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On Some Knowledge Measures of Intuitionistic Fuzzy Sets of Type Two with Application to MCDM

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***Abstract*:** *To overcome the certain limitations of Intuitionistic Fuzzy Sets* (*IFSs*)*, the notion of Intuitionistic Fuzzy Sets of Second Type* (*IFSST*) *was introduced. IFSST is a modified version of IFS for handling some problems in a reasonable manner. Type two Intuitionistic Fuzzy entropy* (*IFSST-entropy*) *measures the amount of ambiguity/uncertainty present in an IFSST. In the present paper, we introduce the concept of dual measure of IFSST-entropy, i.e., IFSST-knowledge measure. We develop some IFSST-knowledge measures and prove some of their properties. We also show the superiority of the proposed IFSST-knowledge measures through comparative study. Further, we demonstrate the application of the proposed knowledge measures in Multi-Criteria Decision-Making* (*MCDM*)*.*

***Keywords*:** *IFSs, IFSST, IFSST-knowledge measure, MCDM.*

1. Introduction

For dealing with vague/ambiguous situations, Zadeh's [45] fuzzy set theory serves as a reasonable tool. In fuzzy set theory, we take into consideration only the membership degree of an element to a set but this provides incomplete information because, in real life, one may assume that to a certain degree an element belongs to the set but probably he may not be so certain about it [6]. Thus, various extensions of fuzzy sets have been introduced by various researchers; one of these extensions is Atanassov's IFSs [1] which considers both membership(μ) as well as non-membership(ν) degree of an element concerning a set along with the condition $\mu+\nu\le 1$. In the available literature, many remarkable results on IFSs both in theory, as well as in application have been given [7, 12, 13, 19, 21, 31, 34, 35, 43, 44]. But in some situations $\mu+\nu\le 1$ doesn't hold. For example, $0.6+0.8>1$ but $0.6^2+0.8^2=1$ or if an expert or decision-maker assigned 0.5 and 0.7 as the degree of membership and degree of non-membership respectively to an element, then IFSs can't handle such kind of situation in a reasonable manner. Atanassov [2] introduced the concept of IFSST and Parvathi and Palaniappan [22] defined new operations on IFSST. In an IFSST $\mu+\nu$ may be ≥ 1, but $\mu^2 + \nu^2 \leq 1$. Some pioneer studies regarding IFSST are [3-5]. Yager and Abbasov [40] and Yager [41] termed the IFSST as Pythagorean fuzzy set (PFS). Intuitionistic Fuzzy Sets of Second Type (IFSsST) are reasonable to handle vague data in some situations. In the literature developed after Yager and Abbasov [40] and Yager [41], most of the papers are using the term "PFS" instead of "IFSST". So, in the following, we present some recent studies regarding IFSST and use the term "PFS" and "IFSST" equivalently. Following the work of Yager and Abbasov [40] and Yager [41], the concept of Pythagorean Fuzzy Number (PFN) and mathematical form of PFSs was introduced by Zhang and Xu [49]. Furthermore, they also presented the score function and basic operational laws for PFNs. For PFNs, the division and subtraction were developed by Peng and Yang [24] and Gou, Xu and Ren [10] developed the derivability, continuity, and differentiability. In the meantime, a chain of various types of Pythagorean fuzzy aggregation operations was introduced by various researchers [9, 18, 25, 41]. Zhang and Xu [49] introduced Pythagorean fuzzy TOPSIS in order to handle Multi-Criteria Decision-Making (MCDM) problems with Pythagorean fuzzy data. PFSs were first time used by Reformat and Yager [29] in managing the collaborative-based recommender system. Peng and Yang [24] introduced Pythagorean Fuzzy Superiority and Inferiority Ranking method (PF-SIR) for solving a MAGDM problem in the Pythagorean fuzzy situation. Zhang [50] proposed similarity measures for PFSs and applied them in solving a MADM problem in Pythagorean fuzzy environment. Peng and Dai [27] introduced a new score function involving the hesitation degree of Pythagorean fuzzy number and also proposed a new axiomatic definition of Pythagorean fuzzy distance measure. Besides, they also proposed two Pythagorean fuzzy Stochastic Multi-Criteria Decision-Making (SMCDM) methods based on regret theory and prospect theory and illustrated these methods by considering the evaluation of a book. G. Wei and Y. Wei [37] proposed some Pythagorean similarity measures based on cosine functions and demonstrated their application in pattern recognition and medical diagnosis. Garg [8] introduced correlation coefficients and weighted correlation coefficients for PFSs along with their applications in medical diagnosis and pattern recognition.

To measure uncertainty in a random process, entropy plays a vital role. Larger entropy corresponds to less information content regarding the process/experiment. The entropy is a theoretical measurement of inherent information and can be contextualized mainly in three senses: conflicting, non-specific and fuzzy. Xue et al. [39] introduced the concept of entropy in the Pythagorean fuzzy situations. Peng, Yuan and Yang [28], Peng and Yang [25] introduced the axiomatic definition of Pythagorean fuzzy information measures (similarity measure, distance measure, inclusion measure, entropy) together with their relationship. Some new entropy measures for PFSs which are based on min-max operation, Pythagorean index, distance, and probability type were introduced by Yang and Hussain [42]. In the Pythagorean fuzzy entropy measures introduced by Xu et al. [39], Peng, Yuan and Yang [28] and Yang and Hussain [42], we have, $E(A)$ is maximum, i.e., $E(A)=1$ if $\mu_A=\nu_A$ for any Pythagorean fuzzy number $A=(\mu_A, \nu_A)$, i.e.,

$E(0.5,0.5)=E(0.3,0.3)=E(0,0)=1$. This means that existing entropy measures for PFSs introduced by researchers [28, 39, 42] do not differentiate between different PFSs. To overcome this limitation, Thao, Ali and Smarandache [35] recently introduced a new entropy measure for PFSs together with the COmplex PRoportional ASsessment (COPRAS) method for solving MCDM problem in which weights are calculated with the help of new proposed entropy. Pythagorean fuzzy entropies due to Thao and Smarandache [36] are not able to differentiate between different PFSs in many situations. In this work, we introduce some knowledge measures for IFSST. Therefore, in the following, we discuss some background related to knowledge measures in various ambiguous situations.

In a fuzzy system, knowledge measure is usually perceived as a dual of entropy measure which implies that a greater amount of knowledge may always accompany less entropy. But in context of Atanassov's Intuitionistic Fuzzy Sets (A-IFSs) in which the corresponding membership and non-membership values are equal, then by some familiar models [16, 23, 31, 32, 46, 48], these A-IFSs will have maximum entropy of one, thereby leading to the lesser amount of knowledge (i.e., zero) associated with them. This is clearly counter-intuitive as there are various distinct possibilities for the membership and non- membership values to be equal. This means that the information content of these A-IFSs may be different and as such, they may be completely different as far as the amount of knowledge is concerned. An analogous situation also occurs in the case of Atanassov Interval-Valued Intuitionistic Fuzzy Set (A-IVIFSs) [17, 38, 47]. Naturally, among these A-IVIFSs in which corresponding membership and non-membership values are equal, the set with greater hesitancy degree seems to be more uncertain from a knowledge point of view because of the high content of unknown information. So, knowledge content associated with this type of A-IFSs should be less. As the existing measures of entropy could not differentiate these A-IFSs, therefore, Szmidt, Kacprzyk and Bujnowski [33] suggested the concept of knowledge measure for IFSs and introduced a knowledge measure involving both hesitation margin and entropy. Guo [11] introduced the axiomatic definition of another information measure known as knowledge measure in an intuitionistic environment. Practically, knowledge measures and entropy measures are the same but structurally they are different. Guo [11] also introduced in detail the numerical relationship between an intuitionistic fuzzy entropy measure and intuitionistic fuzzy knowledge measure. X. T. Nguyen and V. D. Ngyuen [21] introduced a new knowledge measure for IFSs calculating both intuitionism and fuzziness related to the dearth of information. Besides, he also constructed similarity measure and entropy measure based on the new knowledge measure and also applied the new knowledge measure in MADM. Lalotra and Singh [15] introduced a novel knowledge measure in IF-environment with some applications. Recently, to fill the certain research gap, Singh, Lalotra and Sharma [30] conceptualized the notion of knowledge measure as a dual of entropy measure in the fuzzy settings. Now, in the following, we highlight the motivation to consider the present study.

In an MCDM problem, in order to determine the feasible alternative, criteria weights play an important role [14]. For this, entropy/knowledge measures are required. But some of the existing entropy/knowledge measures either give the same criteria weights or the weights with a narrow range, which affects the selection of the best alternative. Also, an information measure is considered to be effective if it can differentiate between different fuzzy/non-standard fuzzy sets. However, various existing fuzzy/non-standard fuzzy information measures give the same amount of ambiguity for different fuzzy/non-standard fuzzy sets in many cases and therefore consider them to be identical, which is not reasonable. However, in Pythagorean Fuzzy Settings (PFSs), Thao and Smarandache [36] provided certain entropy measures to differentiate some PFSs having the same membership and non-membership value. But we have observed that these PF-entropy (IFSST-entropy) measures were not able to handle similarly other cases. This insufficiency of the existing PF-entropy (IFSST-entropy) measures is one aspect that motivated us to develop some new knowledge measures, which are not only consistent with the existing PF-entropy (IFSST-entropy) measures but also outperforms in certain situations. Secondly, the extension of the notion of knowledge measure in Pythagorean fuzzy (IFSST) circumstances also motivated us to consider some extended investigations.

The main contribution of this study is:

- We propose three knowledge measures in the framework of IFSST.
- We investigate the superiority of the proposed IFSST-knowledge measures through a comparative study from the aspect of linguistic hedges, and weight computation.
- We demonstrate the application of the proposed IFSST-knowledge measures in MCDM and investigate comparative performance with the existing PF-entropies (IFSST-entropies) in the Pythagorean (IFSST) logic-based TOPSIS method.

The remainder of this paper is organized as follows.

Section 2 presents some basic definitions. In Section 3, we propose IFSST-knowledge measures and prove their valuation property. Section 4 presents the advantages as well as implications of the proposed IFSST-knowledge measures. The application of the proposed IFSST-knowledge measures in MCDM utilizing TOPSIS method is shown in Section 5. Finally, Section 6 presents the conclusion and future work of the present study.

2. Preliminaries

In this section, we provide basic definitions concerning the present study.

Definition 1 [45]. Let $X=\{x_1, x_2, \ldots, x_n\}$ be a universal set, then a fuzzy subset $\tilde{A}$ of a universal set X is given by $\tilde{A}=\{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}: X \to [0, 1]$ represents membership function and $\mu_{\tilde{A}}(x)$ gives the membership degree of $x \in X$ in $\tilde{A}$.

Definition 2 [1]. Let $X=\{x_1, x_2, \ldots, x_n\}$ be a universal set, then an intuitionistic fuzzy set $\tilde{A}$ on X is defined as $\tilde{A}=\{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x): X \to [0, 1]$ represents the membership degree and $\nu_{\tilde{A}}(x): X \to [0, 1]$ represents the non-membership degree of the element $x \in X$ in $\tilde{A}$ respectively, and they satisfy $0 \le \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \le 1,\ x \in X$. Also, for each IFS $\tilde{A}$ in X, we have $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$, where $\pi_{\tilde{A}}(x)$ is called hesitancy degree of *x* to $\tilde{A}$.

Definition 3 [2]. Let $X = \{x_1, x_2, ..., x_n\}$ be a universal set, then an IFSST $\tilde{A}$ on X is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$, where $\mu_{\tilde{A}}(x): X \to [0, 1]$ represents the membership degree and $\nu_{\tilde{A}}(x): X \to [0, 1]$ represents the non-membership degree of the element $x \in X$ in $\tilde{A}$ respectively, and they satisfy. $\mu_{\tilde{A}}^2(x) + \nu_{\tilde{A}}^2(x) \le 1,\ x \in X$ Also, for each IFSST $\tilde{A}$ in *X*, we have $\pi_{\tilde{A}}(x) = \sqrt{1 - \mu_{\tilde{A}}^2(x) - \nu_{\tilde{A}}^2(x)}$, where $\pi_{\tilde{A}}(x)$ is called hesitancy degree of *x* to *A*.

Definition 4 [42]. Let $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\}$ be a PFS (IFSST) in *X*. Then the modifier for the PFS (IFSST) $\tilde{A}$ is defined as

$$A^n = \left\{\left\langle x, (\mu_{\tilde{A}}(x))^n, \sqrt{1 - (1 - \nu_{\tilde{A}}(x))^n} \right\rangle \mid x \in X\right\},$$

where *n* is any positive real number.

Definition 5 [42]. For a PFS (IFSST) $\tilde{A}$, the $\mathrm{CON}(\tilde{A})$ is defined as

$$\mathrm{CON}(\tilde{A}) = \left\{\left\langle x, \mu_{\mathrm{CON}(\tilde{A})}(x), \nu_{\mathrm{CON}(\tilde{A})}(x) \right\rangle \mid x \in X\right\},$$

where $\mu_{\mathrm{CON}(\tilde{A})}(x) = (\mu_{\tilde{A}}(x))^2$ and $\nu_{\mathrm{CON}(\tilde{A})}(x) = \sqrt{1 - (1 - \nu_{\tilde{A}}(x))^2}$.

Definition 6 [42]. For a PFS (IFSST) $\tilde{A}$, the $\mathrm{DIL}(\tilde{A})$ is defined as

$$\mathrm{DIL}(\tilde{A}) = \left\{\left\langle x, \mu_{\mathrm{DIL}(\tilde{A})}(x), \nu_{\mathrm{DIL}(\tilde{A})}(x) \right\rangle \mid x \in X\right\},$$

where $\mu_{\mathrm{DIL}(\tilde{A})}(x) = (\mu_{\tilde{A}}(x))^{\frac{1}{2}}$ and $\nu_{\mathrm{DIL}(\tilde{A})}(x) = \sqrt{1 - (1 - \nu_{\tilde{A}}(x))^{\frac{1}{2}}}$.

In the next section, we propose the axiomatic definition of knowledge measure of an IFSST and subsequently, introduce three IFSST-knowledge measures.

3. Knowledge measures for intuitionistic fuzzy sets of second type

We propose the following axiomatic definition of knowledge measure of IFSST.

Let IFSST(*X*) be the set of all IFSsST in *X* and $K: \mathrm{IFSST}(X) \to [0, 1]$ be a real function. Then a measure of knowledge *K* on IFSST(*X*) should satisfy the following axioms (KN1)-(KN4).

(KN1) (Maximality). $K(\tilde{A}) = 1$ if and only if $\tilde{A}$ is a crisp set.

(KN2) (Minimality). $K(\tilde{A})=0$ if and only if $\mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ \ x_i\in X$.

(KN3) (Resolution). $K(\tilde{A})\geq K(\tilde{B})$ if $\tilde{A}$ is crisper than $\tilde{B}$, i.e., $x_i\in X$,

$\mu_{\tilde{A}}(x_i)\leq\mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i)\geq\nu_{\tilde{B}}(x_i)$ for $\mu_{\tilde{B}}(x_i)\leq\nu_{\tilde{B}}(x_i)$ or

$\mu_{\tilde{A}}(x_i)\geq\mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i)\leq\nu_{\tilde{B}}(x_i)$ for $\mu_{\tilde{B}}(x_i)\geq\nu_{\tilde{B}}(x_i)$.

(KN4) (Symmetric). $K(\tilde{A})=K(\tilde{A}^{c})$, where $\tilde{A}^{c}$ is the complement of $\tilde{A}$.

In view of the axiomatic definition of the IFSST-knowledge measure, we propose the following IFSST-knowledge measures:

1. $$K_1(\tilde{A})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2},$$

2. $$K_2(\tilde{A})=\frac{1}{n}\sum_{i=1}^{n}\left|\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right|,$$

and

3. $$K_3(\tilde{A})=1-\frac{1}{n}\sum_{i=1}^{n}\frac{2\mu_{\tilde{A}}^2(x_i)\nu_{\tilde{A}}^2(x_i)}{\mu_{\tilde{A}}^4(x_i)+\nu_{\tilde{A}}^4(x_i)}.$$

For establishing the validity of these IFSST-knowledge measures, we prove the Theorems 1-3.

Theorem 1. $K_1(\tilde{A})$ is a valid IFSST-knowledge measure.

Proof **(KN1):** First, suppose that $\tilde{A}$ is a crisp set. We have to show that $K(\tilde{A})=1$.

Case 1. When $\mu_{\tilde{A}}(x_i)=0,\ \nu_{\tilde{A}}(x_i)=1$ then

$$K_1(\tilde{A})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}=1.$$

Case 2. When $\mu_{\tilde{A}}(x_i)=1,\ \nu_{\tilde{A}}(x_i)=0$ then

$$K_1(\tilde{A})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}=1.$$

$\therefore\ K_1(\tilde{A})=1$ when $\tilde{A}$ is a crisp set.

Conversely, suppose $K_1(\tilde{A})=1$.

We show that $\tilde{A}$ is a crisp set.

Now, $K_1(\tilde{A})=1$

$$\Rightarrow\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}=1\ \Rightarrow\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2=n.$$

This is possible only if $\tilde{A}$ is a crisp set.

Therefore, $K_1(\tilde{A})=1$ if and only if $\tilde{A}$ is a crisp set.

(KN2) First, suppose that $\mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ \ x_i\in X$. We have to show that $K_1(\tilde{A})=0.$

Now, $K_1(\tilde{A})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\mu_{\tilde{A}}^2(x_i)\right)^2}=0.$

$\therefore K_1(\tilde{A})=0$ when $\mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ x_i\in X$.

Conversely, suppose that $K_1(\tilde{A})=0$. We have to show that $\mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ \ x_i\in X$.

Now, $K_1(\tilde{A})=0$

$$\Rightarrow \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}=0,$$

$$\Rightarrow \sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2=0,$$

$$\Rightarrow \mu_{\tilde{A}}^2(x_i)=\nu_{\tilde{A}}^2(x_i)\ \forall x_i,$$

$$\Rightarrow \mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ x_i\in X.$$

Therefore, $K_1(\tilde{A})=0$ if and only if $\mu_{\tilde{A}}(x_i)=\nu_{\tilde{A}}(x_i),\ \ x_i\in X$.

(KN3) Since $\mu_{\tilde{A}}(x_i)\le\mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i)\ge\nu_{\tilde{B}}(x_i)$ for $\mu_{\tilde{B}}(x_i)\le\nu_{\tilde{B}}(x_i)$
$\Rightarrow \mu_{\tilde{A}}(x_i)\le\mu_{\tilde{B}}(x_i)\le\nu_{\tilde{B}}(x_i)\le\nu_{\tilde{A}}(x_i),$ then we have

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}\ge\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{B}}^2(x_i)-\nu_{\tilde{B}}^2(x_i)\right)^2}.$$

Again from axiom (KN3), we have
$\mu_{\tilde{A}}(x_i)\ge\mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i)\le\nu_{\tilde{B}}(x_i)$ for $\mu_{\tilde{B}}(x_i)\ge\nu_{\tilde{B}}(x_i),$ then we have

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}\ge\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{B}}^2(x_i)-\nu_{\tilde{B}}^2(x_i)\right)^2}.$$

In both cases, we have

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}\ge\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{B}}^2(x_i)-\nu_{\tilde{B}}^2(x_i)\right)^2}.$$

Therefore, $K_1(\tilde{A})\ge K_1(\tilde{B})$.

$\therefore\ K_1(\tilde{A})\ge K_1(\tilde{B}),$ if $\tilde{A}$ is crisper than $\tilde{B}$.

(KN4) Since $x_i\in X$,

$$K_1(\tilde{A})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i)-\nu_{\tilde{A}}^2(x_i)\right)^2}=$$

$$=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(-\left(\nu_{\tilde{A}}^2(x_i)-\mu_{\tilde{A}}^2(x_i)\right)\right)^2}=$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\nu_{\tilde{A}}^2(x_i) - \mu_{\tilde{A}}^2(x_i)\right)^2} =$$

$$= K_1(\tilde{A}^c).$$

$$\therefore \quad K_1(\tilde{A}) = K_1(\tilde{A}^c).$$

Hence, $K_1(\tilde{A})$ is a valid IFSST-knowledge measure.

Theorem 2. $K_2(\tilde{A})$ is a valid IFSST-knowledge measure.

P r o o f: Similar to Theorem 1.

Theorem 3. $K_3(\tilde{A})$ is a valid IFSST-knowledge measure.

P r o o f: Similar to Theorem 1.

Theorem 4. Let $K_1(\tilde{A})$ and $K_1(\tilde{B})$ be IFSST-knowledge measures of IFSsST $\tilde{A}$ and $\tilde{B}$, respectively, such that either $\tilde{A} \subset \tilde{B}$ or $\tilde{B} \subset \tilde{A}$ then

$$K_1\left(\tilde{A} \cup \tilde{B}\right) + K_1\left(\tilde{A} \cap \tilde{B}\right) = K_1\left(\tilde{A}\right) + K_1\left(\tilde{B}\right).$$

P r o o f: We prove the result for two cases.

Case 1. When $\mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i) \ge \nu_{\tilde{B}}(x_i)$ then

$$K_1(\tilde{A} \cup \tilde{B}) + K_1(\tilde{A} \cap \tilde{B}) =$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}\cup\tilde{B}}^2(x_i) - \nu_{\tilde{A}\cup\tilde{B}}^2(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}\cap\tilde{B}}^2(x_i) - \nu_{\tilde{A}\cap\tilde{B}}^2(x_i)\right)^2} =$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{B}}^2(x_i) - \nu_{\tilde{B}}^2(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i) - \nu_{\tilde{A}}^2(x_i)\right)^2} =$$

$$= K_1(\tilde{B}) + K_1(\tilde{A}).$$

Case 2. When $\mu_{\tilde{A}}(x_i) \ge \mu_{\tilde{B}}(x_i)$ and $\nu_{\tilde{A}}(x_i) \le \nu_{\tilde{B}}(x_i)$ then

$$K_1(\tilde{A} \cup \tilde{B}) + K_1(\tilde{A} \cap \tilde{B}) =$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}\cup\tilde{B}}^2(x_i) - \nu_{\tilde{A}\cup\tilde{B}}^2(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}\cap\tilde{B}}^2(x_i) - \nu_{\tilde{A}\cap\tilde{B}}^2(x_i)\right)^2} =$$

$$= \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{A}}^2(x_i) - \nu_{\tilde{A}}^2(x_i)\right)^2} + \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_{\tilde{B}}^2(x_i) - \nu_{\tilde{B}}^2(x_i)\right)^2} =$$

$$= K_1(\tilde{A}) + K_1(\tilde{B}).$$

Therefore, $K_1\left(\tilde{A} \cup \tilde{B}\right) + K_1\left(\tilde{A} \cap \tilde{B}\right) = K_1\left(\tilde{A}\right) + K_1\left(\tilde{B}\right).$

Theorem 5. Let $K_2(\tilde{A})$ and $K_2(\tilde{B})$ be IFSST-knowledge measures of IFSsST $\tilde{A}$ and $\tilde{B}$, respectively, such that either $\tilde{A} \subset \tilde{B}$ or $\tilde{B} \subset \tilde{A}$, then

$$K_2\left(\tilde{A} \cup \tilde{B}\right) + K_2\left(\tilde{A} \cap \tilde{B}\right) = K_2\left(\tilde{A}\right) + K_2\left(\tilde{B}\right).$$

P r o o f: Similar to Theorem 4.

Theorem 6. Let $K_3(\tilde{A})$ and $K_3(\tilde{B})$ be IFSST-knowledge measures of IFSsST $\tilde{A}$ and $\tilde{B}$ respectively, such that either $\tilde{A} \subset \tilde{B}$ or $\tilde{B} \subset \tilde{A}$, then

$$K_3\left(\tilde{A} \cup \tilde{B}\right) + K_3\left(\tilde{A} \cap \tilde{B}\right) = K_3\left(\tilde{A}\right) + K_3\left(\tilde{B}\right).$$

Proof: Similar to Theorem 4.

Now, to justify the proposal of new IFSST-knowledge measures in various scenarios, we consider some comparative studies in the next section.

4. Comparative study

4.1. Comparative analysis based on linguistic hedges

In this section, we present the illustrative examples with linguistic hedges to investigate the performance and behavior of some of the existing PF-entropy (IFSST-entropy) measures and the proposed IFSST-knowledge measures.

Example 1. Consider an IFSST $\tilde{A}$ in the universal set $X = \{1, 2, 3, 4, 5\}$ given as

$$\tilde{A} = \{\langle 1, 0.0, 0.8\rangle, \langle 2, 0.1, 0.7\rangle, \langle 3, 0.6, 0.5\rangle, \langle 4, 0.9, 0.0\rangle, \langle 5, 1.0, 0.0\rangle\}.$$

From Definitions 5 and 6, the dilation and concentration of $\tilde{A}$ are given as Dilation: $\mathrm{DIL}(\tilde{A}) = \tilde{A}^{1/2}$ and Concentration: $\mathrm{CON}(\tilde{A}) = \tilde{A}^2$. Now, we utilize IFSST $\tilde{A}$ to explain the strength of the linguistic variable ($\tilde{A}$ in $X = \{1, 2, 3, 4, 5\}$) using the linguistic variable characterization. By using the here-mentioned operator, we may consider the following $\tilde{A}^{1/2}$ is considered as "More or Less Large", $\tilde{A}$ is considered as 'Large', $\tilde{A}^{3/2}$ is considered as 'Quite Large' A^2 is considered as "Very Large", $A^{5/2}$ is considered as "Quite Very Large" and A^3 is considered as "Very Very Large".

By utilizing the above-mentioned operators, we may generate the following IFSsST

$$\tilde{A}^{1/2} = \left\{ \begin{matrix} \langle 1, 0.0, 0.6325\rangle, \langle 2, 0.3162, 0.5347\rangle, \\ \langle 3, 0.7746, 0.3660\rangle, \langle 4, 0.9487, 0.0\rangle, \langle 5, 1.0, 0.0\rangle \end{matrix} \right\};$$

$$\tilde{A}^{3/2} = \left\{ \begin{matrix} \langle 1, 0.0, 0.8854\rangle, \langle 2, 0.0316, 0.7974\rangle, \\ \langle 3, 0.4648, 0.5920\rangle, \langle 4, 0.8538, 0.0\rangle, \langle 5, 1.0, 0.0\rangle \end{matrix} \right\};$$

$$\tilde{A}^{2} = \left\{ \begin{matrix} \langle 1, 0.0, 0.9330\rangle, \langle 2, 0.0100, 0.8602\rangle, \\ \langle 3, 0.3600, 0.6614\rangle, \langle 4, 0.81, 0.0\rangle, \langle 5, 1.0, 0.0\rangle \end{matrix} \right\};$$

$$\tilde{A}^{5/2} = \left\{ \begin{matrix} \langle 1, 0.0, 0.9603\rangle, \langle 2, 0.0032, 0.9024\rangle, \\ \langle 3, 0.2789, 0.7161\rangle, \langle 4, 0.7684, 0.0\rangle, \langle 5, 1.0, 0.0\rangle \end{matrix} \right\};$$

and

$$\tilde{A}^{3} = \left\{ \begin{matrix} \langle 1, 0.0, 0.9764\rangle, \langle 2, 0.001, 0.9313\rangle, \\ \langle 3, 0.216, 0.7603\rangle, \langle 4, 0.729, 0.0\rangle, \langle 5, 1.0, 0.0\rangle \end{matrix} \right\}.$$

Now, from an intuitive viewpoint, a good entropy measure of IFSST should satisfy the following requirement

(1) $$E(\tilde{A}^{1/2}) > E(\tilde{A}) > E(\tilde{A}^{3/2}) > E(\tilde{A}^{2}) > E(\tilde{A}^{5/2}) > E(\tilde{A}^{3}).$$

In terms of knowledge, a good knowledge measure of IFSST should satisfy the following requirement:

(2) $$K(\tilde{A}^{1/2}) < K(\tilde{A}) < K(\tilde{A}^{3/2}) < K(\tilde{A}^{2}) < K(\tilde{A}^{5/2}) < K(\tilde{A}^{3}).$$

In Table 1, we compare the results of the following PF-entropy (IFSST-entropy) measures and our proposed IFSST-knowledge measures for a given IFSST $\tilde{A}$:

$$e_{\mathrm{PI}}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\mu_{\tilde{A}}^{2}(x_i)-\nu_{\tilde{A}}^{2}(x_i)\right)$$ (Yang and Hussain [42]);

$$e_{\mathrm{min/max}}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\min\left(\mu_{\tilde{A}}^{2}(x_i),\ \nu_{\tilde{A}}^{2}(x_i),\ \pi_{\tilde{A}}^{2}(x_i)\right)}{\max\left(\mu_{\tilde{A}}^{2}(x_i),\ \nu_{\tilde{A}}^{2}(x_i),\ \pi_{\tilde{A}}^{2}(x_i)\right)}$$ (Yang and Hussain [42]);

$$E_{\mathrm{Xue}}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n}\left[1-\left(\mu_{\tilde{A}}^{2}(x_i)+\nu_{\tilde{A}}^{2}(x_i)\right)\left|\mu_{\tilde{A}}^{2}(x_i)-\nu_{\tilde{A}}^{2}(x_i)\right|\right]$$ (Xue et al. [39]);

$$E_{\mathrm{T}}(\tilde{A}) = \frac{1}{n}\sum_{i=1}^{n}E_{\mathrm{T}}^{i}(\tilde{A}),$$ where $E_{\mathrm{T}}^{i}(\tilde{A}) = 1-\left|\mu_{\tilde{A}}^{2}(x_i)-\frac{1}{3}\right|-\left|\nu_{\tilde{A}}^{2}(x_i)-\frac{1}{3}\right|$, for all $i = 1, 2, ..., n$ (Thao and Smarandache [36]).

Table 1. Fuzziness values corresponding to different entropy and knowledge measures of PFSs (IFSST)

PFS	$e_{\mathrm{PI}}(\tilde{A})$	$e_{\mathrm{min/max}}(\tilde{A})$	$E_{\mathrm{Xue}}(\tilde{A})$	$E_{\mathrm{T}}(\tilde{A})$	$K_1(\tilde{A})$	$K_2(\tilde{A})$	$K_3(\tilde{A})$
$\tilde{A}^{1/2}$	0.3160	0.0772	0.5232	0.3906	0.6666	0.5904	0.7903
$\tilde{A}$	0.2880	0.1322	0.5254	0.3920	0.6794	0.6080	0.8044
$\tilde{A}^{3/2}$	0.2567	0.1000	0.4747	0.3435	0.7165	0.6565	0.8207
$\tilde{A}^{2}$	0.2333	0.0593	0.4180	0.2852	0.7524	0.7148	0.8910
$\tilde{A}^{5/2}$	0.2165	0.0303	0.3762	0.2476	0.7812	0.7524	0.9407
$\tilde{A}^{3}$	0.2046	0.0161	0.3449	0.2233	0.8032	0.7767	0.9679

By comparing the results obtained in Table 1, we see that only the PF-entropy (IFSST-entropy) measure e_{PI} satisfies the requirement (1) and the proposed IFSST-knowledge measures K_1, K_2 and K_3 satisfy (2) and thus perform better than some existing PF-entropy (IFSST-entropy) measures.

Now, we compare only e_{PI}, K_1, K_2, and K_3. For this, we consider another IFSST $\tilde{B}$ of $X = \{1, 2, 3, 4, 5\}$ defined as

$$\tilde{B} = \{\langle 1, 0.1, 0.8\rangle, \langle 2, 0.4, 0.7\rangle, \langle 3, 0.6, 0.5\rangle, \langle 4, 0.9, 0.0\rangle, \langle 5, 1.0, 0.0\rangle\}.$$

After doing computation as done for set $\tilde{A}$ in Table 1, we see that for set $\tilde{B}$, the PF-entropy (IFSST-entropy) measure e_{PI} does not satisfy the requirement (1) and only our proposed IFSST-knowledge measures K_1, K_2 and K_3 satisfy (2) and thus perform better than PF-entropy (IFSST-entropy) e_{PI}. Therefore, the performance of our proposed IFSST-knowledge measures is encouraging from the point of view of structured linguistic variables in IFSST logic.

4.2. Comparative analysis based on weight computation

In a decision-making problem, the criteria weights play a vital role in selecting the best alternative. Usually, the decision-maker provides the criteria weights, but in certain situations, the criteria weights are computed by using some fuzzy information measures (knowledge/entropy). Here, we demonstrate the superiority of the proposed IFSST-knowledge measures with the help of the following illustrative examples concerning the computation of criteria weights.

In Examples 2-5, we consider different IFSST fuzzy decision matrices concerning the ratings of three criteria C_1, C_2 and C_3 for three available alternatives $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$, and compute the objective weights of the criteria:

$$\mathrm{DM}_2 = \begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \qquad C_2 \qquad & \qquad C_3 \quad \end{array} \\ \begin{bmatrix} (0.7000, 0.1000) & (0.5830, 0.4000) & (0.6030, 0.5140) \\ (0.5000, 0.4999) & (0.6782, 0.1999) & (0.3959, 0.6900) \\ (0.6500, 0.2780) & (0.4999, 0.5000) & (0.8090, 0.2010) \end{bmatrix} \end{array}.$$

Example 2. Consider the IFSST decision matrix DM_1.

Then the criteria weight vectors computed by using e_{PI}, K_1, K_2, and K_3 are (0.3333, 0.3333, 0.3333), (0.3983, 0.2581, 0.3436), (0.3983, 0.2581, 0.3436), (0. 0.4813, 0.1894, 0.3293), respectively. We observe that e_{PI} gives the same weights to all criteria C_1, C_2 and C_3, whereas our proposed IFSST-knowledge measures give different weights.

Example 3. Consider the IFSST decision matrix DM_2.

$$\mathrm{DM}_2 = \begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \qquad C_2 \qquad & \qquad C_3 \quad \end{array} \\ \begin{bmatrix} (0.7000, 0.1000) & (0.5830, 0.4000) & (0.6030, 0.5140) \\ (0.5000, 0.4999) & (0.6782, 0.1999) & (0.3959, 0.6900) \\ (0.6500, 0.2780) & (0.4999, 0.5000) & (0.8090, 0.2010) \end{bmatrix} \end{array}.$$

Then the criteria weight vectors computed by using $e_{\min/\max}$, K_1, K_2, and K_3 are (0.3027, 0.3027, 0.3947), (0.3357, 0.2441, 0.4202), (0.3357, 0.2441, 0.4202), (0. 0.4019, 0.2646, 0.3335), respectively. We observe that $e_{\min/\max}$ gives the same weights to both criteria C_1 and C_2, whereas our proposed IFSST-knowledge measures give different weights.

Example 4. Consider the IFSST fuzzy decision matrix DM_3.

$$\mathrm{DM}_3 = \begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \quad C_2 \quad & \quad C_3 \quad \end{array} \\ \begin{bmatrix} (0.5000, 0.4999) & (0.6533, 0.5999) & (0.7001, 0.4999) \\ (0.6500, 0.4595) & (0.5352, 0.6898) & (0.6999, 0.4948) \\ (0.4779, 0.5608) & (0.5350, 0.6899) & (0.4965, 0.5100) \end{bmatrix} \end{array}.$$

Then the criteria weight vectors computed by E_{Xue}, K_1, K_2, and K_3 are (0.3333, 0.3333, 0.3333), (0.2385, 0.3576, 0.4039), (0.2385, 0.3576, 0.4039), (0.2775, 0.2753, 0.4473), respectively. We observe that E_{Xue} gives the same weights to all criteria C_1, C_2 and C_3, whereas our proposed IFSST-knowledge measures give different weights.

Example 5. Consider the IFSST decision matrix DM_4.

$$\mathrm{DM}_4 = \begin{array}{c} \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad C_1 \quad & \quad C_2 \quad & \quad C_3 \quad \end{array} \\ \begin{bmatrix} (0.5028, 0.4999) & (0.6533, 0.5999) & (0.5725, 0.4949) \\ (0.6500, 0.4595) & (0.5352, 0.6898) & (0.6999, 0.4948) \\ (0.4779, 0.5608) & (0.5350, 0.6899) & (0.4965, 0.5100) \end{bmatrix} \end{array}.$$

Then the criteria weight vectors computed by E_{T}, K_1, K_2, and K_3 are (0.3333, 0.3333, 0.3333), (0.2761, 0.4100, 0.3139), (0.2761, 0.4100, 0.3139), (0.3373, 0.3345, 0.3281), respectively. We observe that E_{T} gives the same weights to all criteria C_1, C_2 and C_3, whereas our proposed IFSST-knowledge measures give different weights.

Therefore, from Examples 2-5, we observe that the existing PF-entropy (IFSST- entropy) measures give the same criteria weights, whereas our proposed HF-knowledge measures give different weights for different criteria. This demonstrates the superiority of the proposed IFSST-knowledge measures over some of the existing PF-entropy (IFSST-entropy) measures.

In the next section, we demonstrate the application of our proposed IFSST-knowledge measures in MCDM using the TOPSIS method in the IFSST environment.

5. Multi-criterion decision-making TOPSIS method based on new IFSST- knowledge measures

In this section, we resolve the problem of MCDM with IFSST data through the TOPSIS method (Yang and Hussain [42]).

Scenario: Consider a set of *m*-alternatives $A = \{A_1, A_2, ..., A_m\}$ and a set of *n*-criteria $C = \{C_1, C_2, ..., C_n\}$. Let $w_j = (w_1, w_2, ..., w_n)$ be the given weight vector of criteria with $0 \le w_j \le 1,\ \ j = 1,\ 2,\ ...,\ n$ and $\sum_{j=1}^{n} w_j = 1$.

Aim: To select the best alternative out of *m*-available alternatives.

Algorithm

The key steps for the Pythagorean fuzzy (or IFSST) TOPSIS method concerning the proposed IFSST-knowledge measures are summarized in six steps.

Step 1. IFSST decision matrix construction. Construct the IFSST decision matrix $D = [d_{ij}],\ i = 1, ..., m,\ j = 1, ..., n,$ where $d_{ij} = (\mu_{ij},\ \nu_{ij},\ \pi_{ij})$ represent the Intuitionistic Fuzzy Number of Second Type (IFNST). μ_{ij} is the fulfilment degree, ν_{ij} is the non-fulfilment degree and π_{ij} is the hesitation degree of the alternative A_i satisfying the criteria C_j.

Step 2. Criteria weights computation. In MCDM, the determination of criteria weights plays a key role. There are two ways to assign weight to each criterion, first is, weights are given by the decision-maker to each criterion, these are called subjective weights and the second is, weights are calculated with the help of some model, these are called objective weights.

Suppose the information about criteria weights is completely unknown, then we compute the criteria weights with the help of proposed IFSST-knowledge measures as

$$w_j = \frac{K_j}{\sum_{j=1}^{n} K_j},\ \ j = 1, 2, ..., n.$$

Here, w_j is the objective weight of the *j*-th criterion c_j.

Step 3. Determination of IFSST Positive Ideal Solution (IFSSTPIS) and IFSST Negative Ideal Solution (IFSSTNIS). In general, in a TOPSIS method, it is necessary to calculate Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS). By the TOPSIS method and IFSST principle, the IFSSTPIS is given as

$$A^+ = \left\{ \left\langle C_j,\ \left(\mu_j^+, \nu_j^+, \pi_j^+\right) \right\rangle \mid j = 1,\ 2,\ ...,\ n \right\},\ \text{where}\ \left(\mu_j^+, \nu_j^+, \pi_j^+\right) = (1, 0, 0).$$

In a similar way, the IFSSTNIS is given as

$$A^- = \left\{ \left\langle C_j,\ \left(\mu_j^-, \nu_j^-, \pi_j^-\right) \right\rangle \mid j = 1,\ 2,\ ...,\ n \right\},$$

where, $\left(\mu_j^-, \nu_j^-, \pi_j^-\right) = (0, 1, 0)$.

Step 4. Computation of weighted distance measures from IFSSTNIS and IFSSTPIS. Compute the weighted distance D_i^+, D_i^- of alternative A_i i = 1, 2,*...,* *m*, from the IFSSTPIS and IFSSTNIS by using the following equations:

$$D_i^+ = \sqrt{\frac{1}{2} \sum_{j=1}^{n} w_j \left[\left(1 - \mu_{ij}^2\right)^2 + \left(\nu_{ij}^2\right)^2 + \left(1 - \mu_{ij}^2 - \nu_{ij}^2\right)^2 \right]},$$

$$D_i^- = \sqrt{\frac{1}{2}\sum_{j=1}^{n} w_j \left[\left(\mu_{ij}^2\right)^2 + \left(1-\nu_{ij}^2\right)^2 + \left(1-\mu_{ij}^2-\nu_{ij}^2\right)^2\right]}.$$

Step 5. Computation of closeness coefficient. The closeness coefficient of each alternative A_i with respect to IFSSTNIS and IFSSTPIS is given as

$$M(A_i) = \frac{D_i^-}{D_i^+ + D_i^-}.$$

Step 6. Alternatives ranking. The alternative with the highest closeness coefficient is the best alternative.

Now, we implement the above-mentioned IFSST knowledge measure based TOPSIS algorithm in the following illustrative MCDM problem.

Problem Statement. Suppose a student has to choose the best college out of the three colleges A_i, $i = 1, 2, 3$ for taking admission after completion of his higher secondary studies. For choosing the best college, the student considers the following three criteria: C_1 = Quality of education; C_2 = Placement; C_3 = Institution ranking. The decision-maker provides the evaluation values of the three colleges A_i, $i = 1, 2, 3$, in terms of IFSST as shown in the decision matrix given in Table 2.

Table 2. IFSST decision matrix

Alternative	C_1	C_2	C_3
A_1	(0.5000, 0.4999)	(0.6533, 0.5999)	(0.7001, 0.4949)
A_2	(0.6500, 0.4595)	(0.5352, 0.6898)	(0.6999, 0.4948)
A_3	(0.4779, 0.5608)	(0.5350, 0.6899)	(0.4965, 0.5100)

Compute the criteria weights with the help of Step 2. The computed values of the criteria weights are shown in Table 3.

Table 3. Criteria weights using, PF-entropy (IFSST-entropy) and proposed IFSST-knowledge measures

Measure	w_1	w_2	w_3
K_1	0.2385	0.3576	0.4039
K_2	0.2385	0.3576	0.4039
K_3	0.2775	0.2753	0.4473
e_{PI}	0.2811	0.3875	0.3314
$e_{\mathrm{min/max}}$	0.3333	0.3333	0.3333
e_{Xue}	0.2032	0.3841	0.4128
E_{T}	0.3039	0.3023	0.3938

From Table 3, we observe that the PF-entropy (IFSST-entropy) $e_{\mathrm{min/max}}$ is assigning equal weights to all the criteria, so, it is not reasonable to use this

PF-entropy (IFSST-entropy) for the MCDM problem considered here. Now, with the help of Step 3 and Step 4, we determine the IFSST positive ideal and IFSST negative ideal solution and calculate the weighted distance D_i^+, D_i^- of alternative A_i, (i = 1, 2, ..., m), from the IFSSTPIS and IFSSTNIS. The calculated values are given in Table 4.

Table 4. Distance of alternatives from IFSSTPIS and IFSSTNIS

Measure	D_i^+, D_i^-	A_1	A_2	A_3
K_1	D_i^+	0.5227	0.5307	0.6534
	D_i^-	0.5998	0.5717	0.5783
K_2	D_i^+	0.5227	0.5307	0.6534
	D_i^-	0.5998	0.5717	0.5783
K_3	D_i^+	0.5272	0.5171	0.6565
	D_i^-	0.6048	0.5859	0.5919
e_{PI}	D_i^+	0.5340	0.5387	0.6528
	D_i^-	0.6021	0.5724	0.5701
e_{Xue}	D_i^+	0.5159	0.5337	0.6522
	D_i^-	0.5966	0.5648	0.5752
E_T	D_i^+	0.5347	0.5239	0.6558
	D_i^-	0.6060	0.5852	0.5853

With the help of Step 5, we compute the closeness coefficient of each alternative $M(A_i),\ i=1,\ 2,\ 3$ and the evaluated values are given in Table 5.

Table 5. Closeness coefficients of alternatives

Measure	$M(A_1)$	$M(A_2)$	$M(A_3)$
K_1	0.5344	0.5186	0.4695
K_2	0.5344	0.5186	0.4695
K_3	0.5343	0.5312	0.4741
e_{PI}	0.5300	0.5151	0.4622
e_{Xue}	0.5363	0.5142	0.4686
E_T	0.5313	0.5276	0.4716

In view of the Table 5, we rank the alternatives in the decreasing order of closeness coefficients. The ranking of alternatives using all the considered entropy and knowledge measures is found to be same, i.e., $A_1 > A_2 > A_3$ and A_1 is the best alternative. Therefore, the performance of our proposed IFSST-knowledge measures is consistent with the existing Pythagorean fuzzy (IFSST) entropies in the hypothetical example considered here. But, in view of the distinctive features of our proposed IFSST-knowledge measures from the aspect to linguistic hedges, and weight computation, these seem to outperform the existing PF-entropy (IFSST-entropy) measures on a larger dataset.

6. Conclusion

In this paper, we have introduced some IFSST-knowledge measures along with some of their properties. Through comparative study, we have shown the effectiveness of the proposed IFSST-knowledge measures over some of the existing PF-entropy (IFSST-entropy) measures while dealing with linguistic hedges, criteria weight computation in MCDM problems. However, certain situations may exist in which none of our proposed IFSST-knowledge measures is suitable but, some existing PF-entropy (IFSST-entropy) may be suitable. Still, the utility of our proposed measures cannot be undermined, as, these are also handling many ambiguous situations in which existing PF-entropies (IFSST-entropy) fail (refer to Sections 4.1 and 4.2) from various viewpoints. We have also demonstrated the application of the proposed IFSST-knowledge measures in the MCDM problem with IFSST data.

In the context of our present work, our future study includes:

1. One parametric and two parametric generalizations of the proposed IFSST-knowledge measures in IFSST environment.

2. Development of distance, similarity and accuracy measures for IFSST with their applications in image segmentation and pattern recognition.

3. Development of knowledge/generalized knowledge measures for hesitant IFSST.

4. Development of knowledge/generalized knowledge measures for dual hesitant IFSST.

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