, 5, 0) is exactly 27,310. None of them is regular. Table 3 summarizes the data. The column *# Spreads* lists the number of spreads which are line-disjoint from the spreads contained in the fixed structure. The order of the stabilizer of the fixed structure inside *N*=*N*(*H*) is given as $|N_{P_F}|$. The next columns present the number of parallelisms with a definite order of the automorphism group, and the whole number of nonisomorphic parallelisms. Some of the parallelisms obtained with the 5th fixed structure are isomorphic to parallelisms with the 6th fixed structure, and that is why the total sum in the column for automorphism group order 8 of Table 3 is smaller than the sum of the entries. There are 38 self-dual parallelisms.

Part (b) contributes further evidence to the validity of Bamberg's result that there are no regular parallelisms in PG(3, 4).

Table 3. Parallelisms of PG(3, 4) with 5 regular spreads invariant under H

Case	# Spreads	$\|N_{P_F}\|$	Order of the automorphism group				All
			2	4	8	16	
1	30,707	2	8798	–	–	–	8798
2	30,328	4	4004	76	–	–	4080
3	30,322	4	4032	44	–	–	4076
4	29,928	8	1778	26	3	–	1807
5	31,579	8	2742	366	410	16	3534
6	30,863	8	2384	498	366	8	3256
7	29,928	8	1778	26	3	–	1807
Total			25,516	1036	734	24	27,310

3. Construction method

Next, we will describe the method to construct and classify the parallelisms invariant under H described in Lemma 4. This entails creating the 776,928 fixed structures from Lemma 3 and lifting the 8268 possible fixed structures of type (5, 0, 0) from Table 2, row 1 as well as the 7 fixed structures of type (0, 5, 0) from Table 2, row 6. Because of the computational challenge, parallel computing will be used. To ensure better reliability of the results the three authors offer three different approaches. In what follows, we will outline the main ideas and point out differences.

In order to construct and classify the fixed structures, we need to compute the orbits of the normalizer N of the assumed group H on the subsets of spread orbits of type F_1 and F_2 of size 5. The first author uses an algorithm called poset classification, using his own computer algebra package Orbiter [4]. The second and third authors use a backtrack method based on canonical forms. All authors confirm that there are 776,928 fixed structures, as noted in Lemma 3.

The remaining task is to complete the parallelisms P from the fixed structure. Each fixed structure needs to be considered case-by-case. For a given fixed structure P_F, the orbits of type L which are line-disjoint from P_F need to be considered. The first author precomputes these orbits and reduces the remaining problem of finding all parallelisms P to a clique problem. For each fixed structure P_F, a certain graph $\Gamma(P_F)$ is defined. The vertices of the graph $\Gamma(P_F)$ are the orbits of type L which are line-disjoint from the spreads in P_F. The cliques of size 8 in $\Gamma(P_F)$ correspond to parallelisms P containing P_F and every parallelism containing P_F is of this form. In terms of computational complexity, the search for cliques is the dominant part in this step. Parallel computing is used to speed up the search. Regarding part (a) of Lemma 4, we break the 8268 cases into 8 batches. Each batch in turn is split into either 32 or 8 second order batches, and each second order batch is run on one machine and with multiple cores in parallel. The results are collected in files. In a final step, the files are read in one-by-one and the resulting parallelisms are tested for isomorphism, using a graph theoretic approach, based on the notion of canonical forms and canonical labelings. The clique search was done on two machines, one departmental server with 54 cores and one personal machine with 12 cores. In total, the clique searching step was done in 24 hours. The isomorphism testing took another 12 hours. It is perhaps interesting to note that each graph $\Gamma(P_F)$ has around 10,000 vertices and that one clique search can be done in under *5* minutes of computing time on a single processor. Regarding part (b) of Lemma 4, the graphs are much larger than before but there are fewer of them. The number of spreads, which are listed in the second column of Table 3 is the same as the number of vertices in the graphs $\Gamma(P_F)$. The clique search is a lot more challenging that for the graphs associated with part (a). Finding the cliques in one of these graphs takes between 8 and 12 hours per graph. For this problem, we first apply the stabilizer of the fixed structure in $N=N_G(H)$ to the set of parallelisms in one case. This is very fast and rules out some isomorphic copies in each class. After that, we use the canonical forms to decide the isomorphism problem for parallelisms of all seven cases. This leads to the bottom row of Table 3.

The second and third authors use their own software written for this purpose in C++. The third author does not use graphs, but in general her approach is similar to the first author's. She constructs in advance all the possible L-orbits too. The main difference is that the search is applied on all these possible L-orbits, not on the smaller set of L-orbits disjoint with the currently extended fixed structure. Instead of constructing the possible L-orbits in advance, the second author applies backtrack search on the lines of the projective space. For other related problems [6, 7] and for part (b) of Lemma 4, this is almost as fast as the other approach, but in the case of subregular parallelisms at least 13 lines have to be added to the current spread before we can establish that it is not subregular, and this makes the computation need more time.

The second and the third authors both define a lexicographic order on the obtained by backtrack search parallelisms and apply to each of them a normalizer-based minimality test which rejects solutions that can be mapped to lexicographically smaller ones by an element of the normalizer of the predefined group *H*. This way normalizer-nonisomorphic parallelisms are obtained. The approach is described in more details in [6].

The third author's software needs about an hour (on a Intel(R) Core(TM) i7-6500U CPU @2.50GHz PC) to extend one fixed subregular structure to normalizer-nonisomorphic parallelisms, and the second author's several hours. That is why the particular computational problem is solved by parallelization of the sequential algorithms using MPI. The different fixed structures are consequently assigned to different processes. If p is the number of processes and m the number of fixed structures to extend, each process tries to extend only m/p of them. The process with rank $\boldsymbol{i}$ will extend fixed structures with indices $i, p+i, 2p+i, \ldots$, etc. Each process writes the constructed parallelisms in its own file. At the end p files are obtained and have to be combined. The different fixed structures need slightly different times to be processed and therefore some small imbalance of load appears. This way the access to Avitohol (**http://nchdc.acad.bg/en/resources/iict/avitohol/**) made the problem easy to solve.

The second and third authors construct 8253 normalizer-nonisomorphic subregular parallelisms and calculate for each of them the invariants described in [23]. All the invariants are different. This shows that the parallelisms are nonisomorphic. The calculation and comparison of the invariants takes several minutes. In a similar way 27,422 normalizer-nonisomorphic parallelisms with regular fixed parts are obtained. The invariants [23] partition them to 27,310 classes. A further full isomorphism test shows that parallelisms from one and the same invariant class are isomorphic.

4. Conclusion

All subregular parallelisms with automorphisms of order 2 agree with earlier results of the authors on parallelisms of PG(3, 4). They are available online at **http://www.moi.math.bas.bg/moiuser/~stela**. We remark that the parallelisms from Lemma 4 that possess a full automorphism group order bigger than two are known from earlier work (see Tables 1 and 3).

The classification of all parallelisms of PG(3, 4) which possess automorphisms of order 2 is a challenging open problem. The case of aregular parallelisms (with fixed part (0, 0, 5) – Table 2, last row) is more complex than the subregular case, but seems to be within reach.

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